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Diffractive origin of fractal resonator modes

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Abstract

The modes of unstable optical resonators possess fractal character. In this paper, the fundamental question of how and why fractals originate in one of the simplest linear optical systems is addressed. The answer is related to the fact that unstable resonator modes consist of a superposition of Fresnel diffraction patterns with effectively random phases. A connection is established between the mode eigenvalues and their fractal dimensions, and the consequent prediction that higher-order modes should exhibit lower fractal dimension is confirmed by numerical demonstration. © 2001 Elsevier Science B.V. All rights reserved.

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When two spherical mirrors face each other on a common axis to form an unstable optical resonator, a system is formed whose modes possess fractal structure [1,2]. No specialised hardware is required and no fractal boundary conditions are involved. This remarkable finding may have practical implications for novel laser designs and applications in the probing of natural fractal structures; it may also lead to a better understanding of phenomena such as chaotic scattering and optical pattern formation.

Most laser systems are based on stable optical cavities in which light rays are trapped forever between the mirrors and the cavity modes are concentrated near the cavity axis. In contrast, the rays within an unstable cavity run away from the axis and ultimately escape; correspondingly the cavity modes fill the entire resonator volume and energy spills out at the sides. The mode profiles are formed from repeated diffraction of the field circulating in the cavity at the transverse outer boundary, and the shape of the mirrors (or whatever component defines the aperture of the system) is therefore critical [3].

In previous work [1–3], we demonstrated the fractal nature of unstable resonator modes and studied their detailed properties. The fractal dimensions D of typical mode profiles generated in a high Fresnel number (upper branch) confocal slit resonator were calculated, and the dynamic range of the fractal structure was shown to extend over many decades [1]. Mode patterns were subsequently

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computed in two transverse dimensions [2,3] and, although calculations of D were not practical at the lower Fresnel number used in this case, the patterns clearly exhibited the general self-similar features associated with fractal structures.

In the present paper, the fundamental question regarding the origin of the fractal features is addressed. How is it that one of the simplest linear optical systems imaginable possesses such an unexpected property? The answer to this question may have an important bearing on fractal formation in other areas of physical science.

It was argued in Ref. [1] that self-similarity arises from the inherent round-trip magnification M of an unstable resonator; the fact that the beam is expanded by a factor M each time it cycles the cavity implies that the mode is constructed from multiple patterns of different sizes. This picture, which is of course based on ray optics in which diffraction is neglected, may perhaps be applicable under certain very restrictive conditions, at particular planes in self-imaging unstable resonators for instance [4]; this is the case treated by Courtial and Padgett [5]. But in all other circumstances diffraction plays a central role; indeed it is almost certainly diffractive effects that render the modes not rigorously but only statistically self-similar [1,2].

In this paper, we present a detailed interpretation of the origin of the fractal structure in which full account is taken of diffraction; the analysis is not limited to particular positions within the cavity. It is shown on the basis of unstable resonator theory how the appropriate spatial frequency spectrum for fractal formation originates, a feature that cannot be explained using geometrical optics. Semi-analytic methods are employed to relate the fractal dimension to the mode eigenvalue; the argument suggests that the fractal dimension should be somewhat smaller for higher-order modes and we demonstrate that this is indeed the case by presenting a specific example.

Fractal structures have a particular signature in the Fourier (spatial frequency) domain. A profile is fractal if the angular power spectrum follows a power law of the form $|F(k)|^2 \sim k^{-b}$ where k is the angular frequency, and the phases of the Fourier components are random. In this case, one expects that the fractal dimension D will be given by $D = (5 - b)/2$ [6]. If $D = 1.5$ for instance (close to the value observed in Ref. [1]), $b = 2$ and $|F(k)|^2$ obeys an inverse square law. Our key objective here is to establish the diffractive origin of fractal mode structure by explaining how these conditions originate in an unstable resonator. The argument is based on three main considerations:

(A) In Southwell's virtual source (VS) method [7], [1] a mode of an unstable resonator is expressed as a superposition of the Fresnel diffraction patterns cast by a set of virtual images of the defining aperture of the system. The mode profile is duly represented as a series containing a constant term and terms representing the edge waves from each virtual aperture (see Eq. (5) below). The mode eigenvalue determines the relative amplitudes and phases within the summation.

(B) The spatial frequency power spectrum of a Fresnel intensity profile can be shown to exhibit an underlying inverse square law characteristic (see Eq. (2) below). This is not to imply that a *single* Fresnel diffraction pattern exhibits fractal character, but rather that a randomly phased superposition of many such patterns can.

