Dark spatial soliton break-up in the transverse plane

G.S. McDonald [1], K.S. Syed [2] and W.J. Firth

Department of Physics and Applied Physics, University of Strathclyde, 107 Rottenrow, Glasgow G4 0NG, UK

We report on numerical simulations investigating the instabilities which arise when dark soliton solutions of the two-dimensional (2D) nonlinear Schrödinger equation are allowed to propagate in a 3D self-defocussing medium. Firstly, propagation of fully 3D gaussian beams is studied. Then, on plane background, small sinusoidal and random perturbations are considered. We demonstrate break-up of the dark soliton profile into patterns of dark spots which are identified as phase singularities.

1. Introduction

Nonlinear systems are common in nature and often require approximations, computational methods or both to extract information. In a wide class of systems, solitons have been found to be simple exact solutions and in an even greater number of related systems their characteristics are manifest and can dominate at least part of the nonlinear evolution. The context of this paper is that of transverse effects in passive nonlinear optics and the consequences of allowing the second (transverse) dimension in the diffractive laplacian [1,2]. Although considerations are tailored to current research in that field, results from this simple model may be relevant to a wide range of physical problems. More specifically, we study the nonlinear Schrödinger equation (NLS) in both two and three dimensions.

2. One transverse dimension

Beam propagation in a Kerr medium under waveguide geometry (i.e. with restriction to one transverse dimension) is described by the following 2D NLS equation [3]

$$i\frac{\partial F}{\partial z} + \frac{\alpha}{2}\frac{\partial^2 F}{\partial x^2} + \eta|F|^2 F = 0 , \tag{1}$$

where x is the coordinate orthogonal to the propagation direction (z). α defines the scale of the transverse coordinates and thus parameterises diffraction. η is positive for a self-focussing medium and negative in the self-defocussing case. The electric field, F, is scaled such that η has unit magnitude and its sign flags whether the local nonlinear refractive index increases or decreases with $|F|^2$, the intensity of light. It is the defocussing nonlinearity we are concerned with here and thus the dark soliton solutions [4] to eq. (1).

For $\alpha = 1$, the initial value problem with $F(x, z=0) = \tanh(x)$ is a "kink" in the real part of the electric field (odd in x) and observable as a dip in the otherwise flat background of the light intensity. Most importantly, it initiates the stable propagation of a dark soliton. To realise this in experiment or computation one requires a uniform light background of infinite extent. We either make a numerical approximation and investigate propagation within spatially periodic lattices or consider the manifestation of such structures on a modulated background – a gaussian beam. To ensure spatial periodicity in the former cases, it is convenient to examine not the evolution of one structure (with odd symmetry) but instead to simultaneously simulate *pairs*. The separation of each member of the pair can be chosen so as not to affect the important qualita-

[1] Present address: Optique Non-Linéaire Théorique, Université Libre de Bruxelles, Campus Plaine CP 231, 1050 Bruxelles, Belgium.
[2] Present address: Department of Physics, Imperial College, London SW7 2AZ, UK.

tive features of the central phenomena, except when interaction effects are themselves under study.

In fig. 1 we illustrate the stability of such a dark soliton pair and demonstrate that, by virtue of their sharp and strongly localised profile, they are in fact weakly interacting. The initial condition used here is constructed from two kinks of equal amplitude and opposite sense and is given by

$$F(x, z=0) = \tanh(x-\Delta), \qquad \text{if } x>0,$$
$$= -\tanh(x+\Delta), \quad \text{if } x<0. \qquad (2)$$

The choice of kink amplitudes and their separation parameter, Δ, allows $F(x, z=0)$ to be common to both structures and easily and obviously defined.

The general dark soliton solution has a single parameter which determines both the soliton intensity and its transverse velocity and corresponds to a real discrete eigenvalue from the inverse scattering problem [4,5]. In fig. 1, there is an absolute zero in the electric field at the centre of each soliton and this feature is associated with the soliton solution which has zero transverse velocity. Other eigenvalues imply a non-stationary soliton whose dip in intensity does not reach zero. These "grey solitons" maintain the usual invariance or "soliton area" (dip intensity times the square of the soliton width) by being wider than their dark counterparts.

For a focussing nonlinearity (quasi-)bound states of, individually attracting, bright solitons exist lead-

ing to the so-called higher order solitons [6]. There is no known equivalent for dark solitons – a fact that may be attributed to their mutual repulsion [5]. Performing a series of initial value problems in which the initial kink amplitudes are varied one observes dark soliton fission and interactions. A particular example is shown in fig. 2 where

$$F(x, z=0) = 2 \tanh(x-\Delta), \qquad \text{if } x>0,$$
$$= -2 \tanh(x+\Delta), \quad \text{if } x<0. \qquad (3)$$

In this case each initial dark profile breaks up into three solitons, thus preserving the system symmetry [7]. Clean (non-radiative) propagation and interactions at the periodic boundaries can be clearly seen.

