MULTI-TURING INSTABILTIIES IN NONLINEAR CAVITIES: IMPLICATIONS FOR PATTERN FORMATION

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Introduction

Alan Turing's seminal analysis of morphogenesis (Turing, 1952) was ahead of its time, and it laid the foundations for a modern understanding of the origins of pattern and form in Nature. He discovered that when a reaction-diffusion system is sufficiently stressed, arbitrarily-small disturbances to its stationary states may give rise to spontaneous self-organization into a simple pattern. Moreover, the emergent pattern in this winner-takes-all process has a dominant scalelength $\lambda_0 \equiv 2\pi/K_0$ that is directly related to the most unstable spatial frequency K_0 in the system (see Fig. 1).
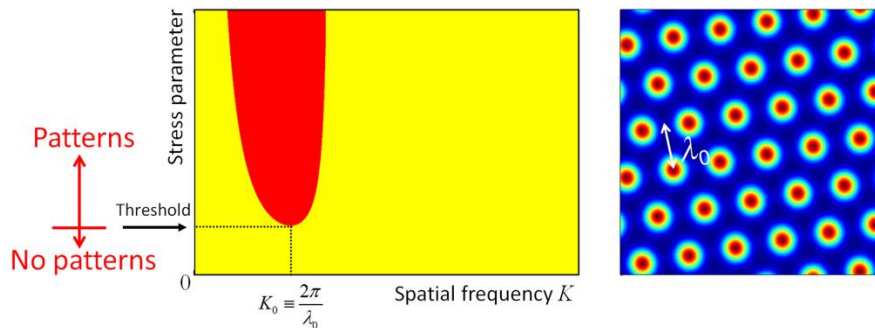

Figure 1: Illustration of the relationship between a single Turing minimum and the dominant scalelength of the emergent pattern. The hexagonal array is static (i.e., it does not change in time).

There also exists a class of system whose Turing instability spectrum comprises many comparable minima. Previously, our group has proposed that such a structure can provide a mechanism for the generation of spontaneous spatial fractal patterns (Huang & McDonald, 2005) – that is, patterns that contain proportional levels of detail across decimal orders of spatial scale. This mechanism was proposed to be truly universal (independent of the physical nature of the system), and to have independence with respect to the nonlinearity (which may be dispersive, absorptive, or both).

The first analysis of nonlinear spatial optical fractals considered the classic Kerr slice with a single feedback-mirror, we have subsequently identified three further candidate systems that possess multi-Turing spectra – the Fabry-Pérot (FP) cavity, and a ring cavity containing either a thin slice or a slab of nonlinear material. All three systems capture the interplay between diffraction (in the free-space path), nonlinearity (in the slice), and a host of cavity effects (round-trip time delays, interferomic mistuning, periodic pumping (light re-injected on every transit), and losses at the mirrors as indicated – see Fig. 2]. A low-pass filtering element has been inserted into the cavity to allow the effective control of pattern formation by limiting the maximum spatial frequency K that can propagate around the system.
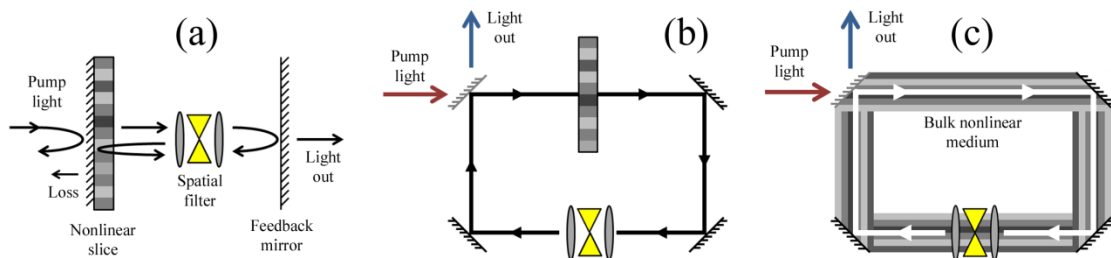

