FROM MAXWELL'S EQUATIONS TO NEW FAMILIES OF HELMHOLTZ SOLITONS

(Invited paper)

Pedro Chamorro-Posada[1], Julio Sánchez-Curto[1], James M. Christian[2], Graham S. McDonald[2]

[1] *Departamento de Teoría de la Señal y Comunicaciones e Ingeniería Telemática, Universidad de Valladolid, ETSI Telecomunicación, Campus Miguel Delibes s/n, Valladolid 47011, Spain.*
Phone: +34-983-42185545, Fax: +34-983-423667, e-mail: pedcha@tel.uva.es
[2] *Joule Physics Laboratory, School of Computing, Science and Engineering, Institute of Materials Research, University of Salford, Salford M5 4WT, UK*
Phone : +44-161-295-5079, Fax: +44-161-295-5575, e-mail: g.s.mcdonald@salford.ac.uk

Abstract **–** In this presentation, we give an overview of new results in Helmholtz soliton theory. Firstly, fundamental considerations are made in terms of new contexts for Helmholtz solitons that arise directly from Maxwell's equations. We will then explore applications involving a variety of different material interfaces and the role of Helmholtz solitons in these configurations. Finally, specific new families of solutions arising from the generalisation of the Manakov equation will be reported.
Keywords**:** Spatial optical solitons, nonlinear Helmholtz equation, non-paraxiality, time-domain simulation

INTRODUCTION

Previous research on the Non-Linear Helmholtz (NLH) equation has permitted the generalization of both bright [1] and dark [2] spatial solitons in Kerr media to the finite-angle regime, where oblique beam propagation may be at an *arbitrarily large angle* relative to the longitudinal axis. In this approach, the intrinsic angular limitations of conventional Non-Linear Schrödinger (NLS) analyses, imposed by the assumption of beam paraxiality, are eliminated. Our analytical investigations are complimented by well-tested numerical techniques, developed specifically for the accurate solution of the NLH equation [3].

The NLH equation is fully equivalent [1] to the Non-paraxial Non-Linear Schrödinger (NNLS) equation

$$\kappa\frac{\partial^2 u}{\partial\zeta^2}+i\frac{\partial u}{\partial\zeta}+\frac{1}{2}\frac{\partial^2 u}{\partial\xi^2}\pm|u|^2u=0\,, \tag{1}$$

which describes the evolution of the normalized complex envelope u of an optical beam. Here, $\zeta=z/L_D$, $\xi=2^{1/2}x/w_0$, $L_D=kw_0^2/2$, $\widetilde{E}(x,z)=E_0u(x,z)\exp(ikz)$ and $k=2\pi/\lambda$. n_0 is the linear refractive index, $\lambda=\lambda_0/n_0$ the optical wavelength, $E_0=(n_0/k|n_2|L_D)^{1/2}$, n_2 the Kerr coefficient and $\kappa=1/(k^2w_0^2)$ is the non-paraxiality parameter. The ± sign flags a focusing or defocusing Kerr non-linearity. Equation (1) retains the full spatial symmetry of the NLH model, and is a more convenient framework for comparing new results with those obtained from paraxial calculations. The NLS equation can be recovered from Eq. (1) when the Helmholtz term $\kappa\partial^2u/\partial\zeta^2$ is neglected.

In this presentation, we give an overview of some recent results in Helmholtz soliton theory. Three topics have been selected for this purpose: The modeling of the propagation properties of Helmholtz solitons directly using the full 2D Maxwell equations [4], the behaviour of solitons incident on non-linear interfaces at oblique angles [5], and new exact analytical vector solitons arising from the Helmholtz-Manakov (H-M) equation [6].

0-7803-9130-6/05/$20.00 © 2005 IEEE

The use of the full 2D non-linear Maxwell equations for analyzing the propagation properties of Helmholtz solitons provides a more general framework free of the restrictions encountered in other approaches. The results support investigations based on the scalar NNLS equation for TE-polarized optical beams in a quasi-2D medium, and allow us to extend previous work on Helmholtz solitons to non-paraxial regimes other than those arising solely from angular considerations.

