Fast Algorithm for Studying the Evolution of Optical Solitons under Perturbations

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***Abstract*— A numerical algorithm is proposed for computing the eigenvalues of the linear scattering problem associated with the nonlinear Schrödinger equation. The numerical method allows a fast calculation of the dynamical evolution of the properties of optical solitons under a wide range of perturbations.**

***Index terms*—Nonlinear optics, nonlinear wave propagation, optical fibers, optical solitons, solitons.**

I. Introduction

The nonlinear Schrödinger equation (NSE) provides a model for studying nonlinear pulse propagation in optical fibers, as well as a description of the self-lensing of optical beams in two-dimensional nonlinear Kerr media [1]. Since they were first proposed as a basis for high bit rate transoceanic links [2], temporal solitons in optical fibers have been the subject of intense study. Spatial solitons have also received much attention, having potential applications in optical switching and memory devices [3]–[5]. However, in practice, soliton propagation can be affected by a wide variety of effects in addition to the dispersive/diffractive and self-phase modulation effects that are already accounted for in the NSE. These new effects can be introduced into the nonlinear evolution equation as higher-order terms and lead to modifications of the propagation characteristics of optical solitons.

Computation of the discrete spectrum of the direct scattering problem can reduce the full complexity of the nonlinear evolution problem to the analysis of only the solitonic components. In this paper, we propose a novel method for computing the evolution of this scattering data. The algorithm is suitable for studying a wide range of physical problems which typically arise in experiments dealing with spatial or temporal optical soltions. Our method offers a very low computational load when compared to previous approaches. A key difference in our approach is that the evolution of the scattering data is tracked by employing a refinement technique.

Manuscript received June 1, 1998.
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This work was partly supported by CICYT PB97-0487

In section II, we give a brief overview of background theory and outline the inverse scattering technique associated with the NSE. We focus on the numerical analysis of the direct scattering spectrum in section III. Finally, applications of the new algorithm, to problems in the spatial and temporal domains, are presented in section IV.

II. Inverse Scattering Theory

The nonlinear Schrödinger equation (NSE),

$$i\frac{\partial u}{\partial Z}+\frac{1}{2}\frac{\partial^2 u}{\partial T^2}+|u|^2u=0, \tag{1}$$

describes the evolution of a normalized electric field envelope $u(T,Z)$, where T is the temporal (or transverse spatial) coordinate normalized to the initial pulse duration (or beam width) τ. Z is the propagation coordinate normalized to the dispersion (or diffraction) length. Equation (1) belongs to a class of integrable nonlinear partial differential equations which can be solved using inverse scattering theory. This analytical technique can be thought of as a generalization of the Fourier transform method for solving linear partial differential equations.

The linear eigenvalue problem associated with the NSE[6],

$$\begin{aligned} i\frac{d\psi_1}{dT}+u_0(T)\psi_2&=\zeta\psi_1\\ -i\frac{d\psi_2}{dT}-u_0(T)^*\psi_1&=\zeta\psi_2, \end{aligned} \tag{2}$$

where the superscript $*$ denotes the complex conjugate, defines a spectral problem in which ζ is the complex wavenumber and the initial field envelope $u_0(T)=u(T,Z=0)$ plays the role of a scattering potential. $\int_{-\infty}^{+\infty}|u_0|dT<\infty$ is assumed for the initial condition and $|u_0|$ is required to decrease faster than any power function when $|T|\to\infty$. The NSE is recovered from the compatibility condition $\Psi_{ZT}=\Psi_{TZ}$ [6] when an appropriate Z evolution is defined for the complex wave function $\Psi=(\psi_1,\psi_2)^t$ (superscript t denotes transposition). Once the solution Ψ is given, the linearly independent solution $\bar{\Psi}=(\psi_2^*(\zeta^*),-\psi_1^*(\zeta^*))^t$ completes a system of solutions for (2). Jost functions are solutions of (2) which are defined by the following asymptotic behaviours [7]

$$\Phi(T;\zeta) \to \begin{pmatrix}1\\0\end{pmatrix} \exp(-i\zeta T),\ T \to -\infty$$
$$\Psi(T;\zeta) \to \begin{pmatrix}0\\1\end{pmatrix} \exp(i\zeta T),\ T \to \infty \quad (3)$$

