# Exact analytical Helmholtz bright and dark solitons

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- Type I scenarios

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Scalar full Helmholtz theory (Kerr media)

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$$\zeta = \frac{z}{L_D} \qquad \xi = \frac{\sqrt{2}x}{w_0} \qquad u(\xi,\zeta) = \sqrt{\frac{kn_2L_D}{2}}A(\xi,\zeta)$$

$L_D = \dfrac{kw_0^2}{2}$ (diffraction length) $\kappa = \dfrac{1}{k^2w_0^2}$ (nonparaxiality parameter)

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# Bright soliton solution

- The general bright soliton solution is given by the expression

$$u(\xi,\zeta) = \eta \,\mathrm{sech}\left[\frac{\eta(\xi + V\zeta)}{\sqrt{1+2\kappa V^2}}\right] \exp\left[i\sqrt{\frac{1+2\kappa\eta^2}{1+2\kappa V^2}}\left(-V\xi + \frac{\zeta}{2\kappa}\right)\right] \exp\left[-i\frac{\zeta}{2\kappa}\right]$$

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- In the multiple limit $\kappa \to 0$, $\kappa V^2 \to 0$, $\kappa\eta^2 \to 0$, we recover the NLS fundamental soliton

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- The general $V \neq 0$ solutions can be obtained by transforming the $V = 0$ soliton.

# Galilei invariance vs rotational invariance

- The NLS is invariant under the Galilei transformation

$$\xi = \xi' + V\zeta' \qquad \zeta = \zeta'$$

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$$\xi \quad = \frac{\xi' + V\zeta'}{\sqrt{1+2\kappa V^2}} \qquad \zeta = \frac{-2\kappa V\xi' + \zeta'}{\sqrt{1+2\kappa V^2}}$$

$$u'(\xi',\zeta') = \exp\left[-i\left(\frac{V\xi'}{\sqrt{1+2\kappa V^2}} + \frac{1}{2\kappa}\left(1 - \frac{1}{\sqrt{1+2\kappa V^2}}\right)\zeta'\right)\right]u(\xi,\zeta)$$

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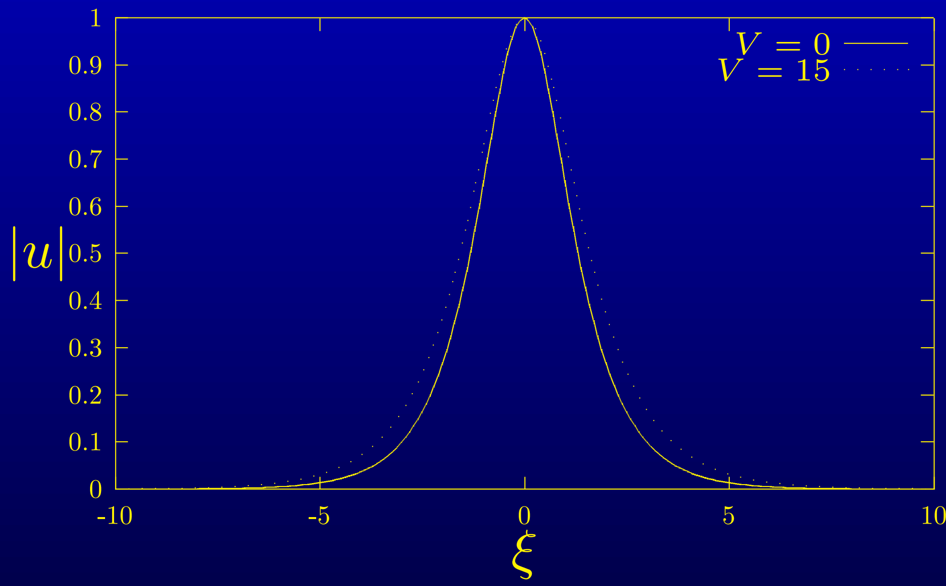
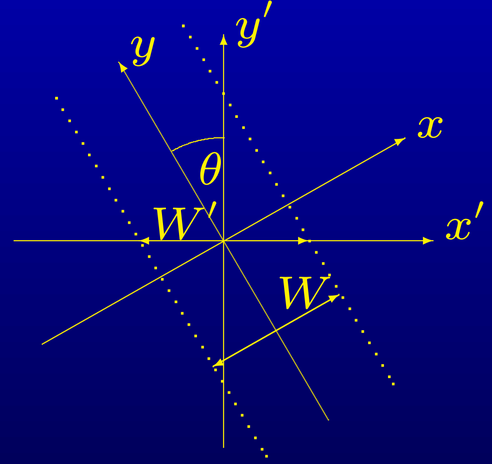
- The additional phase term is introduced due to the *phase reference* used to obtain the normalised equation.