The reflection and refraction properties of soliton beams at non-linear interfaces have been analyzed extensively using the NLS equation [7]. We present new results of the non-linear reflection and refraction properties of optical solitons at arbitrary incidence angles using an NLH model. Our work highlights the limitations of previous studies based on the NLS equation, which are restricted by the paraxial approximation to considering vanishingly-small incidence angles.

The propagation of spatial vector soliton beams is often described by the Manakov equation. We will report the Helmholtz generalization of the Manakov model and present its exact analytical soliton solutions, derived for both focusing and defocusing Kerr media. These results will be accompanied by an overview of the dynamical properties of the new solutions. Helmholtz-Manakov solitons are found to exhibit non-trivial features that are absent from the corresponding paraxial-based descriptions.

MODELING OF HELMHOLTZ SOLITONS USING MAXWELL'S EQUATIONS

The evolution of a TE-polarized optical field propagating in a non-magnetic two-dimensional medium with electric field $\mathbf{E}(x,z,t)=\hat{\mathbf{y}}E_y(x,z,t)$ is described by the 2D Maxwell equations

$$\frac{\partial E_y}{\partial x}=-\mu_0\frac{\partial H_z}{\partial t}, \qquad \frac{\partial E_y}{\partial z}=\mu_0\frac{\partial H_x}{\partial t} \qquad \text{and} \qquad \frac{\partial H_x}{\partial z}-\frac{\partial H_z}{\partial x}=\varepsilon_0 n^2\frac{\partial E_y}{\partial t}, \tag{2}$$

where propagation takes place in the x-z plane. In a Kerr medium, the refractive index is

$$n^2 \approx n_0^2 + 2n_0 n_2 E_y^{\,2}. \tag{3}$$

For a continuous-wave (CW) beam, the NLH equation can be derived from Eqs. (2) and (3). It governs the evolution of the complex amplitude $\tilde{E}(x,z)$ of the optical field, where $E_y(x,z,t)=\mathrm{Re}\left[\tilde{E}(x,z)\exp(-i\omega_0 t)\right]$. The NNLS equation (1) is then obtained as the corresponding evolution equation for the field envelope.

The behaviour of multi-soliton solutions of the NLS equation can be strongly influenced by the presence of perturbations. Helmholtz-type non-paraxiality acts as such a perturbative contribution during the initial focusing stages of the periodic evolution [8]. The NNLS equation predicts that the Helmholtz operator $\kappa\partial^2/\partial\zeta^2$ modifies the soliton period [8], and this has been confirmed by numerical solution of the full Maxwell equations [9].

When even stronger non-paraxiality is present, a launched high-order soliton can become unstable and undergo a fission effect, whereby the quasi-bound state breaks up into its individual components [8]. Figure 1.a illustrates the splitting of a third-order soliton beam into three fundamental solitons when κ=0.005 in the NNLS equation [8]. Figure 1.b displays the electric field amplitude at a given time t_0, obtained by solving the full Maxwell equations, and verifies the expected behaviour. Using Maxwell's equations as an analytical tool, we have found that the evolution of the third-order soliton can, in fact, exhibit a rich dynamical behavior. A full account of these dynamics will be presented at the conference.

(a) (b)