(C) The spatial frequency content of the intensity profile diffracted by a particular virtual aperture is found to lie within a clearly delineated frequency band whose limits are readily determined (see Eq. (4) below). Within the band, the frequency dependence varies essentially as k^{-2} (see B above). Although bands associated with different virtual apertures will not necessarily align so as to maintain the inverse square law dependence, it turns out that for typical low-order modes, the overall power law dependence is preserved to a good approximation. The contributions from different virtual apertures are uncorrelated in phase.

Each of these points is now elaborated in turn. We take as our example the confocal unstable resonator of Fig. 1, concentrating on the one-dimensional (slit resonator) case for simplicity; a non-confocal cavity can be represented by its equivalent confocal cavity [8]. The focal lengths of the convex and concave mirrors are respectively f

[1] Note that γ is real and β is complex in this paper.

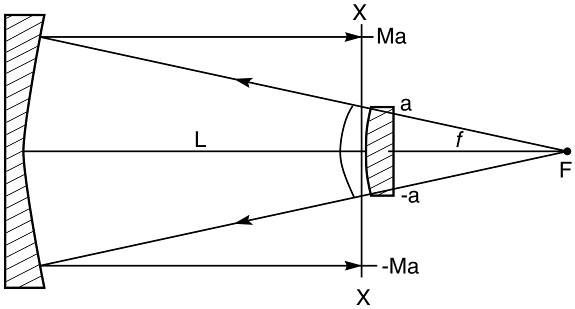

Fig. 1. Schematic diagram of a confocal unstable resonator with magnification M. The aperture of the system is determined by the smaller convex mirror, whose boundaries lie at $x = \pm a$. The rightward component of the lowest order mode approximates to a plane wavefront X–X shown just as it reaches the convex mirror

and $(L + f)$ where L is the mirror separation. The resonator is characterised by two key parameters: the magnification $M = 1 + L/f$, and the equivalent Fresnel number $N_{\rm eq} = (1/2)(M-1)N_{\rm F}$ in which the basic Fresnel number $N_{\rm F} = a^2/L\lambda$ and a is the half-width of the convex ("feedback") mirror that defines the aperture of this system.

An observer looking to the left from the reference plane X–X adjacent to the feedback mirror sees a corridor of virtual images of the mirror edges located at lateral and longitudinal coordinates [7]

$$(x_j, z_j) = (\pm a_j, f(M^{2j} - 1)) \quad (1)$$

where $a_j = aM^j$, the index j is 1 for the nearest aperture, and the centre of the feedback mirror is

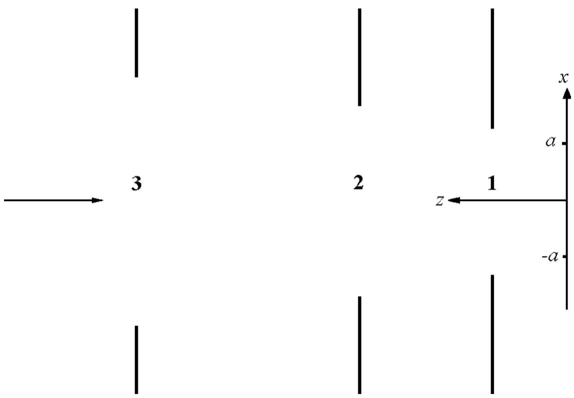

Fig. 2. The last three virtual apertures (j = 1–3) in the Southwell sequence, drawn roughly to scale for $M = 1.3$.

at $x = z = 0$. Fig. 2 shows the last three apertures in the sequence, drawn roughly to scale for $M = 1.3$. In the VS approach, a plane wave propagates through the corridor, and each virtual slit becomes the source of a pair of edge waves; it is these "virtual sources" that give Southwell's method its name. The resonator mode is formed on the $z = 0$ plane in Fig. 2 from the interference of the edge waves with the original plane wave and with each other. The mode profile at other planes in the resonator can be determined by subtracting the appropriate constant distance from z_j.

To understand how the spatial power spectrum of the complete mode is constructed, it is first necessary to consider the contribution of individual apertures. The power spectrum of the Fresnel intensity pattern formed a distance L from a single slit of width $2a$ can be shown to be

$$|P(k)|^2 = \begin{cases} \left(\dfrac{\sin\{ka(1 - k/k_{\rm crit})\}}{k\pi} \right)^2 & (k < k_{\rm crit}) \\ 0 & (k > k_{\rm crit}) \end{cases} \quad (2)$$

where $k_{\rm crit} = 4\pi a/L\lambda$; in the case of aperture j, a is replaced by a_j and L by z_j. $|P(k)|^2$ thus has an underlying inverse square law dependence on k, and a sharp cut-off at $k = k_{\rm crit}$.

