FRESNEL DIFFRACTION AT SNOWFLAKE APERTURES: SYMMETRIES & BOUNDARY CONDITIONS

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Introduction

Diffraction plays a fundamental role in our understanding of wave physics in general, and it is perhaps at its most fascinating when the system under consideration incorporates some degree of fractality (where proportional levels of spatial structure persist over many decades of scale). For example, Berry examined “diffractals” arising from plane waves scattered from fractal phase objects (Berry, 1979). Those scattered waves acquire multi-scale characteristics in their statistical properties.

In this theoretical study, we present a detailed and systematic analysis of the Fresnel (near-field) patterns produced when a plane wave illuminates a fractal aperture that has a shape tending toward the classic von Koch snowflake (an iterated function system involving self-similar sequences of superimposed equilateral triangles). Related analyses by other authors have considered mainly Fraunhofer (far-field) patterns (Uozumi *et al*., 1991). Our mathematical description is based upon line integrals and Young’s edges waves. Key results will be presented for several stages of iteration, progressing from the initiator (a single triangle) through four subsequent applications of the generator algorithm (see Fig. 1).

Figure 1: Left to right: initiator (n = 0) and first four iterations toward creating a fractal snowflake (n = 1, 2, 3 and 4). The n^{th} iteration has $N = 3 \times 4^n$ sides (N = 3, 12, 48, 192, and 768, respectively), so one might expect the computation times for calculating Fresnel patterns to diverge geometrically.

Attention is paid not only to snowflake apertures but also, by way of Babinet’s Principle (Born & Wolf, 1980), the patterns produced by their complementary stops. Our results in the optics domain provide the basis for modelling a new generation of unstable-resonator systems, where the feedback mirror has the shape of a fractal rather than the regular polygon associated with proposed “kaleidoscope” laser cavities (McDonald *et al.*, 2000).

Line Integral Method and Babinet’s Principle

When a plane monochromatic wave of complex amplitude U_0 is incident on a simple closed aperture with area Ω in the (ξ, η) plane, the diffraction light pattern $U(\mathbf{p})$ at a distance L downstream can be well-described by the two-dimensional Huygens-Fresnel integral

$$\frac{U(\mathbf{p})}{U_0} = \frac{k}{i2\pi L} \iint_{\Omega} d^2\mathbf{q} \, \exp\left(i\frac{k}{2L}|\mathbf{p}-\mathbf{q}|^2\right), \tag{1}$$

where $\mathbf{p} \equiv x + iy$ and $\mathbf{q} \equiv \xi + i\eta$ are vectors (and coordinates) in the image and aperture planes, respectively, while $k = 2\pi/\lambda$ and λ is the optical wavelength (see Fig. 2). Here, we consider Ω to be progressing toward a fractal shape. Over recent decades, many authors have analysed the Fraunhofer patterns from the von Koch snowflake, both experimentally (Horváth *et al.*, 2010) and theoretically (Uozumi, Kimura, & Asakura, 1991). More recently, the Fresnel patterns have received some attention (Lisicki *et al.*, 2008) but there are, to the best of our knowledge, almost no published works providing a full detailed theoretical account of such spatial structures.

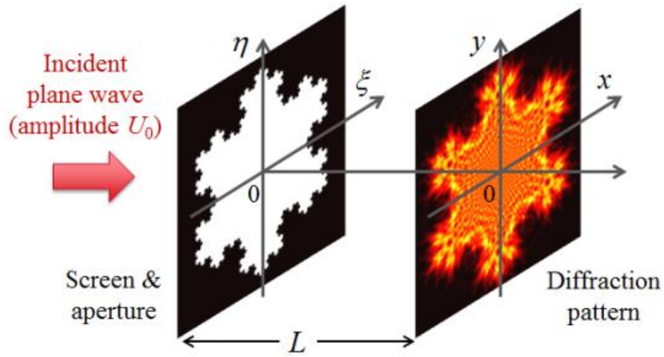

Figure 2: Schematic diagram of the typical setup for diffraction at an aperture. For Eq. (1) to be valid, it must be that $L >> (kb^4/8)^{1/3}$ (with b denoting the largest characteristic length-scale of the aperture).

