DIFFRACTING FRACTALS:
NEW PARADIGMS IN LINEAR WAVE PHYSICS

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Introduction

Berry's seminal work from over three decades ago established that plane waves (waveforms with 'flat' profiles) scattering from a complex object (e.g., a transparent mask with a random fractal phase modulation) may acquire fractal characteristics in their statistics (Berry, 1979). Here, we consider the diametrically-opposing paradigm for complexity: the diffraction of a *fractal wave* from a *simple object*. Surprisingly, this rich and (potentially) highly fertile research ground has received almost no attention in the literature to date.

In this presentation, we will report on very recent research results concerning the scattering of fractal light from simple apertures. Attention is first paid to two historic Fresnel diffraction contexts (Born & Wolf, 1980): (i) a single infinite edge (see Fig. 1); (ii) a single slit (constructed from a pair of parallel edges – see Fig. 2). While classic analyses considered normally-incident planar (i.e., uniform) illumination, the novelty of our approach lies in accommodating an incident wave that possesses a very broad spatial bandwidth (i.e., a waveform whose Fourier spectrum extends over decimal orders of pattern scale-length).

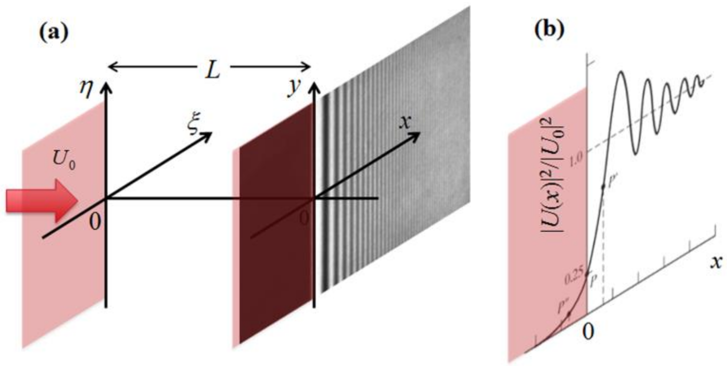

Figure 1: Schematic diagram illustrating the geometry of a single infinite edge aligned along the η axis and illuminated with a normally-incident plane wave with complex amplitude U_0. (a) The experimental setup and observed diffraction pattern. (b) Typical edge-wave pattern predicted by theory.

Exact mathematical descriptions of near-field (Fresnel) diffraction patterns have been obtained using a prescription based on Young's edge waves (Siegman, 1986). These preliminary calculations have formed the basis for analysing the diffraction of fractals at more sophisticated two-dimensional apertures such as squares, rectangles, and circles. The interaction of fractal waves with arrangements of slits and fully two-dimensional apertures (e.g., regular polygons) has implications, for instance, in novel unstable-cavity laser designs (Huang, Christian, & McDonald, 2006, McDonald *et al.*, 2000, Karman & Woerdman, 1998).

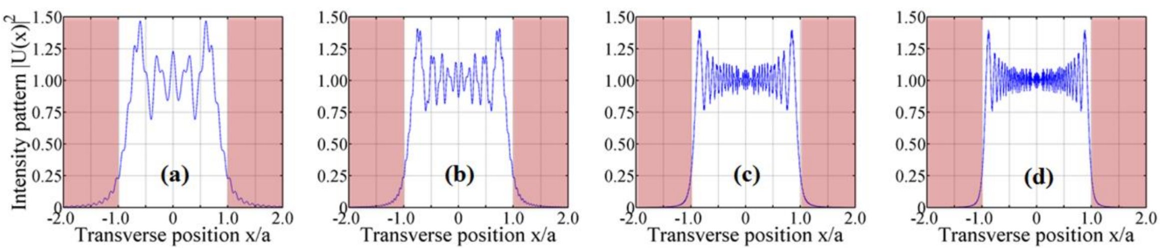

Figure 2: Diffraction patterns from a slit (of width 2*a*) illuminated by a normally-incident plane wave (with $U_0 = 1.0$) for increasing aperture Fresnel number $N_F \equiv a^2/\lambda L$ (e.g., observing closer to the aperture). (a) $N_F = 5$, (b) $N_F = 10$, (c) $N_F = 30$, (d) $N_F = 50$. Shaded areas denote geometrical shadow regions.