Since $\{\Psi, \bar{\Psi}\}$ form a complete set for (2), one can express $\Phi(T;\zeta)$ and $\bar{\Phi}(T;\zeta)$ as a linear combination of these solutions

$$\Phi(T;\zeta) = a(\zeta)\bar{\Psi}(T;\zeta) + b(\zeta)\Psi(T;\zeta)$$
$$\bar{\Phi}(T;\zeta) = \bar{b}(\zeta)\bar{\Psi}(T;\zeta) - \bar{a}(\zeta)\Psi(T;\zeta). \quad (4)$$

$a(\zeta)$, $b(\zeta)$, $\bar{a}(\zeta)$ and $\bar{b}(\zeta)$ are the *scattering coefficients.* They verify $a\bar{a} + b\bar{b} = 1$, $\bar{a}(\zeta) = a^*(\zeta^*)$ and $\bar{b}(\zeta) = b^*(\zeta^*)$, and define the scattering matrix

$$A = \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix} \quad (5)$$

such that $[\Phi, -\bar{\Phi}] = [\Psi, \bar{\Psi}]A$.

The roots $\{\zeta_n\}_{n=1}^{n=N}$ of $a(\zeta)$ with $\mathrm{Im}\{\zeta\} > 0$ define the discrete spectrum and eigenfunctions which are bounded. Then,

$$\Phi(T;\zeta_n) = b(\zeta_n)\Psi(T;\zeta_n) \quad (6)$$

and Φ and Ψ approach zero as $|T|$ goes to infinity.

The *discrete scattering spectrum*, corresponding to the *soliton* components of the initial field envelope, $u_0(T)$, is defined as

$$\Sigma_d(Z=0) = \{\zeta_n, C_n(0)\}_{n=1}^{N} \quad (7)$$

where $C_n(0) = b_n(0)/a'_n(0)$ are the normalization constants for the bound states, $b_n = b(\zeta_n)$ and $a'_n = (\partial a/\partial\zeta)(\zeta_n)$. The *continuous spectrum*, corresponding to the *radiation* field, is

$$\Sigma_c(0) = \{r(\zeta;0) \text{ for } \zeta \text{ real}\} \quad (8)$$

where $r(\zeta;0) = b(\zeta;0)/a(\zeta;0)$.

The scattering potential $u(T,Z)$ can be recovered from the scattering spectrum $\Sigma_d \cup \Sigma_c$ by means of an integral equation [6]. The inverse scattering technique is completed with a description of the evolution of the scattering data

$$a(\zeta, Z) = a(\zeta, Z=0)$$
$$b(\zeta, Z) = b(\zeta, Z=0)\exp\left(-2i\zeta^2 Z\right). \quad (9)$$

In summary, the field envelope $u(T,Z)$ can be determined in three stages: (i) calculation of the scattering data for the initial condition $u_0(T)$, (ii) Z evolution of the scattering spectrum and (iii) reconstruction of the scattering potential at Z.

III. Numerical Analysis

The soliton eigenvalues, ζ_n, of the linear scattering problem are Z invariant when $u(T,Z)$ evolves according to the NSE. These eigenvalues can be written as

$$\zeta_n = \frac{\kappa_n + i\eta_n}{2}, \quad (10)$$

where η_n is the *soliton energy* and κ_n is the *soliton velocity.* κ_n is related to a shift of the optical carrier frequency in the temporal case and a transverse velocity in the spatial case. The discrete spectrum of the *direct scattering transform* (DST) contains all the information required for studying the solitonic components of the optical field.

The NSE describes the basic dispersive/diffractive and nonlinear effects that give rise to soliton formation, but one often needs to include additional physical effects to model accurately any particular experiment. For instance, when considering picosecond pulse propagation in optical fibers, it is sometimes also necessary to consider linear losses. For ultrashort pulse propagation, higher-order dispersion, pulse self-steepening and intra-pulse Raman scattering may become important [8]. In the description of spatial solitons in Kerr-like media, nonlinear loss due to two-photon absorption [9], or nonparaxial effects [10], [11], may be needed in the modeling. These additional effects give rise to higher-order terms in the evolution equation. Under such conditions, the soliton eigenvalues are no longer Z independent. Nevertheless, it is found that the evolving field envelope can still demonstrate soliton-like behaviour even when the characteristic soliton parameters are not constant. Thus, even though the perturbed evolution equation may no longer be integrable, the DST data can still prove to be an invaluable tool for interpreting the underlying structure of the nonlinear wave propagation.