In this research, we consider a systematic transition from a triangular aperture toward a (pre-fractal) snowflake (see Fig. 1). Analysis is based on converting the conventional *integration over the aperture area* [c.f. Eq. (1)] to a *circulation around the aperture edge* (Hannay, 2000). If the apexes of the initiator (that is, the triangular aperture in Fig. 1) lie on a circle of radius a and all transverse spatial distances are measured in units of a, then one can transform Eq. (1) (by deployment of a low-dimensional divergence theorem) to be in the form

$$\frac{U(\mathbf{p})}{U_0} = \varepsilon - \frac{1}{2\pi} \oint_\sigma dl\, \mathbf{n} \cdot \frac{\mathbf{p}-\mathbf{q}}{|\mathbf{p}-\mathbf{q}|^2} \exp(i\pi N_F |\mathbf{p}-\mathbf{q}|^2), \tag{2}$$

where σ denotes the aperture boundary, dl is an element along σ, and $\mathbf{n}$ is an outward-normal unit vector. Also, $\varepsilon = 1$ (or 0) if the observation point $\mathbf{p}$ lies inside (or outside) the geometrical projection of the aperture.

The diffraction pattern from an aperture is thus parametrized solely by the Fresnel number $N_F \equiv a^2/\lambda L$. Analysis proceeds by breaking-up Eq. (2) into a piecewise superposition of edge waves (Huang, Christian, & McDonald, 2006). One can also calculate the pattern from the complementary aperture – in this case, a snowflake stop that has the same size and shape as the snowflake aperture "cut out of" the screen shown in Fig. 2. If the pattern from the stop is denoted by $U_{\text{stop}}(\mathbf{p})$, then Babinet's Principle states that $U_{\text{stop}}(\mathbf{p}) = U_0 - U(\mathbf{p})$. Thus, a single numerical calculation yields both aperture and stop patterns.

Patterns from Snowflakes: Low N_F

Fresnel patterns with low but finite N_F numbers are shown in Fig. 3. The patterns are relatively smooth, having a narrow spatial frequency content and they are tending toward their Fraunhofer counterparts. Hence, for low Fresnel numbers the presence of smaller-scale edge-related aperture/stop details has a less noticeable effect on the scattered patterns.

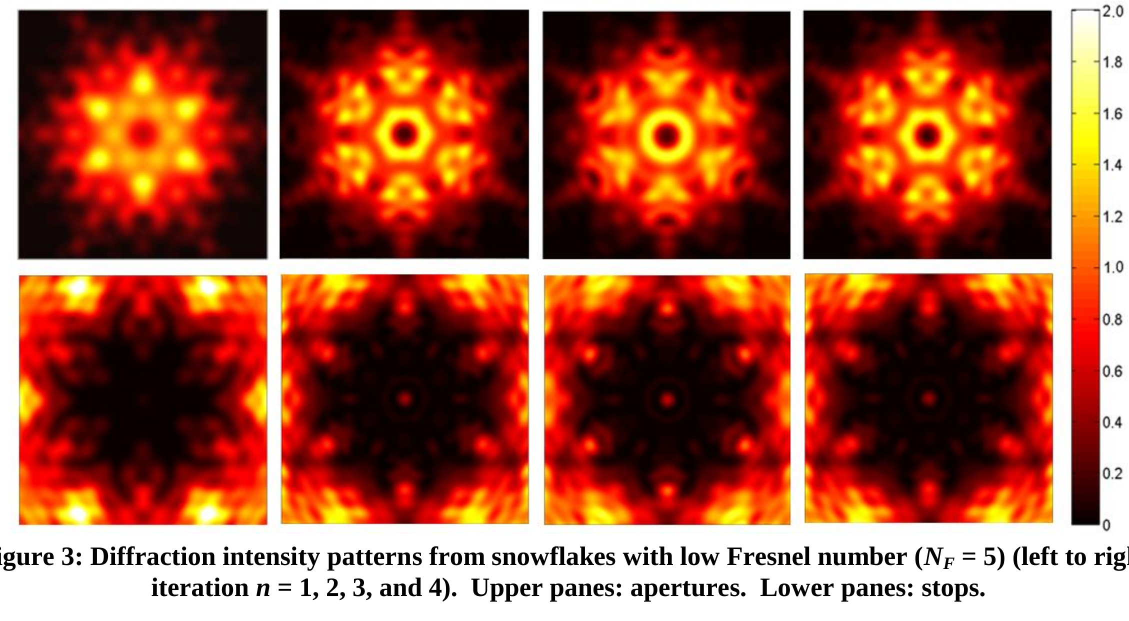

Figure 3: Diffraction intensity patterns from snowflakes with low Fresnel number (N_F = 5) (left to right: iteration n = 1, 2, 3, and 4). Upper panes: apertures. Lower panes: stops.