From Plane Waves to Weierstrass Fractals

Rather than plane wave illumination, we instead consider an incident field that is *fractal* – comprising proportional levels of detail spanning decimal orders of spatial scale. For definiteness, an analytical form for the input field $U_{in}(\xi)$ is constructed from a plane wave plus a Weierstrass fractal with finite (real) amplitude ε:

$$\frac{U_{\text{in}}(\xi)}{U_0} = 1 + \frac{\varepsilon}{2}\sum_{n=0}^{+\infty}\frac{1}{\gamma^{(2-D)n}}\left\{\exp\left[i\left(K_n\xi+\phi_n\right)\right]+\exp\left[-i\left(K_n\xi+\phi_n\right)\right]\right\}, \tag{1}$$

where $K_n = (2\pi/\Lambda)\gamma^n$ and $\gamma > 1$. The Weierstrass fractal belongs to a class of mathematical functions that are *continuous everywhere but differentiable nowhere*. The set of spatial frequencies $\{K_n\}$ shows that the largest scalelength in U_{in} is $\Lambda_0 \equiv 2\pi/K_0 = \Lambda$, and that there is no small-scale cut-off. The parameter $1 < D \leq 2$ is the Hausdorff-Besicovich dimension (which always exceeds the topological dimension of unity), and it quantifies the capacity of U_{in} to fill space (the upper limit of $D = 2$ corresponds to an area-filling pattern – see Fig. 3). The set of phases $\{\phi_n\}$ may be either deterministic (e.g., $\phi_n = n\mu$, where μ is a real constant) or randomly chosen. An alternative input field might be the Weierstrass-Mandelbrot function, which is a truly scale-free function that can be used to model various physical processes and phenomena (Berry & Lewis, 1980).

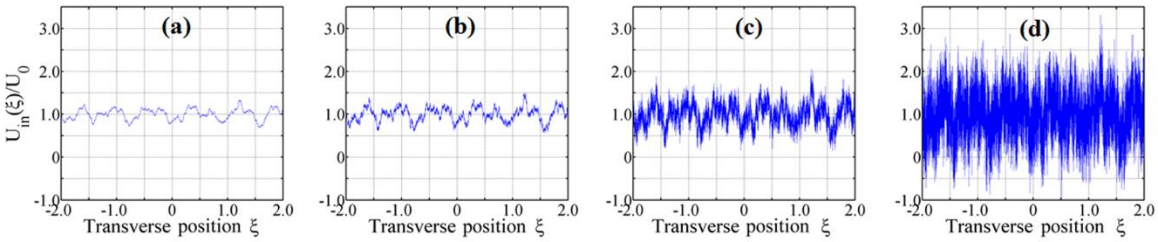

Figure 3: The Weierstrass fractal function with random phases for different values of the dimension *D*. (a) *D* = 1.2, (b) *D* = 1.5, (c) *D* = 1.8, (d) *D* = 1.99. Other parameters: γ = 1.2, Λ = 1.0, and ε = 0.1.

Fractal Illumination: Knife Edges and Slits

The Weierstrass function in Eq. (1) may be realized experimentally (e.g., in an optical context) by interfering sets of equal-amplitude obliquely-incident plane waves with opposite spatial frequencies ($+K_n$ and $-K_n$, respectively). With that in mind, we now consider the knife-edge geometry [see Fig. 1(a)] and address non-uniform illumination considerations in the form of a Weierstrass fractal.

Results are shown below in Fig. 4, where easy comparison can be made with the corresponding pattern from a uniform illuminating field. Note that the fractal field does not decay monotonically into the shadow region – there is pronounced oscillatory spatial structure in the intensity pattern (which may also be fractal). We have also considered diffraction of a Weierstrass fractal at a slit (c.f. Fig. 2). Preliminary calculations suggest that the Fresnel number of the aperture plays a strong role in moderating the diffraction of the fractal input wave. For example, the patterns shown in Fig. 5 all correspond to an input fractal with D = 1.99.