Figure 2: Schematic diagram of three distinct nonlinear cavity configurations. (a) Fabry-Pérot cavity with a thin slice of near-negligible thickness L. Ring cavity with (b) a thin slice, and (c) a slab of material.
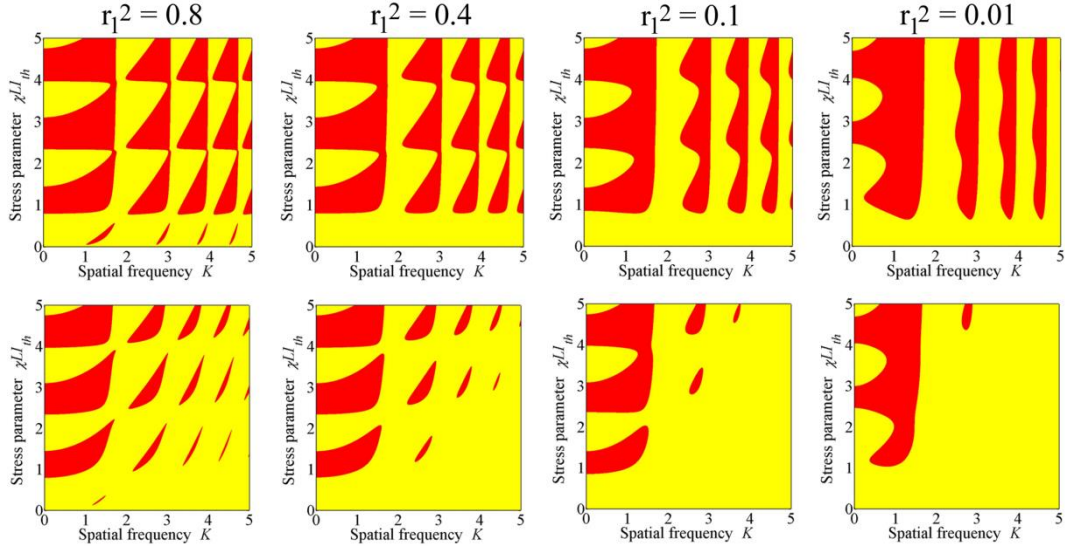
Figure 3: Typical multi-Turing threshold spectra ($\chi LI \rightarrow \chi LI_{th}$) for an FP cavity containing a thin slice of focusing ($\chi L = +1$) Kerr material for local (upper row) and diffusive (bottom row) photoexcitations. The feedback mirror losses have been set to $r_2^2 = 0.9$.

Nonlinear Fabry-Pérot Cavity: Slice

The normalized equations governing the forward and backward electric field envelopes (denoted by *F* and *B*, respectively) and the photoexcitation density *n* are

$$\frac{\partial F}{\partial z} = i\chi LnF, \quad \frac{\partial B}{\partial z} = -i\chi LnB, \quad \text{and} \quad -l_D^2 \nabla_\perp^2 n + \tau\frac{\partial n}{\partial t} + n = |F|^2 + |B|^2. \tag{1}$$

The longitudinal coordinate within the slice (which has near-negligible thickness *L*) is $0 \leq z \leq 1$, χ parametrizes the nonlinearity (positive for self-focusing, negative for self-defocusing), and *n* has diffusion length l_D and relaxation time τ. The model must also be supplemented by appropriate boundary conditions (McDonald, Stephen, & Firth, 1990). Analysis yields a Turing spectrum that has a multiple-minimum structure and whose stress parameter turns out to be the nonlinear phase shift χLI, where *I* is related to the threshold intracavity light intensity (see Fig. 3). The spectra can be investigated by fixing the mistuning, feedback-mirror losses, free-space path length, etc. and varying just the reflectivity of the slice. As $r_1^2 \rightarrow 0$, the classic prediction for the single feedback-mirror (Firth, 1990, D'Alessandro & Firth, 1991) system can be recovered.