Patterns from Snowflakes: Moderate and High N_F

Results for snowflakes with moderate and high Fresnel numbers are shown in Figs. 4 and 5, respectively. As expected, full six-fold rotation symmetry is always evident in the pattern (as it must be). When N_F increases, the patterns become 'sharper' and resemble more strongly the shape of the aperture. A recurrent trend is that for fixed N_F, one always reaches a point where additional substructure in the pattern can no longer be discerned as the aperture becomes 'more fractal' (i.e., as n increases). To that end, we have also magnified the central region of the aperture patterns (see Fig. 6).

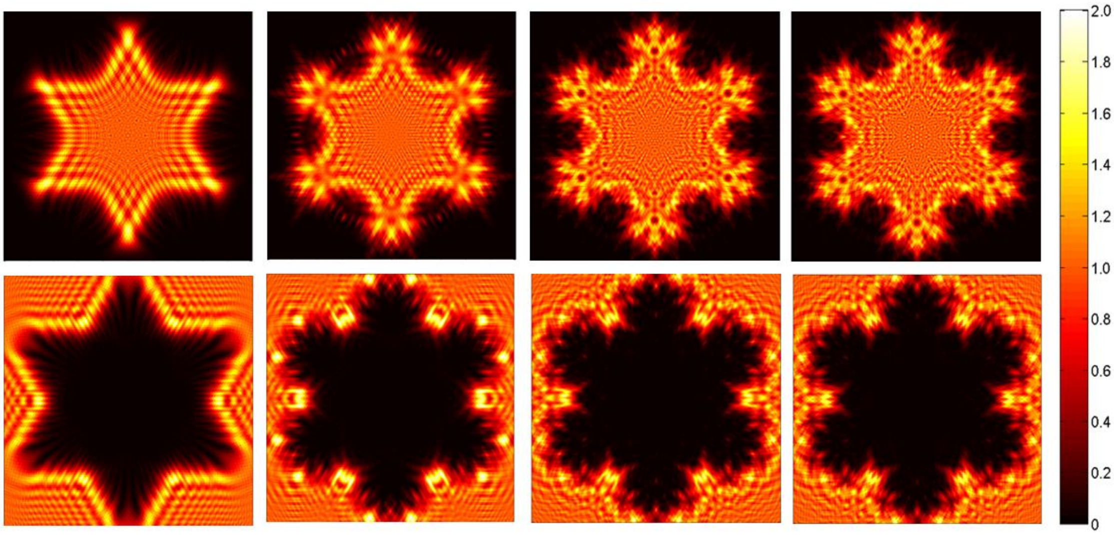

Figure 4: Diffraction intensity patterns from snowflakes with moderate Fresnel number (N_F = 50) (left to right: iteration n = 1, 2, 3, and 4). Upper panes: apertures. Lower panes: stops.