3. Two transverse dimensions

Allowing two transverse dimensions, $\eta > 0$ may lead to filamentation (solution "blow-up" on propagation) but recently much interest [8-10] has centred around the case $\eta < 0$ and the stability of dark soliton "stripe" solutions. While the Kerr effect remains local, including a further transverse dimension is allowing diffraction to occur in both the x- and y-directions and leads to consideration of the 3D NLS in which the field evolves according to

$$i\frac{\partial F}{\partial z} + \frac{\alpha}{2}\left(\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2}\right) + \eta |F|^2 F = 0. \qquad (4)$$
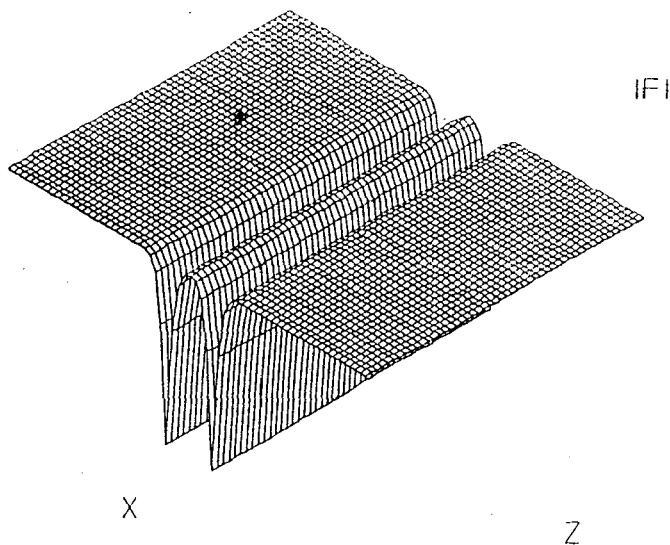

Fig. 1. Isometric plot of the evolution of field modulus of two adjacent dark solitons. X is the transverse dimension while propagation is from $z=0$ to 20.
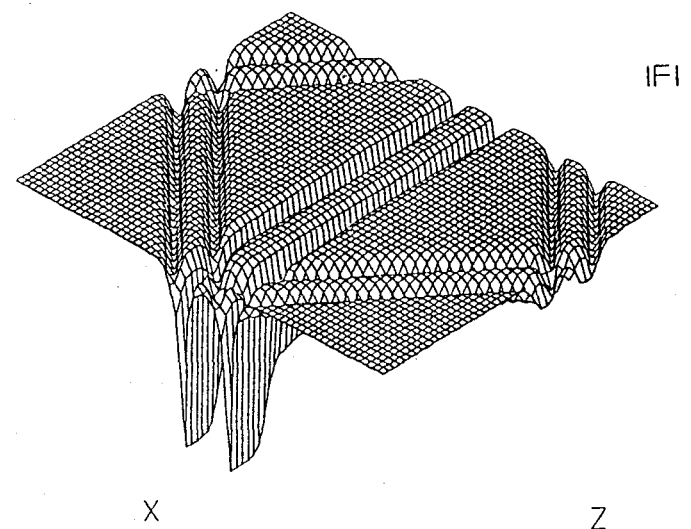

Fig. 2. Dark soliton fission and interactions. Parameters are as in fig. 1 except that the initial amplitudes are doubled.

Most recent attempts to experimentally realise dark spatial solitons have involved two transverse dimensions. In particular, the propagation of transverse patterns of dark stripes and grids through cells filled with a defocussing medium such as sodium vapour have been examined [8] and the persistence of the initial patterns attributed to soliton effects. Further to this, good agreement of the spatial velocities of such structures with those predicted from soliton theory has been obtained [9]. In those studies dark spatial stripes were initialised by masking in the transverse plane, i.e. amplitude modulation of the laser beam before it enters the nonlinear cell. Such masking produces "even" initial conditions which break up into an even number of solitons [5]. An alternative method is that of phase modulation where all one may require is a glass slide to obscure half of the beam and thus, for example, creating the necessary spatial phase difference for a single dark soliton stripe. This method has been adopted for studies of propagation through bulk semiconductors [10] and may create "odd" initial conditions. In that work monitoring of the constancy of soliton area gave additional evidence for solitonic phenomena.