Numerical calculations for spontaneous pattern formation in the FP cavity with plane-wave pumping are in good agreement with the threshold predictions (note that the simulation requires the lowest threshold to occur at a non-zero value of *K*). Simple patterns can grow from the perturbed plane-wave solution (initialized above threshold)
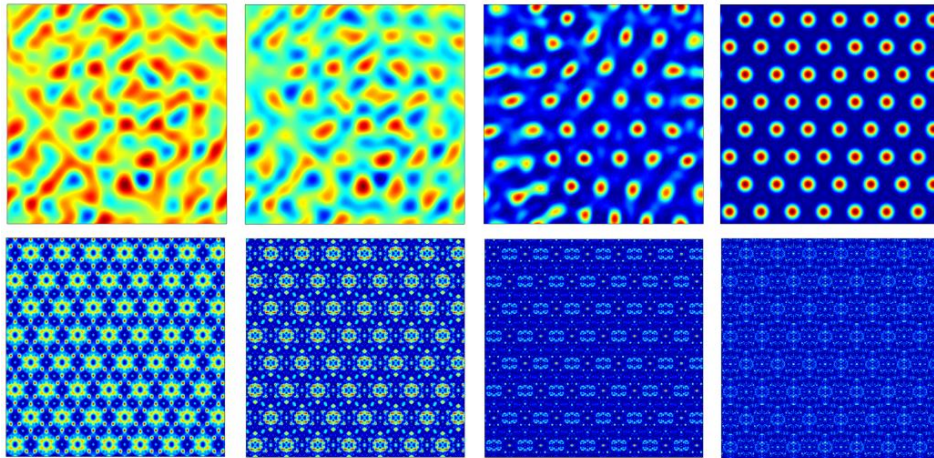

Figure 4: Upper row: evolution of perturbed stationary state towards a Turing (static) hexagon pattern for (left to right) $t/t_R = 32, 96, 112,$ and 256 (at which point the spatial filter is removed). Lower row: evolution towards a fractal pattern for (left to right) $t/t_R = 257, 258, 259, 260$.
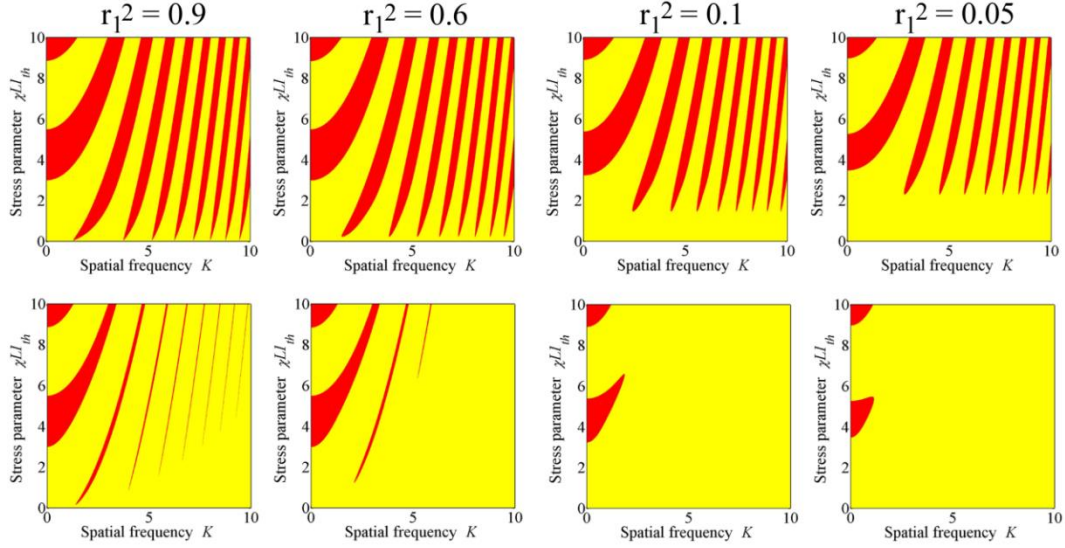
Figure 5: Multi-Turing threshold spectra for a ring cavity with a thin slice of focusing ($\chi L = +1$) Kerr material for instantaneous (upper row, $l_D = 0$) and diffusive (bottom row, $l_D = 1$) photoexcitations. The discrete-band structure is qualitatively similar to that of the single feedback-mirror system (Firth, 1990).

when the cut-off spatial frequency in the filter is chosen so that all waves outside the first instability band are attenuated (see Fig. 4, top tow). Once the single-scale stationary state has established itself (in this case, a hexagonal array), the filter is set so that all physically-meaningful waves are free to propagate and interact with each other through intrinsic nonlinear dynamical processes (e.g., spatial harmonic generation and four-wave mixing). The pattern subsequently evolves towards a fractal (see Fig. 4, bottom row).

Nonlinear Ring Cavity: Slice

The ring-cavity equations can be obtained from model (1) by setting $B = 0$ (no backward field in the uni-directional system) and modifying the boundary conditions accordingly (McLaughlin, Moloney, & Newell, 1985). For completeness, we mention that linear analysis predicts a Turing spectrum with a multi-minimum structure (see Fig. 5), and extensive simulations have shown that this system – and its purely-absorptive counterpart (Huang, Christian, & McDonald, 2011) – can support both simple and fractal pattern formation (Huang, Christian, & McDonald, unpublished).