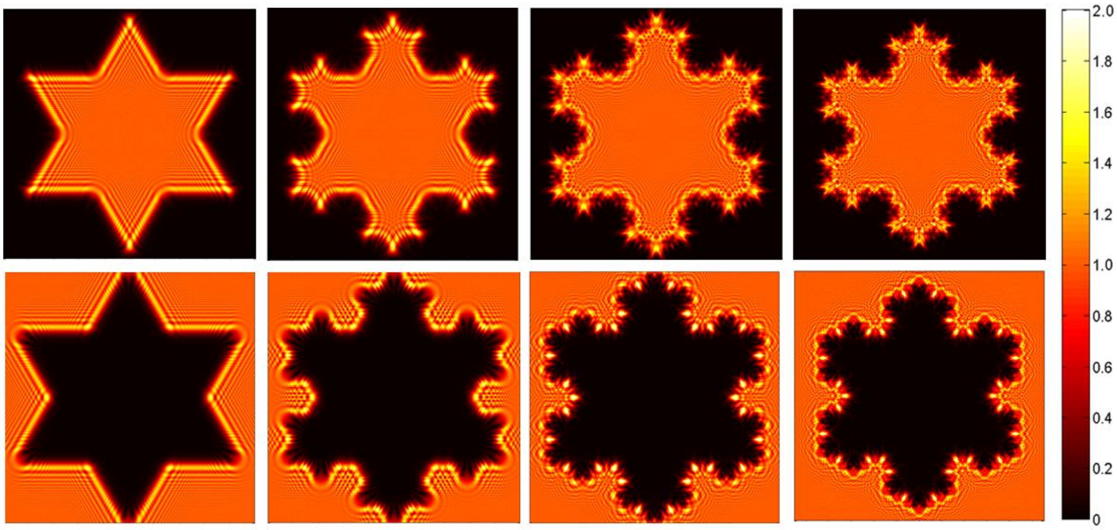

Figure 5: Diffraction intensity patterns from snowflakes with high Fresnel number (N_F = 200) (left to right: iteration n = 1, 2, 3, and 4). Upper panes: apertures. Lower panes: stops.

Conclusions

We have presented, to the best of our knowledge, the first systematic description of diffraction patterns from the first four applications of the von Koch snowflake initiator-generator algorithm. A semi-analytical approach has been deployed, based upon a line integral formulation of Fresnel diffraction and Young's edge waves. Both far-field (low-N_F) and near-field (high-N_F) regions have been considered, and patterns computed for a series of apertures and their corresponding stops. Since accurate numerical evaluation of fully-two-dimensional (2D) edge-waves are exceptionally time-consuming and resource-hungry, our calculations have so far not gone beyond the fifth iteration (where the snowflake comprises 3072 edges). However, we expect to be able to compute and analyze 1D cross-sections through fully-2D patterns from apertures with much higher n values

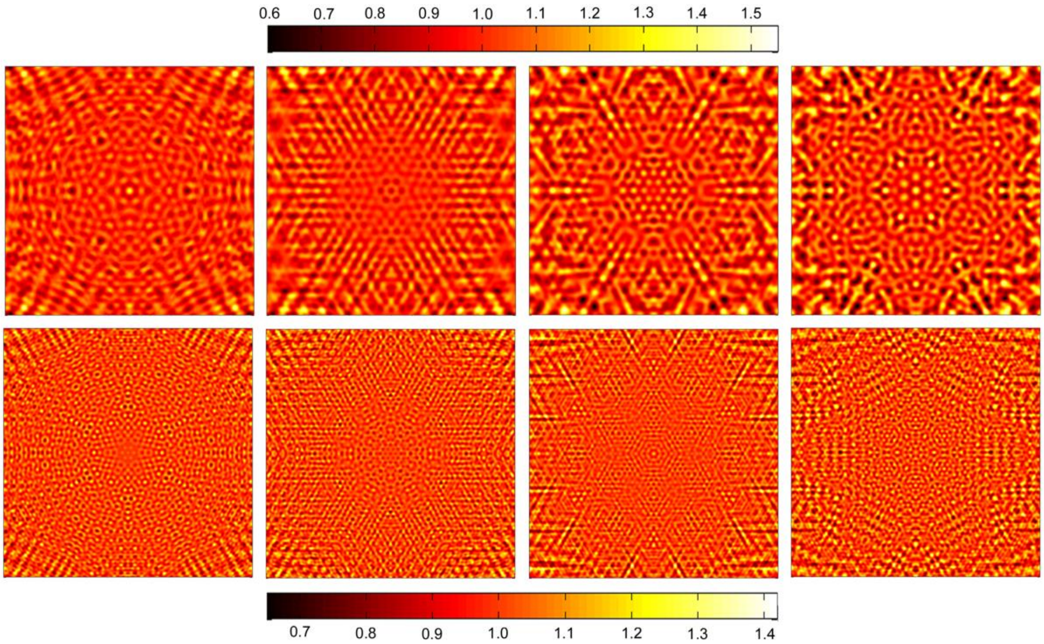

Figure 6: Magnification of diffraction intensity patterns from snowflake apertures (left to right: iteration *n* = 1, 2, 3, and 4) in a central square region with side 0.6*a*. Six-fold rotation symmetry is fully preserved. Upper panes: N_F = 50. Lower panes: N_F = 100. Note that the use of different colour mappings for the two N_F values helps to bring out finer pattern details.

(e.g., to assess the potential for a possible fractal dimension $1 < D \leq 2$ related to n). Our suite of codes is also readily adaptable for 2D virtual-source modelling of novel designs of unstable resonators (Huang, Christian, & McDonald, 2006, Southwell, 1986, 1981).

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