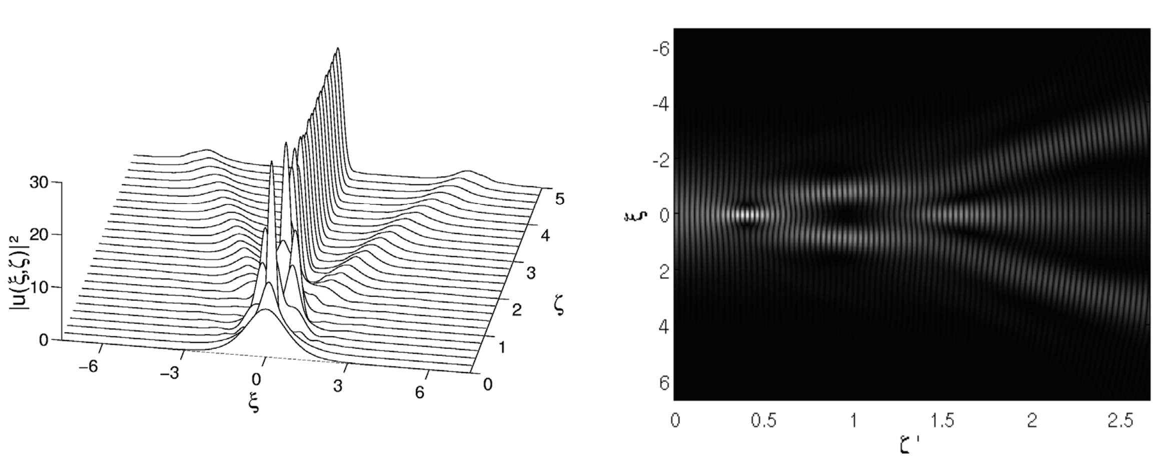

Fig.1 –Splitting of a third-order soliton. The result obtained from the numerical integration of the NNLS equation (a) is compared with the E-field magnitude $|E_y(\xi,\zeta,t_0)|$ computed from the time-domain Maxwell equations (b). In the calculations, equal scalings have been used for the $\xi=2^{1/2}\,x/w_0$ and $\zeta=2^{1/2}\,z/w_0$ coordinates. The ζ coordinate in plot (b) has been adjusted to that of (a) by using $\zeta'=(2\kappa)^{1/2}\zeta$.

HELMHOLTZ SOLITONS AT NON-LINEAR INTERFACES

The evolution of Helmholtz solitons at the interface separating two Kerr-type media can be described by a generalized NNLS equation [5]. Numerical simulations show that when there is a mismatch only in the linear part of the refractive index, the incident solitons are governed by Snell's law. In general, it has been found that the reflection and refraction characteristics of optical solitons possess key features that cannot be adequately described by paraxial theory [7].

In this presentation, we will focus primarily on the analysis of soliton behaviour when the linear refractive index is continuous across the interface. The general solution shows that when a soliton enters a medium with a weaker non-linearity, the outgoing beam may suffer diffractive spreading without limit unless the input power exceeds some critical value (see Fig2.a). On the other hand, when the second medium is characterized by a stronger non-linearity, any excess power associated with the incident soliton causes the input beam to break up into a distribution of narrower solitons (see Fig2.b).

(a) (b)

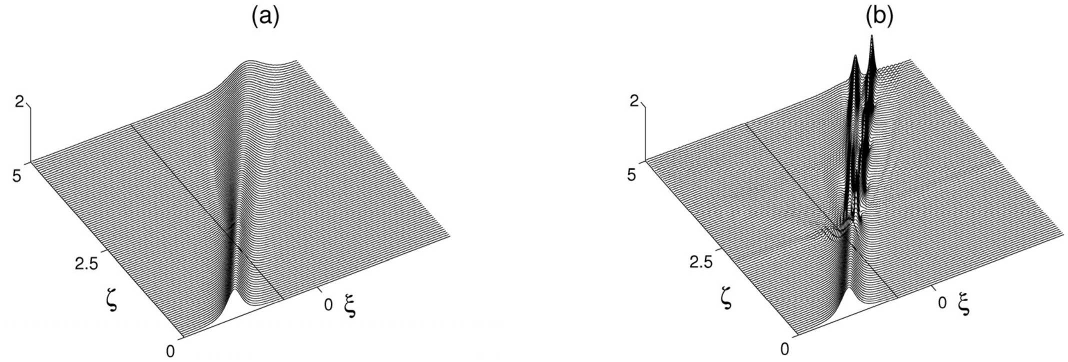

Fig. 2 – Numerical results corresponding to soliton evolution at the inteface between two Kerr-type media when the magnitude of the non-linear refractive index in the second medium is lower (a) and higher (b) than in the first one.