We have found that in the full 3D problem both figs. 1 and 2 can be essentially replicated over the same propagation distance. At each point in the propagation, structure in x is that of the 2D simulations and it is reproduced uniformly across the x–y plane. That the dark soliton solutions are also a solution of the 3D NLS can be seen by trivially setting the additional laplacian term to zero. We thus here state further evidence to suggest the relevance of 2D soliton structures in full 3D propagation. In particular, the ability to reproduce not only 2D dark soliton fission but also 2D interaction sequences uniformly (in y) across the transverse plane.

4. Gaussian beam effects

We have performed both 2D and 3D calculations to determine the evolution of dark soliton stripes on gaussian beams. In this section, only results for the 3D case will be shown.

Generally, we define initial conditions of the form

$$F(x, y, z=0) = A \exp[-(x^2+y^2)/r_0^2]$$
$$\times \tanh[(y-y_0)/y_s] , \tag{5}$$

where the initial gaussian beam waist and dark stripe width are parameterised by r_0 and y_s, respectively. For the 2D problem it has been shown [7] that here the dark structures are, strictly, not solitons as one no longer finds real eigenvalues in the scattering problem. On the other hand, as the relative size of the beam (to the soliton width) approaches infinity then the eigenvalue returns smoothly to the real axis [7]. Thus, depending on relative length scales, we expected soliton-like behaviour to influence the evolution to some extent. Numerical work has already been performed on this problem [11] but, firstly, only in the 2D case and, secondly, initial off-axis dark soliton profiles were not considered – only grey solitons which were generated at beam centre with an implicit finite transverse velocity. Two transverse dimensions have been allowed in related work [12] but in this case propagation effects were removed and instead the system was periodically driven.

This problem is a further departure from the plane wave background case as, in addition to local background intensity curvature and diffractive beam spreading, the gaussian beam shape alone leads to spatial "chirping" i.e. a variation across the beam arising from self phase modulation [11].

Our findings here apply to both the 2D and 3D problems where a full range of (A, y_0, y_s, r_0) parameters have been investigated. In fig. 3 the transverse field profile is shown after propagation of a centrally defined dark stripe $(y_0=0)$. In this case, and with the above considerations, there is a remarkable maintenance of the soliton profile. Note that here there is a background gaussian variation both traversely *and* along the stripe. On propagation, the dark profile simply spreads while retaining the essential phase character of a truly dark soliton.

In fig. 4 an off-centre dark stripe is seeded, initially defining a line of exactly zero electric field. The result is that the off-axis structure rapidly becomes close to a grey soliton by broadening to a finite intensity dip. That these cases become grey does not seem to depend on the amplitude of the background beam but, instead, happens as a consequence of the local intensity curvature of the host beam. Considering the potentials that arise from the nonlinear re-
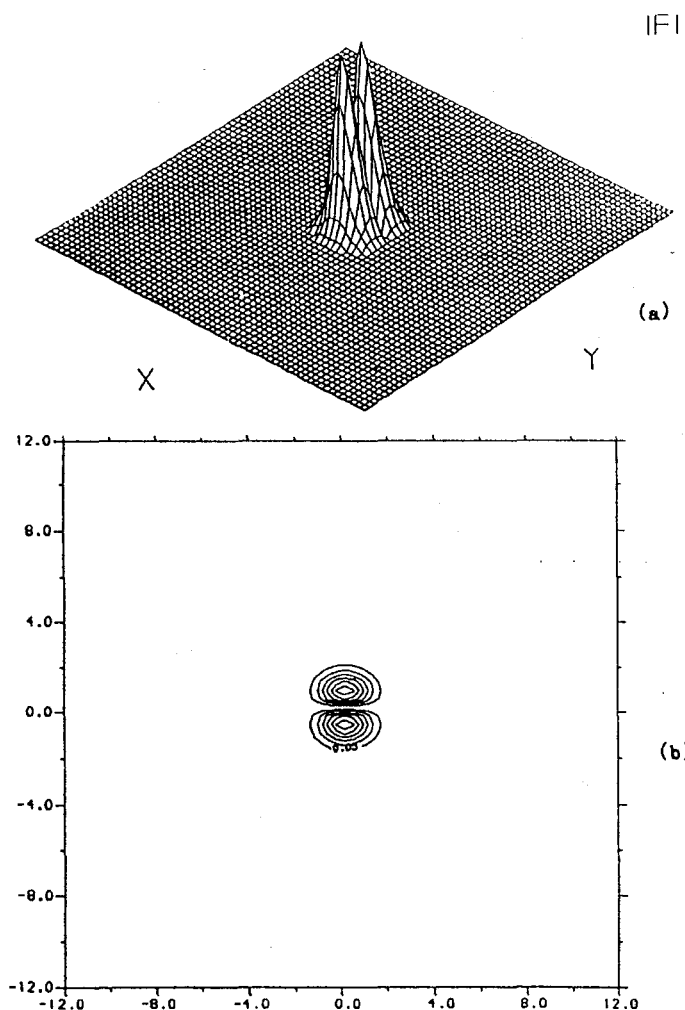
Fig. 3. Evolution of a dark soliton profile centrally positioned on the beam ($A = 1$). Both the surface (a) and contour (b) representations of field modulus correspond to $z = 0.2$.