Nonlinear Ring Cavity: Slab

Most recently, we have turned our attention to a ring cavity containing a slab of dispersive (typically instantaneous Kerr-type) material. Going beyond the thin-slice approximation (with all its advantages and disadvantages) necessarily introduces a finite light-medium interaction length. One must hence be able to accommodate the evolution of a potentially broad-spatial spectrum nonlinear beam inside the host medium. The spirit of our analysis follows the paraxial Schrödinger-type model with a 'lumped' boundary condition that captures periodic pumping, losses, and mistuning (McLaughlin, Moloney, & Newell, 1985, Ouarzeddini, Adachihara, & Moloney, 1988). The governing equation for the propagation of the cavity field during the nth cavity transit is now allowed to be of the Helmholtz type (Feit & Fleck, 1988), facilitating the more complete (nonparaxial) description of high spatial frequencies.

Linear analysis has identified the threshold condition for Turing patterns in the Helmholtz ring cavity for both focusing and defocusing nonlinearities (see Fig. 6). The multi-Turing spectrum is most pronounced when the mirror losses are low (i.e., where the intensity reflectivity r_1^2 is closest to unity). The strongest quantitative corrections to conventional (paraxial) theory appear, as might be anticipated, in regimes involving high stress and high-K (i.e., precisely those regimes where one tends to look for the signatures of nonlinear fractal patterns).

Conclusions

We have investigated three distinct classes of passive optical cavity containing a dispersive host medium of the Kerr type. While each configuration has been found to possess multi-Turing instability characteristics (Huang & McDonald, 2005), more traditional cavity models based on mean-field theory (Lugiato & Lefever, 1988) cannot describe this profoundly new regime of pattern formation. Research efforts are currently focusing on two
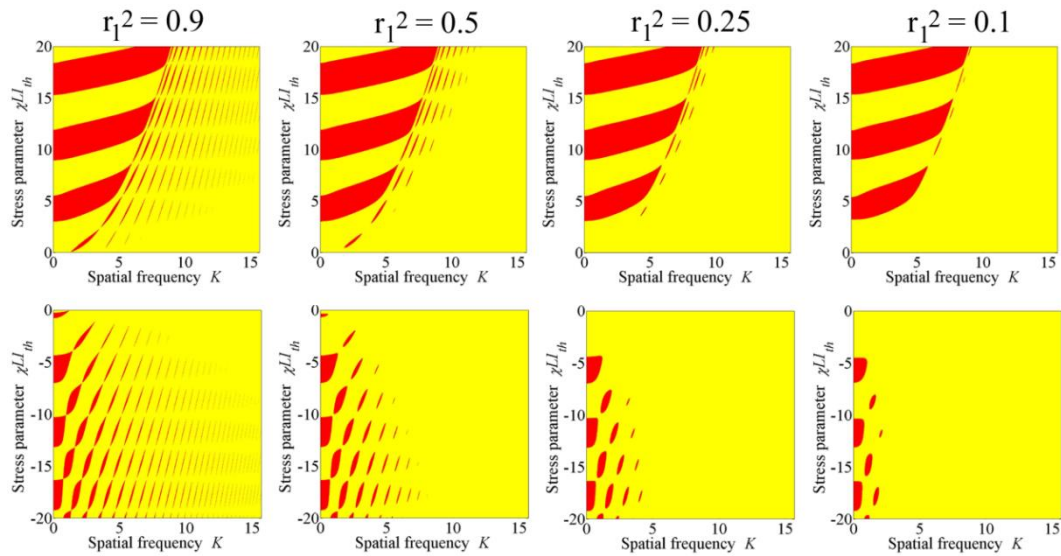
Figure 6: Multi-Turing threshold spectra for a nonparaxial ring cavity filled with a slab of instantaneous Kerr material of the focusing (top row, $\chi L = +1$) and defocusing (bottom row, $\chi L = -1$) type. The plots are qualitatively similar to the discrete-island structure of the FP cavity with a diffusive thin slice.

longer-term key objectives: (i) simulations of the ring cavity + slab configuration, and (ii) generalizing results to allow for a generic system nonlinearity.

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