To a first approximation, each position within a Fresnel pattern diffracted by a slit is associated with just two spatial frequency components determined by the angle between the respective edge waves and the axis. Angle θ corresponds to spatial frequency $2\pi\theta/\lambda$, and it follows from simple geometry that the two spatial frequencies present at coordinate x are

$$k_{\pm}(x) \cong \frac{2\pi}{\lambda} \left| \frac{x \pm a_j}{z_j} \right| \quad (3)$$

The power spectrum of the entire diffraction pattern, given by Eq. (2), contains spatial frequencies lying within a band running from $k_-(a_j) = 0$ to $k_+(a_j) = 4\pi a_j/\lambda z_j$ which is the value of $k_{\rm crit}$ for aperture j; the band centre lies at $k_j \equiv k_{\pm}(0) = 2\pi a_j/\lambda z_j$. However, the range of spatial frequencies lying *within the much narrower confines of the feedback mirror* $(-a < x < a)$ is obtained by setting $x = a$ in Eq. (3), yielding

$$k_{\pm}(a) = k_j \left| 1 \pm \frac{a}{a_j} \right| \equiv \frac{4\pi N_{eq}}{a} \left| \frac{M^j \pm 1}{M^{2j} - 1} \right| \quad (4)$$

This equation defines the limits of the spatial frequency band arising from virtual aperture j across the entire feedback mirror. As j increases, $k_+(a)$ and $k_-(a)$ converge to k_j which itself decreases as M^{-j}; this reflects the fact that the profiles from distant apertures hardly vary across the feedback mirror. Notice that the relative values of the band centres of different apertures k_j depend only on M, since N_{eq} is merely a multiplicative constant in Eq. (4). Whether (for example) the bands from the first and second apertures overlap, or are separated by a gap, therefore depends solely on the magnification.

According to the VS method, an unstable resonator mode is formed from the superposition of a plane wave and the edge-waves from the set of N virtual sources. The mode profile is given by [7]

$$v_N(x) = 1 + \mu \sum_{j=1}^{N} \alpha^{N-j} D_j(x) \quad (5)$$

where $\mu = (\alpha - 1)/D_{N+1}$ (called u_1 in Ref. [6]) and D_j contains the pair of edge waves from aperture j; under typical conditions, values of N no higher than 20 yield accurate results [7]. The parameter α is defined as $\alpha = \gamma/\beta$ where γ is the (complex) mode eigenvalue which includes the geometrical factor $\beta = M^{-\delta/2}$ where δ (= 1 or 2) is the number of transverse dimensions. Hence α can be regarded as the eigenvalue with the geometrical component removed.

A key feature of Eq. (5) is the factor α^{N-j}, which controls the weights of the different edge wave contributions to the overall mode profile. Since α is close to unity for the lowest-order mode of most resonators [7], the edge waves from all apertures are roughly of equal weight, and it follows that, to a first approximation, the frequency bands from different apertures align according to the inverse square law of Eq. (2). At the same time, the phases of the different edge waves are effectively random. Detailed examination shows that each edge wave component contains a phase factor, that depends on the precise distance from the position on the feedback mirror to the virtual source in question; for a particular coordinate x, this gives a set of uncorrelated values as required for fractality.

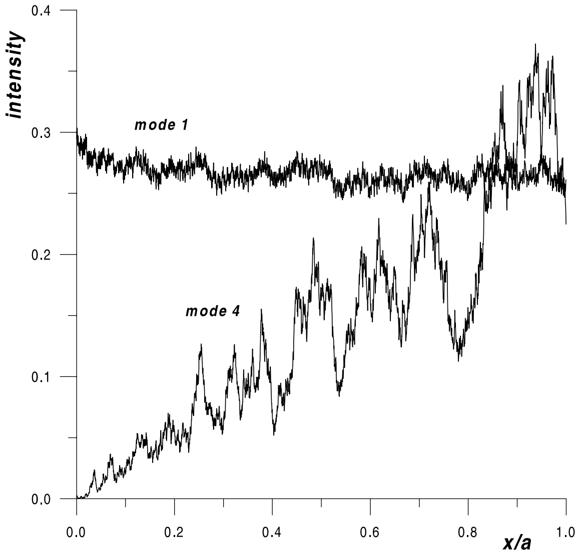

Fig. 3. Intensity profiles for the lowest-order and fourth-order even modes for the cavity of Fig. 1, in the range $x = 0$ to 1 and for $M = 1.9$ and $N_{eq} = 703.3$.