Fig. 4. Evolution of a dark soliton profile initially positioned off beam centre ($A = 1$). Both the surface (a) and contour (b) representations of field modulus correspond to $z = 0.2$.

fractive index, one expects off-axis structures to experience unbalanced transverse "forces" [13]. Acquisition of a finite transverse velocity would be a consequence which is consistent with the initially dark structures becoming grey on propagation. We have also simulated the evolution of large numbers of soliton stripes on such beams. Our findings in this case is that the spreading of the off-centre members leads to overlap and poor definition and contrast.

5. Transverse perturbations

Asymptotically and/or in real systems dark solutions may be subject to inhomogeneities which arise in the medium or input beam. To address the problem of long term stability of dark stripe solutions the
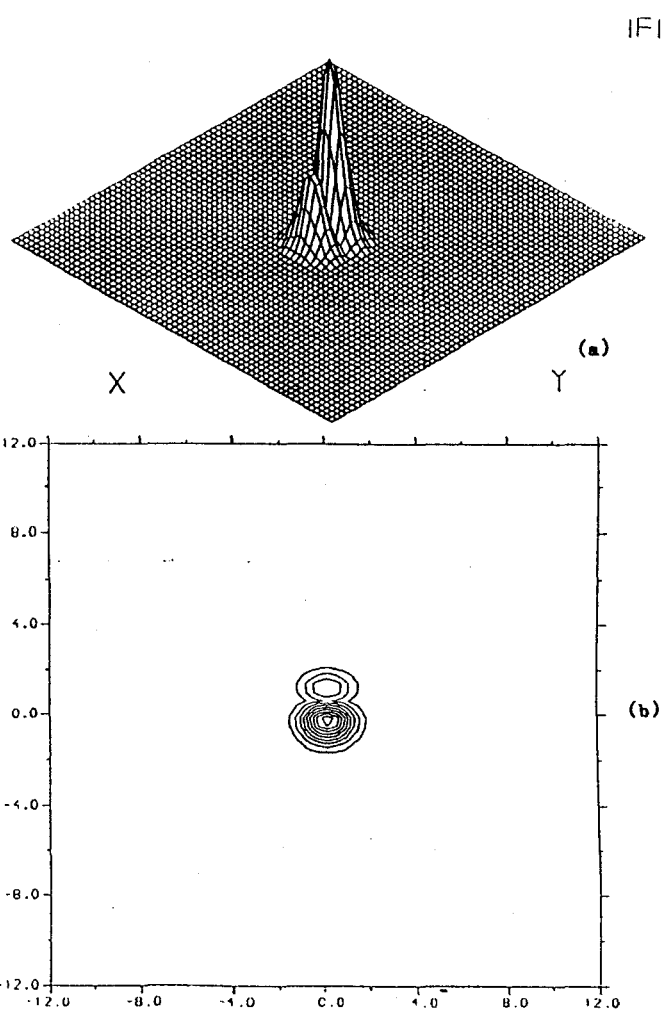
first step is to consider a small and simple perturbative contribution arising from the otherwise dormant laplacian term. For this we examine the case of a sinusoidal variation in the position of the soliton centre as the initial profile is reproduced across the transverse plane. The key question is whether any lateral "disturbances" will be damped and die whereby the dark solution would be an asymptotic fixed point of the system or whether this perturbation will grow and eventually destroy the soliton profile. This issue is clearly of both great practical and fundamental interest since laser beams are normally allowed to evolve with two transverse dimensions of freedom.

A numerical stability analysis into the robustness of the 3D dark solution was carried out by introducing a small x-dependence in the definition of one

of a pair of dark stripes. The component of the initial condition ($y>0$) which gives rise to the perturbed stripe is the following

$$F(x, y, z=0) = \tanh[y-\Delta+\epsilon\sin(k_x x)] , \qquad (6)$$

where ϵ is a small parameter and k_x defines the period of the sine perturbation function. Smallness of ϵ ensures that any radiation shed will not greatly corrupt the overall simulation while the stability of a single dark soliton can be tested by keeping the complementary (unperturbed) neighbour at a sufficient transverse distance.