There are various analytical approaches to calculating the evolution of the discrete scattering spectrum [12]. However, numerical computation of the DST, based on data obtained using a beam propagation method [13], offers a straightforward technique for determining the evolution of the field envelope while simultaneously permitting the analysis of this data in terms of the constituent nonlinear structures.

A. Previous Approaches

The numerical computation of the direct scattering transform starts with a discretization of the eigenvalue problem. In previous works [14], [15], the T coordinate (time or transverse space) is firstly discretized by considering the grid points $\{T_k\}_{k=1}^{K}$. The scattering potential in the interval $I_k = [T_k, T_{k+1}]$ is then approximated by its value at the mid point of this interval, $u_k = u_0((T_k + T_{k+1})/2)$. The corresponding scattering matrix $A_k(\zeta)$ can then be calculated analytically [7], [15],

$$a_k = \exp(i\zeta h_k)\left[\cos d_k - i\zeta h_k \frac{\sin d_k}{d_k}\right]$$
$$b_k = u_k h_k \exp\left(-i\zeta T_{k+1/2}\right)\frac{\sin d_k}{d_k}, \quad (11)$$

where $d_k = h_k\sqrt{\zeta^2+|u_k|^2}$ and $h_k = T_{k+1} - T_k$. The global scattering matrix A is now given by

$$A(\zeta) = \prod_{k=1}^{K} A_k(\zeta). \quad (12)$$

From (12), an approximation for $a(\zeta)$ can be obtained, yielding roots which give the discrete eigenvalue spectrum.

B. Proposed Method

It is commonly the case that the eigenvalues corresponding to the initial scattering potential are known and that one is interested in the subsequent evolution (governed by a perturbed NSE). We propose an algorithm for the fast computation of the evolving eigenvalues which is applicable to this case. At the heart of the method is the calculation of eigenvalues by an efficient refinement technique. The frequency of this refinement is determined by the strength of the perturbation and the resulting rate of change of the soliton characteristics.

Equations (2) are discretized using a finite difference scheme, defining a grid spacing ΔT and an end point T_0 in the T domain. $\psi^m = \psi(T_0 + m\Delta T)$ and $u_0^m = u_0(T_0 + m\Delta T)$ $(m = 1, 2, \ldots, M)$ are the values of the scattering function and the potential, respectively, at the node points. For the T derivatives, we use the approximation

$$\left(\frac{d\psi}{dT}\right)_{T_0+m\Delta T} = \frac{\psi^{m+1} - \psi^{m-1}}{2\Delta T} + O(\Delta T^2) \quad (13)$$

to obtain the following system of equations

$$\psi_1^{m+1} - \psi_1^{m-1} - 2i\Delta T u_0^m \psi_2^m = -2i\Delta T\zeta\psi_1^m$$
$$-\psi_2^{m+1} + \psi_2^{m-1} + 2i\Delta T (u_0^m)^* \psi_1^m = -2i\Delta T\zeta\psi_2^m. \quad (14)$$

Writing $\lambda = -2i\Delta T\zeta$, $\underline{\Psi}_1 = (\psi_1^1, \psi_1^2, \ldots, \psi_1^M)^t$ and $\underline{\Psi}_2 = (\psi_2^1, \psi_2^2, \ldots, \psi_2^M)^t$, we obtain a $(2M)\times(2M)$ matrix eigenvalue problem for λ

$$\begin{bmatrix} U & D \\ D^H & U^H \end{bmatrix}\begin{bmatrix} \underline{\Psi}_1 \\ \underline{\Psi}_2 \end{bmatrix} = \lambda \begin{bmatrix} \underline{\Psi}_1 \\ \underline{\Psi}_2 \end{bmatrix} \quad (15)$$

where the superscript H stands for the transpose complex-conjugate. Matrix U has only an upper and lower diagonal with nonzero elements (values of $+1$ and -1, respectively), whereas D is a diagonal matrix with elements $D_{kk} = -2i\Delta T u_0^k$. In deriving the matrix eigenvalue equation, we have used a finite domain in the T direction. The solution is assumed to be zero outside of this region. Thus, numerical calculations require a sufficiently wide computational window so that the scattering potential is well-resolved from the actual boundaries imposed.