Finally, we present a numerical demonstration based on a confocal slit resonator with $M = 1.9$ and $N_{eq} = 703.3$. Fig. 3 shows the central portions of the profiles for the lowest-order and fourth-order even modes (modes 1 and 4) computed by the VS method. The corresponding power spectra displayed in Figs. 4 and 5 respectively, were obtained by taking the Fourier transforms of the intensity profiles within a much wider window, but with soft apodisation to attenuate the profile for $|x| > a$. The figures are therefore reasonable representations of the spatial spectra of the modes across the feedback mirror. Different components within the spectra are associated with particular apertures in the VS scheme, and the aperture numbers are used as labels in the figure. The low and high frequency limits of the bands from apertures 1 and 2 are marked separately (e.g. 1L, 1H, etc.). Notice that 2H lies to the right of 1L, indicating that these bands overlap; the disturbance in the spectrum in the overlap region is caused by interference. Fig. 4 confirms that, because $\alpha \cong 1$ for the lowest-order mode, the general inverse square law dependence in the power spectrum

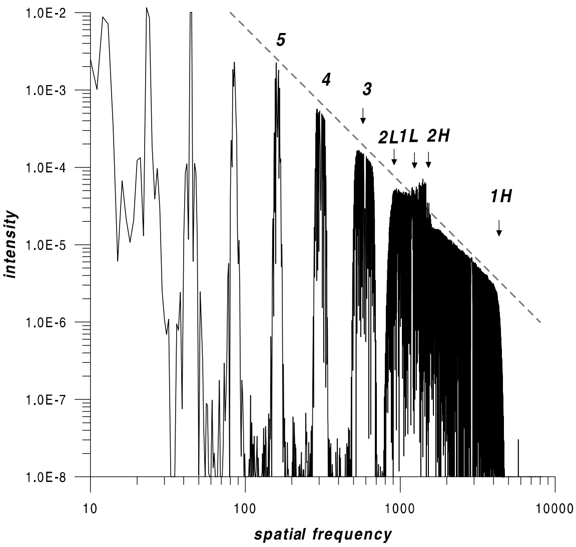

Fig. 4. Computed spatial power spectrum of the intensity profile for the lowest-order mode in Fig. 3. The frequency bands associated with individual virtual apertures are indicated. For apertures 1 and 2, the low- and high-frequency limits (L and H respectively) are shown separately. The broken line has a gradient of −2.

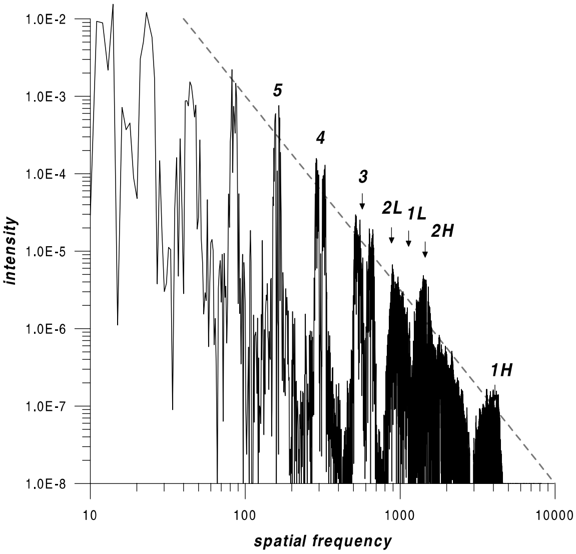

Fig. 5. Computed spatial power spectrum for the fourth-order even mode in Fig. 3. The broken line has a gradient of −2.5.

(as referenced by the broken line) is maintained across a significant dynamic range. On the other hand, for mode 4 (eigenvalue $\alpha = 0.73$), the higher frequency bands (lower j) are proportionately attenuated, and the effect is to tilt the entire spectrum, a feature that is apparent in Fig. 5 in which the gradient of the broken line is now -2.5.

As noted earlier, the expected relationship between the fractal dimension D and the slope of the power spectrum $(-b)$ is $D = (5 - b)/2$. By considering the effect of α on b, it can be deduced that for two modes p and q, the difference in their dimensions is

$$D_p - D_q \cong \frac{\log \alpha_p - \log \alpha_q}{2 \log M} \tag{6}$$

For $\alpha_1 = 1.01$, $\alpha_4 = 0.73$, and $M = 1.9$, this yields $D_1 - D_4 = 0.25$. Direct verification of the relation $D = (5 - b)/2$ is complicated by the fact that, although numerous methods for measuring the absolute value of D are available, different methods tend to give significantly different results when applied to real data [9]. For our profiles, for example, Hurst analysis gives $D_1 = 1.84$ and $D_4 = 1.56$, while the corresponding values obtained by box counting are 1.64 and 1.41. Fortunately, *changes* in D across different data sets are much less dependent on the particular dimension algorithm employed [9]; indeed the values of $(D_1 - D_4)$ are 0.28 and 0.23 for Hurst and box counting respectively, in fairly good agreement with each other and in good agreement with the value of 0.25 predicted by Eq. (6).

In conclusion, we have shown semi-analytically how the spatial frequency spectra of unstable resonator modes acquire the characteristics that correspond to fractal structure. A formula linking mode eigenvalues to the fractal dimensions of their intensity profiles has been obtained, and its predictions confirmed in a numerical demonstration.

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