In fig. 5 we show the evolution of two well-separated, and initially weakly interacting, dark solutions in the x-y transverse plane. A very small perturbation, $\epsilon=10^{-5}$, is amplified on propagation leading to break-up of the dark solution into spots. Note that the instability appears to grow quite independently of the neighbouring stripe. The instability growth rate tends to increase with k_x, but a detailed discussion is beyond the scope of the present work.

Also of interest are the long term states of this system. After both soliton profiles have broken down, the dark spots that are created appear to be more robust than the initial stripes and persist in an approximately stationary manner. A clue to their nature can be seen in fig. 6 where later developments are plotted. Here the dark spots appear to be paired. There also seems to an underlying component in the system which is continually changing. At this stage in the simulation one expects the periodic boundary conditions to have assumed a nontrivial role. Periodicity traps transverse waves within the computational box but even against this form of perturbation the dark spots seem relatively robust.

To verify the validity of this instability we have also tested the stability of the dark soliton profiles with respect to varying levels of gaussian filtered noise – included as part of the initial condition in the y-direction. A particular example is shown in fig. 7a where the randomness of the initial soliton centres was of order 10^{-6}. The final result shows a remarkable similarity with the sinusoidal perturbation simulations.

Since dark solutions are, evidently, unstable one generally seeks fully nonlinear stable structures in the plane. The existence of phase singular ("defect") solutions in 3D NLS-type equations has been known
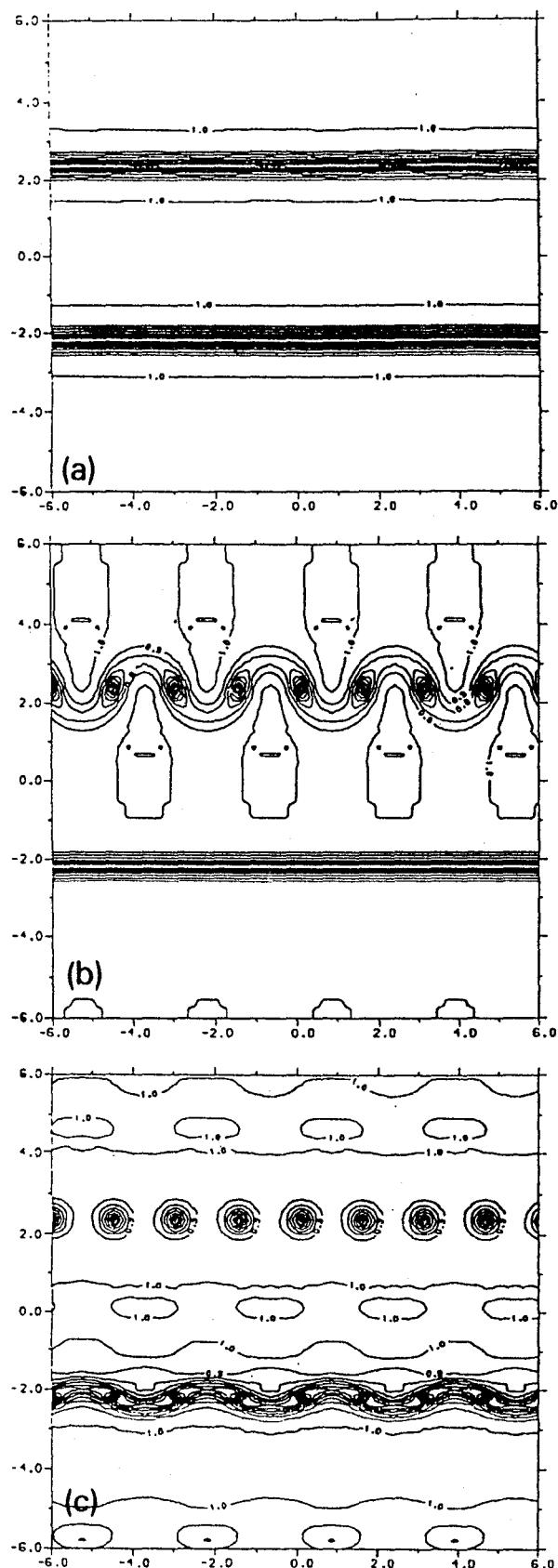