At each propagation distance, Z, one wishes to find the eigenvalues of the matrix $A(Z) = A(U, D(Z))$, as defined in (15). In each case, the solution at a previous propagation point $Z' = Z - p\Delta Z$ is known (p is the number of beam propagation steps between computations of the DST data). We assume a slow variation of the soliton parameters over $p\Delta Z$ and use the *shifted inverse power method* to refine iteratively the eigenvalues at Z'. For each of the eigenvalues of the discrete spectrum, λ^l, we use the iterations

$$\left(A - \lambda_k^l I\right)\underline{y} = \underline{b}_k^l \quad (16)$$

$$\underline{b}_{k+1}^l = \frac{\underline{y}}{||\underline{y}||} \quad (17)$$

$$\lambda_{k+1}^l = \lambda_k^l + \frac{1}{\underline{b}_k^l \cdot \underline{y}}. \quad (18)$$

Fast convergence, within only a few iterations, is guaranteed by taking the initial guesses for λ_0^l and $\underline{b}_0^l$ as the corresponding eigenvalue and eigenvector of the previous DST computation. At each iteration, one is required to solve the sparse linear system (16), where $(A - \lambda_k^l I)$ is a $2M \times 2M$ matrix. We define an $M \times M$ matrix $\tilde{U}_k^l = U - \lambda_k^l I$ and vectors $\underline{y}_1$, $\underline{y}_2$, $\underline{b}_{1,k}$ and $\underline{b}_{2,k}$, such that $\underline{y}^t = [\underline{y}_1^t \underline{y}_2^t]$ and $\underline{b}_k^t = [\underline{b}_{1,k}^t \underline{b}_{2,k}^t]$, and rewrite this system in block matrix notation,

$$\begin{bmatrix} \tilde{U}_k^l & D \\ D^H & (\tilde{U}_k^l)^H \end{bmatrix}\begin{bmatrix} \underline{y}_1 \\ \underline{y}_2 \end{bmatrix} = \begin{bmatrix} \underline{b}_{1,k}^l \\ \underline{b}_{2,k}^l \end{bmatrix}. \quad (19)$$

It then follows that

$$\left(D - \tilde{U}_k^l (D^H)^{-1} (\tilde{U}_k^l)^H\right)\underline{y}_2 = \underline{b}_{1,k}^l - \tilde{U}_k^l (D^H)^{-1}\underline{b}_{2,k} \quad (20)$$

$$\underline{y}_1 = (D^H)^{-1}\left[\underline{b}_{2,k}^l - \left(\tilde{U}_k^l\right)^H \underline{y}_2\right]. \quad (21)$$

The problem is now reduced to the solution of the $M \times M$ band diagonal linear system given by (20). In transforming from (16) to (20), only an inversion of a diagonal matrix is required and the expression of the matrix of the final system is very straightforward to implement.

IV. Applications And Results

Higher-order solitons of the NSE are solutions with initial condition $u(T, 0) = N\text{sech}(T)$, where N is an integer greater than one. The higher-order solutions are multi-soliton bound states in which the pulse (or beam) profile continually evolves during propagation. The period of this evolution is given by $Z_0 = \pi/2$. However, the bound state can be destroyed by the presence of perturbations, leading to fission into individual soliton components. In this section, we consider higher-order soliton break-up in two different contexts: third-order dispersion effects on temporal solitons in optical fibers and two-photon absorption in spatial soliton propagation.

