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Exact soliton solutions of the nonlinear Helmholtz equation: communication

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Exact analytical soliton solutions of the nonlinear Helmholtz equation are reported. A lucid generalization of paraxial soliton theory that incorporates nonparaxial effects is found.

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1. INTRODUCTION

Investigations of the propagation of optical beams in Kerr-type media are usually based on the nonlinear Schrödinger equation (NSE), which is particularly appealing because of its analytical properties. The main drawback of this approach comes from the limitations imposed by the paraxial approximation that is implicit in the NSE. To overcome these difficulties we developed techniques based on a nonparaxial NSE (NNSE) that is derived without the use of the paraxial approximation; exact analytical solutions were derived for bright two-dimensional solitons, and their properties were thoroughly analyzed.[1] Further results were presented in recent publications[2,3] and have been widely disseminated at conferences.[4–7]

In a recent paper,[8] analytical solutions that correspond to soliton solutions of the nonlinear Helmholtz equation were presented. The bright and dark solutions for $N + 1$ dimensions were reported as new, and the authors then concentrated attention on the highly physical two-dimensional solutions. In this Communication we address key questions that were not fully answered in Ref. 8.

2. RESULTS

The two-dimensional nonlinear Helmholtz equation,

$$\frac{\partial^2 E}{\partial z^2} + \frac{\partial^2 E}{\partial x^2} + k^2 E + \gamma |E|^2 E = 0, \tag{1}$$

where $E(x, z)$ is the electric field, $k = n_0 \omega / c$, and $\gamma = 2 n_0 n_2 \omega^2 / c^2$, is transformed into the NNSE:

$$\kappa \frac{\partial^2 u}{\partial \zeta^2} + i \frac{\partial u}{\partial \zeta} + \frac{1}{2} \frac{\partial^2 u}{\partial \xi^2} + |u|^2 u = 0, \tag{2}$$

when it is written in terms of the field envelope $A(x, z)$, where $E(x, z) = A(x, z)\exp(jkz)$ and the normalizations $\zeta = z/L_D$, $\xi = \sqrt{2}x/w_0$, and $u(\xi, \zeta) = (k n_2 L_D / n_0)^{1/2} A(\xi, \zeta)$ have been employed. ω_0 is a reference beam width, $L_D = k\omega_0{}^2/2$ is the corresponding diffraction length, and $\kappa = 1/(k\omega_0)^2$ is a measure of the degree to which an input beam that propagates along the ζ axis deviates from being paraxial. In the paraxial limit, when the first term in Eq. (2) is neglected the NSE is recovered. A detailed account of the physical properties of the NNSE and its solutions can be found in Refs. 1–7. In what follows, we focus only on those results that are relevant to the study presented in Ref. 8.

The fundamental bright-soliton solution of the nonlinear Helmholtz equation, in terms of the normalizations of the NNSE, can be written as a lucid generalization of a paraxial soliton:

$$u(\xi, \zeta) = \eta \operatorname{sech}\left[\frac{\eta(\xi + V\zeta)}{\sqrt{1 + 2\kappa V^2}}\right] \times \exp\left[i\left(\frac{1 + 2\kappa\eta^2}{1 + 2\kappa V^2}\right)^{1/2}\left(-V\xi + \frac{\zeta}{2\kappa}\right)\right] \times \exp\left(\frac{-i\zeta}{2\kappa}\right), \tag{3}$$

where η, V, and κ are the amplitude, transverse velocity, and nonparaxial parameters, respectively. When nonparaxial effects are negligible, i.e., when $\kappa \to 0$, $\kappa\eta^2 \to 0$, and $\kappa V^2 \to 0$, one recovers the well-known paraxial soliton solution. The width of the nonparaxial soliton is given by $\xi_0 = (1 + 2\kappa V^2)^{1/2}/\eta$. The soliton's area is thus proportional to $\xi_0 \eta = (1 + 2\kappa V^2)^{1/2}$ and depends on both the transverse velocity and the actual size of the beam (through κ). The latter property reflects the fact that the solutions of the NNSE are invariant under transformations

$$\xi = \frac{\xi' + V\zeta'}{(1 + 2\kappa V^2)^{1/2}}, \qquad \zeta = \frac{-2\kappa V\xi' + \zeta'}{(1 + 2\kappa V^2)^{1/2}}, \tag{4}$$

$$u(\xi, \zeta) = \exp\left(i\left\{\frac{V\xi'}{(1 + 2\kappa V^2)^{1/2}} + \frac{1}{2\kappa}\left[1 - \frac{1}{(1 + 2\kappa V^2)^{1/2}}\right]\zeta'\right\}\right)u'(\xi', \zeta'), \tag{5}$$

from which the well-known Galilean transformation invariance of the NSE is recovered in the appropriate paraxial limit. The transformations of Eqs. (4) and (5) correspond to a rotation of the solution in the original (unscaled) coordinate system by an angle θ, given by $\sec\theta = (1 + 2\kappa V^2)^{1/2}$ (Refs. 1 and 2); this allows the analysis to be extended to common nonparaxial situations in which off-axis soliton beams are considered. It has further been shown[2] that these transformations can be used to predict the long-term evolution of nonparaxial solitons when they are used in conjunction with analytical paraxial techniques.[9]

With respect to the conservation properties of the NNSE, we have generalized the first conserved quantity of the NSE, $\int_{-\infty}^{+\infty}|u(\xi, \zeta)|^2 \mathrm{d}\xi = C$, where C is a constant, to the framework of the NNSE. This quantity has a correspondence to the conservation of an energy flow that is given by

$$\int_{-\infty}^{+\infty}\left[\frac{1}{2\kappa} + \frac{\partial\phi(\xi, \zeta)}{\partial\zeta}\right]|u(\xi, \zeta)|^2 \mathrm{d}\xi = C', \tag{6}$$

where C' is another constant. This exact result generalizes a previously published (approximate) expression for which only fast on-axis phase variations were taken into account.[10,11] The NSE conservation law is recovered from Eq. (6) in the paraxial limit.

Considering the effect that exact solutions of the NSE have had on a wide range of physical systems in which solitonlike solutions exert a dominant influence, the results given above, along with those of Refs. 1–8, are likely to underpin a broad class of exciting new research topics in nonlinear optics in which higher-order effects and related geometries are considered.

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