Fig. 5. Contour plots of field modulus showing dark soliton breakup in the x-y plane. A small sinusoidal perturbation of amplitude 10^{-5} is included, at $z=0$, to activate the otherwise dormant Laplacian term. x and y vary from -6 to 6 and the spatial frequency of the perturbation, k_x, is $2\pi/3$. (a) $z=30$; (b) $z=50$; (c) $z=80$.
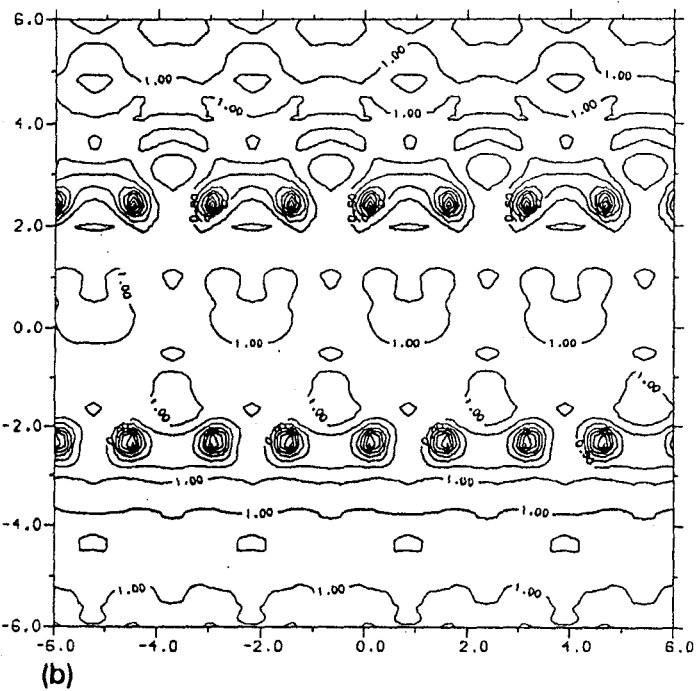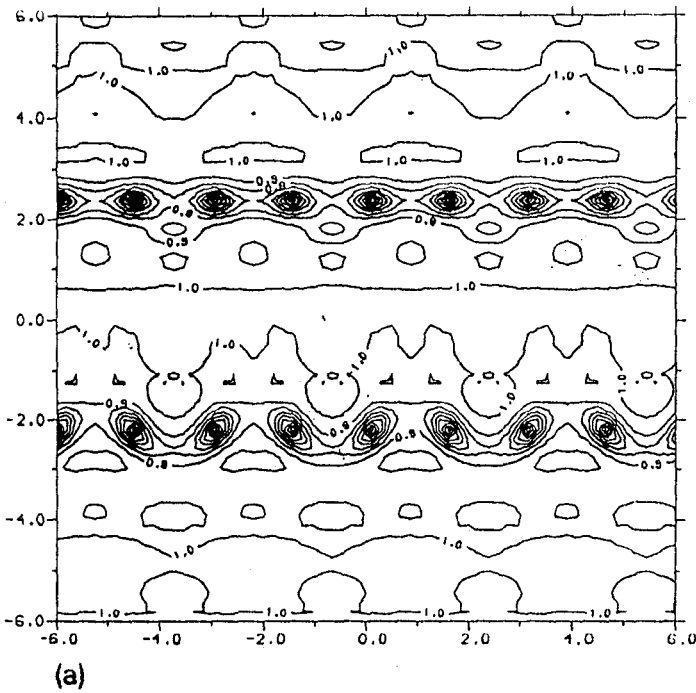
(a)

(b)

Fig. 6. Contours of the field modulus in the $x-y$ plane indicating pairing of the dark spots. Further developments of the simulation shown in fig. 5. (a) $z=100$; (b) $z=140$.

(a)

(b)

Fig. 7. Break-up of the dark soliton solution at $z=100$ – induced through random perturbations (of order 10^{-6}) included in the initial profile. (a) Contour plot of field modulus, (b) zeroes in the real or imaginary part of the field.