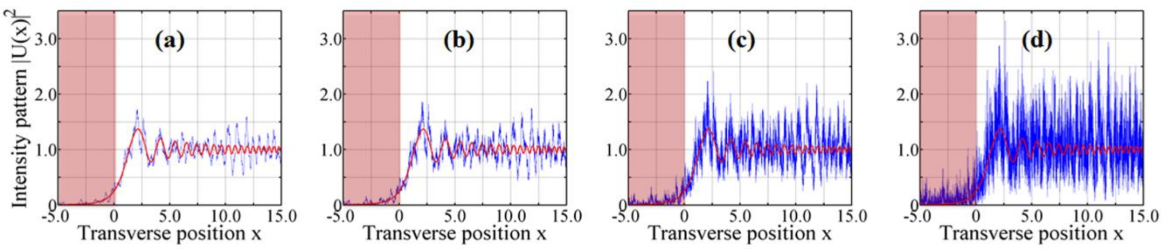

Figure 4: Diffraction of the Weierstrass fractal function [Eq. (1)] with random phases for different values of the dimension *D* at a knife edge. (a) *D* = 1.2, (b) *D* = 1.5, (c) *D* = 1.8, (d) *D* = 1.99. Other parameters: γ = 1.2, Λ = 1.0, *L*/*k* = 1, and ε = 0.1. Shaded areas denote geometrical shadow regions, and the same random phases are used in each pane. Red line: pattern from plane wave illumination.

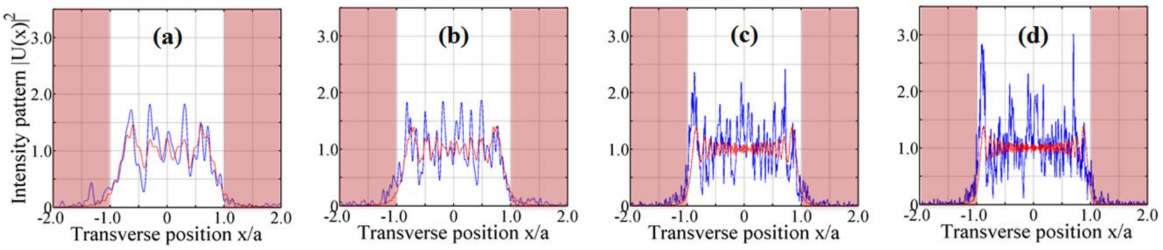

Figure 5: Diffraction of the Weierstrass fractal function [Eq. (1)] with random phases for different values of the dimension D at a single slit when D = 1.99. (a) N_F = 5, (b) N_F = 10, (c) N_F = 30, (d) N_F = 50. Other parameters: γ = 1.2, Λ/a = 1.0, L/k = 1, and ε = 0.1. Shaded areas denote geometrical shadow regions, and the same random phases are used in each pane. Red line: pattern from plane wave illumination.

Future Applications of Fractal Light

In terms of future experimental prospects, this work has greatly widened the scope for potential realization and exploitation of fractal light sources. For example, spatially broadband light may prove to be more efficient for probing, scanning and ablation applications (e.g. "a fractal probe for a fractal world" in various technological and medical contexts). While much previous research has examined the scattering of non-fractal waves by fractal structures, there has been relatively little investigation of the consequences of fractal illumination. However, it has for example been demonstrated (both theoretically and experimentally) that a fractal light probe may improve sensitivity of surface roughness measurements (Nada, Uozumi, & Asakura, 1999). As the number of new Nature-inspired device and system architectures increases, fractal light sources may also play a greater, and perhaps pivotal, role in such developments.

There are further possibilities that such huge spatial optical bandwidths may be harnessed within future information and image processing applications. One can justify this proposal by noting that digital information encoded in the spatial (temporal) frequency-space of a fractal is preserved under the action of linear diffraction (dispersion). Interestingly, such binary data would be able to mix in real space while remaining intact in the Fourier domain. If such a spatial system could be realized, then each temporal information bit could be replaced by a gigantic binary string stored within decades of fractal scales.

Conclusions

We have proposed, to the best of our knowledge, a new branch of diffraction – *scattering of fractal waves from simple apertures*. The mathematical formalism for describing this class of near-field diffraction pattern is based on Young's edge waves. Preliminary calculations have derived a range of exact results for classic 1D apertures such as knife edges and slits. These analyses have recently been generalized to 2D structures such as rectangular and circular apertures (Siegman, 1986). There is also the possibility of considering other shapes such as wedges (Silverman and Strange, 1996) and regular polygons (Huang, Christian, & McDonald, 2006). A pressing matter is to map out the behaviour of the diffraction pattern's fractal dimension in the near-field region. One can also consider the power spectrum associated with the diffraction of fractal waves from simple apertures like the slit.

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