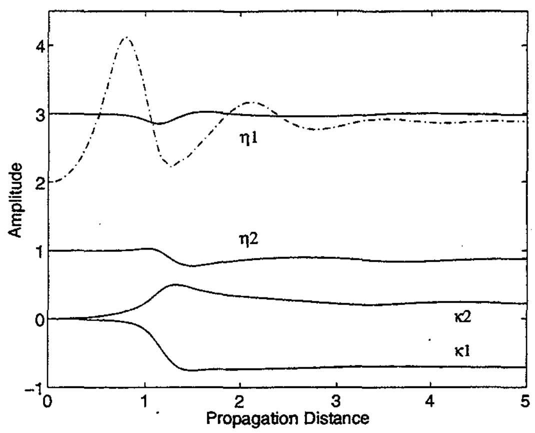

Fig. 1. Evolution of the soliton energies ($\eta 1$,$\eta 2$) and soliton velocities ($\kappa 1$, $\kappa 2$) (solid lines) of a $N = 2$ *temporal* soliton under TOD ($\delta = 0.03$). The dashed-dotted line shows peak *pulse* amplitude.

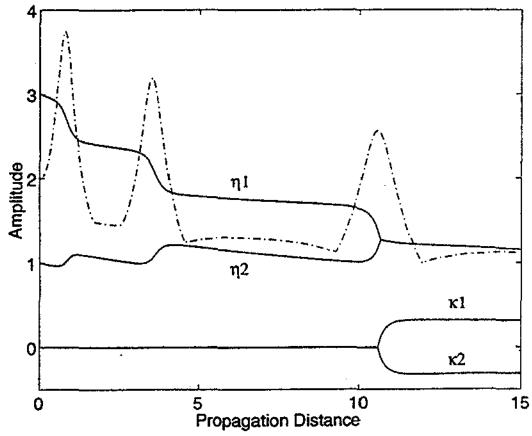

Fig. 2. Evolution of the soliton energies ($\eta 1$,$\eta 2$) and soliton velocities ($\kappa 1$, $\kappa 2$) (solid lines) of a $N = 2$ *spatial* soliton under TPA ($K = 0.01$). The dashed-dotted line shows peak *beam* amplitude.

A. Higher-Order Dispersion

The derivation of the NSE for optical pulse propagation in monomode fibers is based on a Taylor series expansion of the propagation constant β around the optical carrier frequency ω_0. Writing $\beta(\omega) = \beta_0 + \Delta\omega\beta_1 + 1/2\Delta\omega^2\beta_2 + 1/6\Delta\omega^3\beta_3 + \ldots$, where β_n is defined as $d^n\beta/d\omega^n$ evaluated at ω_0 and $\Delta\omega = \omega - \omega_0$, only the first three terms are usually retained. In the vicinity of the low-dispersion wavelength λ_D, where $\beta_2 \simeq 0$, or when ultrashort pulses are concerned, it can be necessary to include third-order dispersion (TOD) effects through the consideration of the contribution of β_3 to β. Then, the extra term $-i\delta(\partial^3 u/\partial T^3)$ has to be included in the evolution equation. The coefficient in this term is given by $\delta = \beta_3/(6|\beta_2|\tau)$ [8]. Higher-order dispersion leads to soliton decay when δ exceeds a threshold value [16] and, for an $N = 2$ soliton, this threshold is given by $\delta = 0.022$. Fig. 1 shows the evolution of the constituent soliton parameters during break-up of an $N = 2$ soliton due to the effect of TOD.

B. Two-Photon Absorption

In many highly nonlinear media, such as ZnSe, the Kerr nonlinearity is accompanied by strong two-photon absorption (TPA). This effect can be included in the modeling by adding the nonlinear loss term $-iK|u|^2u$ to the NSE [9]. Fig. 2 shows the evolution of parameters for an $N = 2$ soliton in the presence of two-photon absorption [9]. TPA is shown to result in higher-order soliton fission, leading to the generation of two escaping solitons with equal amplitudes and opposite transverse velocities.

V. Conclusions

We have developed a fast numerical algorithm for computing the discrete spectrum of the direct scattering transform of the NSE. The method is based on the iterative refinement of the eigenvalues and is particularly suited to the investigation of soliton evolution in the presence of higher-order effects. The application of this efficient alogrithm has been demonstrated in two different physical contexts.

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