The Helmholtz-interface framework yields corrections to paraxial predictions [7] that can exceed 100%. In particular, significant differences between the two descriptions appear when the magnitude of the non-linearity is higher in the second medium. In the paraxial regime, the number of secondary solitons increases depending on the magnitude of the square root of the relation between the non-linear indexes. The Helmholtz model is, however, more restrictive on the number of solitons formed. Moreover, the multi-soliton structure that develops depends not only on the aforementioned relationship but also on the angle of incidence. We will present a full characterization of the soliton pattern generated in the second medium based on extensive numerical simulation.

HELMHOLTZ-MANAKOV SOLITONS

When the electric field confined to a quasi-2D waveguide has only a single transverse field component, the NLH equation (1) provides an accurate description of scalar wave propagation. When the guided field has two orthogonal transverse components, the appropriate model is the Helmholtz-Manakov (H-M) equation [6],

$$\kappa\frac{\partial^2\mathbf{U}}{\partial\zeta^2}+i\frac{\partial\mathbf{U}}{\partial\zeta}+\frac{1}{2}\frac{\partial^2\mathbf{U}}{\partial\xi^2}\pm\left(\mathbf{U}^\dagger\mathbf{U}\right)\mathbf{U}=\mathbf{0}, \tag{4}$$

The physical scalings are identical to those in Eq. (1), but now the wave field **U** is the single-column two-component vector $\mathbf{U}(\xi,\zeta)=\left[A(\xi,\zeta)\quad B(\xi,\zeta)\right]^T$, where T denotes the transpose. As in the scalar case, the familiar (paraxial) Manakov equation [10] can be recovered from Eq. (4) in the limit that the Helmholtz operator is negligible with respect to other terms. The H-M equation possesses U(2) symmetry, and the evolution of the two perpendicular field components involves a non-linear coupling due to the Kerr effect.

Equation (4) admits four new exact analytical soliton solutions, which have been deriving by combining Ansatz approaches and Hirota's method [11] with the physical geometry of the propagation problem [2]. In both focusing and defocusing cases, there are two distinct solution families. In a focusing Kerr medium, we find bright-bright and bright-dark solitons, where the primary component A is always a bright *sech*-type Helmholtz soliton, and the secondary B is a bright and black *tanh*-type structure, respectively. In the defocusing case, we have dark-bright and dark-dark solitons. The new solutions capture all the physical attributes of Helmholtz scalar solitons [1,2], such as angular beam broadening, modifications to the beam phase and non-trivial corrections to intrinsic velocities. An important point to note is that the bright-dark and dark-bright solutions are not equivalent; they have very different stability properties. The known paraxial Manakov solitons [12] can be recovered from the full Helmholtz solutions when an appropriate multiple limit is enforced. This is a physical and mathematical requirement of Helmholtz soliton theory – paraxial solutions must be found when the system behaves paraxially.

We will present an overview of the stability properties of the new H-M solitons against perturbations to their angular spectra. Initial conditions are chosen for Eq. (4) that correspond to exact paraxial solutions of the Manakov equation, propagating off-axis with some non-zero transverse velocity. Geometrical considerations show that this situation is completely equivalent to the evolution of on-axis Manakov solitons whose widths have been decreased by a factor related to the transverse velocity. In general, the input condition transforms asymptotically into an H-M soliton. Figure 3 shows the typical oscillations found in the amplitude and width as the beam undergoes self-reshaping. If a stationary state emerges from the initial condition, we classify the corresponding H-M soliton as a robust fixed-point attractor of the system dynamics.