for some time [14]. The term singularity comes from the phase being undefined at the defect core since at this point the lines of zero real part and zero imaginary part of the field cross thus leading to an absolute zero in the field. To verify whether these dark spots are such singularities one needs to examine the phase of the field. In part (b) of fig. 7 these zero lines are plotted and, indeed, the observed dark spots can
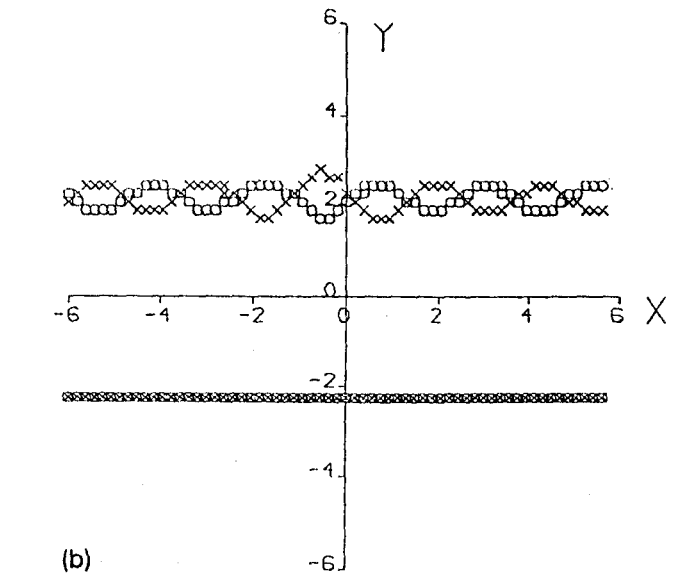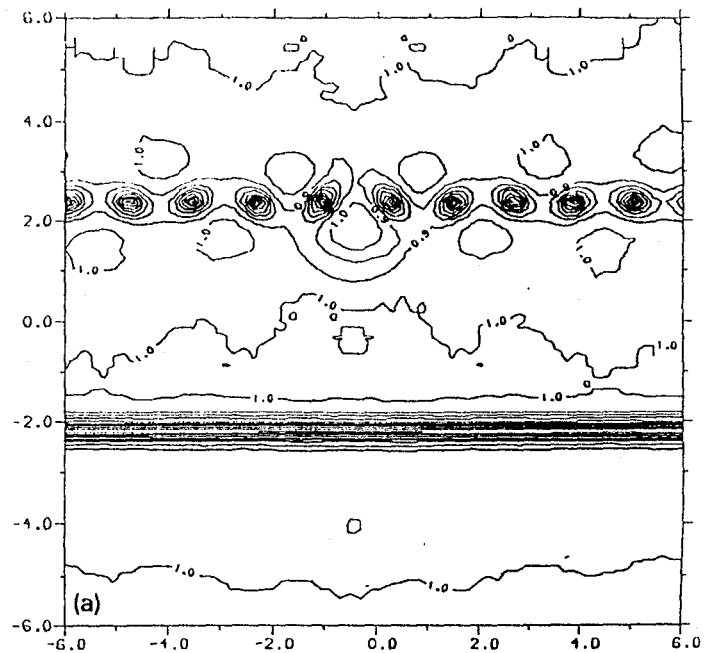
be identified as singularities. In fig. 8 the zero lines for the simulation of fig. 5 are shown. Note here that the lines are continually changing while, overall, the singular points are still retained. Such continual inter-changing is to be expected from the background plane wave solution alone. The topological nature of these defects renders them robust to perturbations, but their positions are not. The pairwise nature of the spots also becomes evident from considerations of the phase. Defects can be seen to have been both
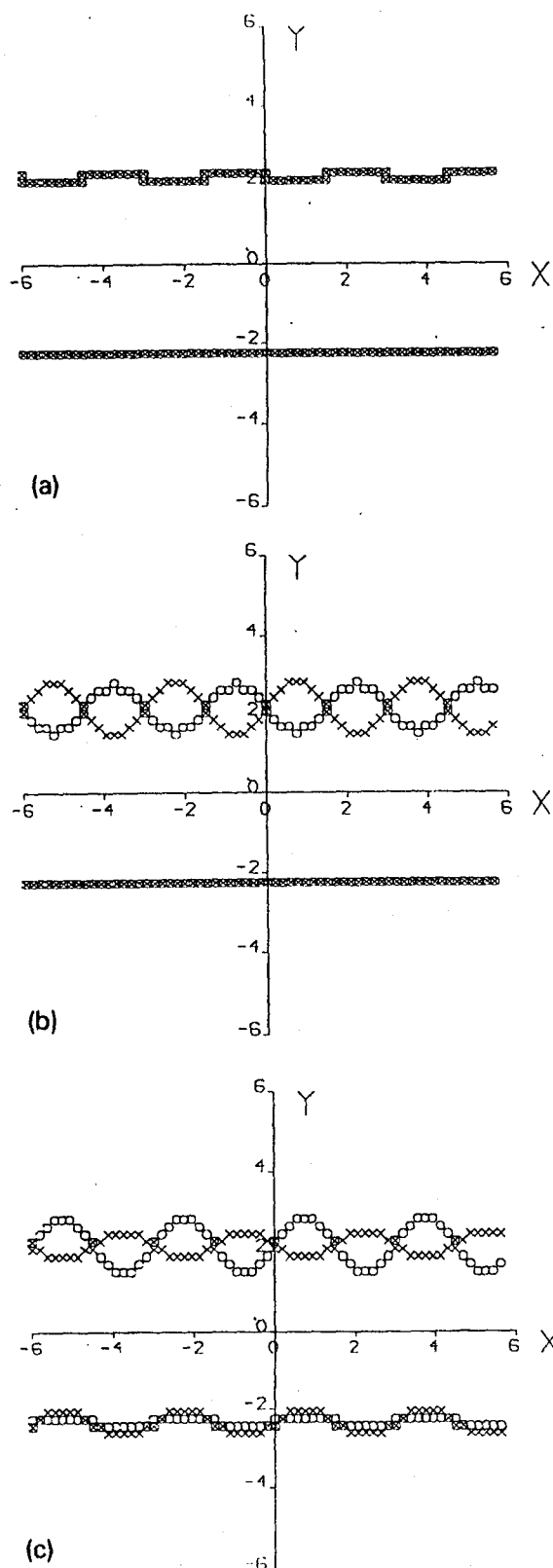
Fig. 8. Representation of dark soliton break-up using two iso-phase contours. Phase singularities in the field occur at the intersection of the given isophase lines. Each contour is composed of points in the transverse plane where either the real or imaginary part of the field passes through zero. Parameters are the same as those in fig. 5.