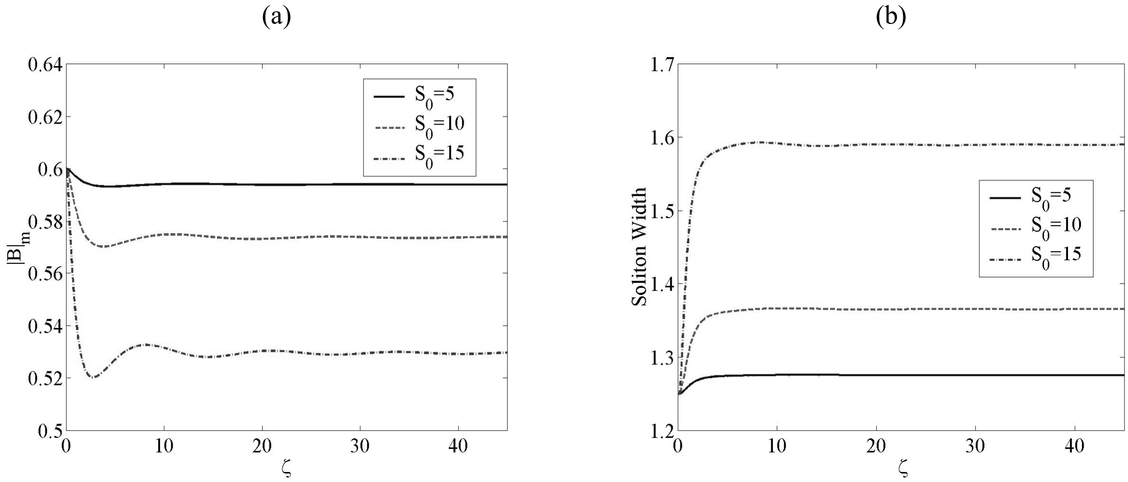

Fig. 4 – Beam self-reshaping oscillations in the (a) peak amplitude, and (b) width, of a perturbed canonical dark-bright soliton. The oscillations vanish in the asymptotic limit $\zeta \rightarrow \infty$, leaving a stationary beam with propagation-invariant characteristics. This is the signature of Helmholtz-Manakov soliton formation.

ACKNOWLEDGEMENT

This work has been funded by MCyT and FEDER, grant number TIC 2003-07020 and JCyL, grant number VA083A05.

REFERENCES

1. P. Chamorro-Posada, G. S. McDonald, G. H. C. New, "Exact soliton solutions of the nonlinear Helmholtz equation: communication," *J. Opt. Soc. Am. B* , vol. 19, p. 1216, 2002.
2. P. Chamorro-Posada and G.S. McDonald, "Helmholtz dark solitons," *Opt. Lett.*, vol. 15, p. 827, 2003.
3. P. Chamorro-Posada, G. S. McDonald and G. H. C. New, "Non-paraxial beam propagation methods," *Opt. Commun.* ,vol. 19, p. 1, 2001.
4. P. Chamorro-Posada and G.S. McDonald, "Analysis of the propagation properties of Helmholtz solitons from Maxwell's equations," *Submitted.*
5. J. Sánchez-Curto, P. Chamorro-Posada and G.S. McDonald, "Helmholtz solitons at nonlinear interfaces," *Submitted.*
6. J.M. Christian, G.S. McDonald and P. Chamorro-Posada, "Helmholtz-Manakov solitons," *Submitted.*
7. A. B. Aceves, J. V. Moloney and A. C. Newell, "Theory of light-beam propagation at nonlinear interfaces. I Equivalent-particle theory for a single interface," *Phys. Rev. A*, vol. 39, p. 1809, 1989.
8. P. Chamorro-Posada, G. S. McDonald and G. H. C. New, "Propagation properties of non-paraxial spatial solitons," *J. Mod. Opt.*, vol. 47, p. 1877, 2000.
9. R.M. Joseph and A. Taflove, "Spatial Soliton Deflection Mechanism Indicated by FD-TD Maxwell's Equations Modeling," *IEEE Photon. Techonol. Lett.*, vol. 6, p. 1251, 1994.
10. S.V. Manakov, "On the theory of two-dimensional stationary self-focusing of electromagnetic waves," *Sov. Phys. JETP*, vol. 38, p. 248, 1974.
11. R. Hirota, "Exact envelope-soliton solutions of a nonlinear wave equation," *J. Math. Phys.*, vol. 14, p. 805, 1973.
12. A.P. Sheppard and Y.S. Kivshar, "Polarized dark solitons in isotropic Kerr media," *Phys. Rev. E*, vol. 55, p. 4773, 1997.