created and sustained as complementary pairs, which means that there are no phase discontinuities at the boundaries.

6. Conclusions

The stability of the dark spatial soliton of the non-linear Schrödinger equation has been numerically investigated when three space dimensions are allowed. Where initially no structure in one transverse dimension is present both fission and interaction of dark solitary stripes have been observed to happen, over reasonable propagation distances, as if the other transverse degree of freedom did not exist. On gaussian beams dark stripes off beam centre become rapidly broad and grey while an initial (odd) stripe defined on beam centre may retain its phase character with an absolute zero in the field and merely broaden in a solitonic manner. These observations have been made in both the 2D and 3D propagation problems. The stability of the dark stripe solutions have been tested against sinusoidal and filtered noise perturbations. In both cases similar results are obtained in which the soliton profile breaks down into more robust phase singularities.

It may be possible to generate laser beams clustered or lined with phase singularities using dark stripes as an intermediate pattern. Creation of the transient stripes may be through amplitude or phase modulation of the input beam. In the latter case the most essential instrument in their initial seeding is simply a glass slide through which part of the beam is passed so that a nonuniform phase profile is attained. There are, in fact, a very wide range of initial disturbances that are expected to evolve into dark solitons [7] and thus quite possibly lead to phase singularities.

Acknowledgements

a Twinning contract of the European Communities.

References

[1] N.B. Abraham and W.J. Firth, J. Opt. Soc. Am. B 7 (1990) 951.

[2] K.S. Syed, Optical Vortices and Spatial Dark Solitons in Passive Nonlinear Media, M.Sc. dissertation (Strathclyde University, October 1991); related independent work was reported by G.A. Swartzlander, Workshop on Computational optics, University of Arizona, March 1992.

[3] R.Y. Chiao, E. Garmire and C.H. Townes, Phys. Rev. Lett. 13 (1964) 479.

[4] V.E. Zakharov and A.B. Shabat, Sov. Phys. JETP 37 (1973) 823.

[5] K.J. Blow and N.J. Doran, Phys. Lett. A 107 (1985) 55.

[6] J. Satsuma and N. Yajima, Prog. Theor. Phys. Suppl. 55 (1974) 284.

[7] S.A. Gredeskul, Yu.S. Kivshar and M.V. Yanovskaya, Phys. Rev. A 151 (1990) 3994.

[8] G.A. Swartzlander Jr., D.R. Anderson, J.J. Regan, H. Yin and A.E. Kaplan, Phys. Rev. Lett. 66 (1991) 1583.

[9] D.R. Anderson, D.E. Hooton, G.A. Swartzlander Jr. and A.E. Kaplan, Optics Lett. 15 (1990) 783.

[10] G.R. Allan, S.R. Skinner, D.R. Andersen and A.L. Smirl, Optics Lett. 16 (1991) 156.

[11] W.J. Tomlinson, R.J. Hawkins, A.M. Weiner, J.P. Heritage and R.N. Thurston, J. Opt. Soc. Am. B 6 (1989) 329.

[12] S.G. Wenden, H.A. Adachichara and J.V. Moloney, in: Nonlinear dynamics in optical systems, OSA Proc. vol. 7, eds. N.B. Abraham, E. Garmire and P. Mantel (Optical Society of America, 1991) p. 188.

[13] G.S. McDonald and W.J. Firth, J. Opt. Soc. Am. B 7 (1990) 1328.

[14] A.L. Fetter, Phys. Rev. 151 (1966) 100.