# Properties of fundamental bright solitons

| Paraxial soliton | Helmholtz soliton |
|---|---|
| ***Soliton area*** $A = \xi_0 \eta = 1$<br>• Conserved during propagation.<br>• Independent of $V$ | ***Soliton area*** $A = \xi_0 \eta = \sqrt{1 + 2\kappa V^2}$<br>• Conserved during propagation.<br>• $V$ dependent ($\sqrt{1 + 2\kappa V^2} = 1/\cos(\theta)$). |
| ***Soliton wave-vector***<br>$\mathbf{k} = \left(-V, \frac{1}{2}\left(\eta^2 - V^2\right)\right)$<br>• $V = 0 \Rightarrow k_z = \frac{1}{2}\eta^2$<br>• $V \rightarrow \infty \Rightarrow \mathbf{k} = (-\infty, -\infty)$ | ***Soliton wave-vector***<br>$\mathbf{k} = \left(-V\sqrt{\frac{1+2\kappa\eta^2}{1+2\kappa V^2}}, \frac{1}{2\kappa}\left(\sqrt{\frac{1+2\kappa\eta^2}{1+2\kappa V^2}} - 1\right)\right)$<br>• $V = 0 \Rightarrow k_z = \frac{1}{2\kappa}\left(\sqrt{1 + 2\kappa\eta^2} - 1\right)$<br>• $V \rightarrow \infty \Rightarrow \mathbf{k} = \left(-\frac{\sqrt{1+2\kappa\eta^2}}{\sqrt{2\kappa}}, -\frac{1}{2\kappa}\right)$ |

# Fundamental bright solitons




$\kappa = 10^{-3}$, $\theta \simeq 34^o$

$W' = W/cos\theta$

# Dark soliton solution

- A general dark solution of the defocusing NNLS is found to be

$$u(\xi,\zeta) = u_0\left(A\tanh\Theta + iF\right)\exp\left[i\left(\frac{1-4\kappa u_0^2}{1+2\kappa V^2}\right)^{1/2}\left(-V\xi + \frac{\zeta}{2\kappa}\right)\right]\exp\left(-i\frac{\zeta}{2\kappa}\right)$$

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where $\Theta = \dfrac{u_0 A\left(\xi + W\zeta\right)}{\left(1+2\kappa W^2\right)^{1/2}}$ and $W = \dfrac{V - V_0}{1+2\kappa V V_0}$ is a net transverse velocity involving $V$ (choice of reference) and $V_0$ (intrinsic grey soliton velocity), given by

$$V_0 = \frac{u_0 F}{\left[1-(2+F^2)2\kappa u_0^2\right]^{1/2}}.$$

# Dark solitons

- $F$ and $A$ are real constants, where $F = \pm(1 - A^2)$.
  - ⋆ $F = 0$: black solitons.
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- In the paraxial limit, the NLS dark soliton is obtained:

$$u(\xi, \zeta) = u_0 \left( A \tanh \Theta + iF \right) \exp \left( -iu_0^2 \zeta - i\frac{1}{2}V^2 \xi + \frac{\zeta}{2\kappa} \right)$$

where $\Theta = u_0 A \left[ \xi + (V - Fu_0)\zeta \right]$.

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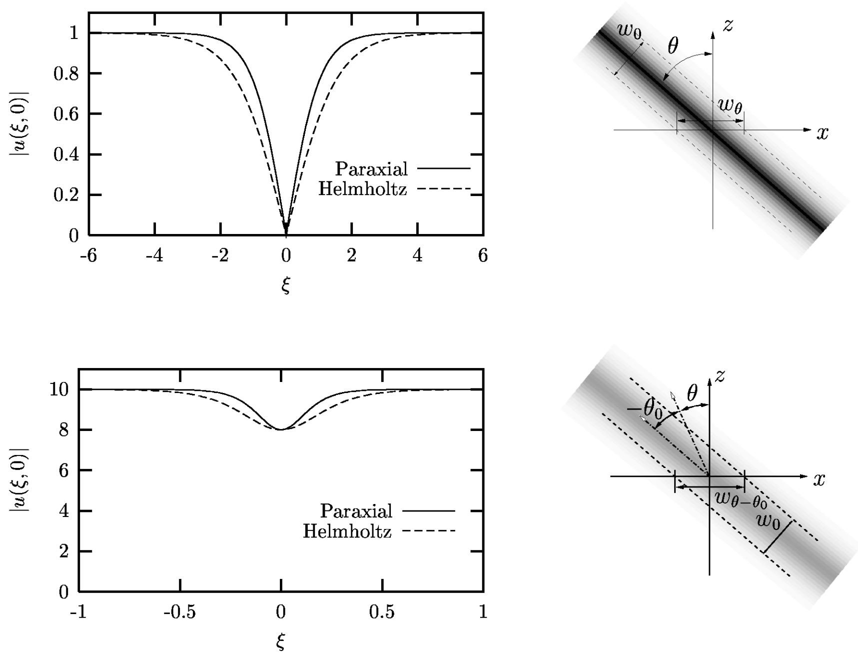
- The beam enlargement factor is $1/\sqrt{1 + 2\kappa W^2} = \sec(\theta - \theta_0)$, where $\theta_0 = \sec^{-1}\sqrt{1 + 2\kappa V_0^2}$ and $\theta = \sec^{-1}\sqrt{1 + 2\kappa V^2}$.

# Properties of dark solitons




$\kappa = 0.001$. Black solitons (top) with $V = 25$ and $u_0 = 1$. Grey solitons (bottom) with $V = 10$ and $F = 0.8$.

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- When $\kappa\eta^2 << 1$

$$S_0 = V\sqrt{\frac{1+2\kappa\eta^2}{1+2\kappa V^2}} \simeq \frac{V}{\sqrt{1+2\kappa V^2}} = \frac{\sin\theta}{\sqrt{2\kappa}},$$

the value of $V$ is fixed by the initial condition and the asymptotic value of soliton area is $\sec\theta=\sqrt{1+2\kappa V^2}$

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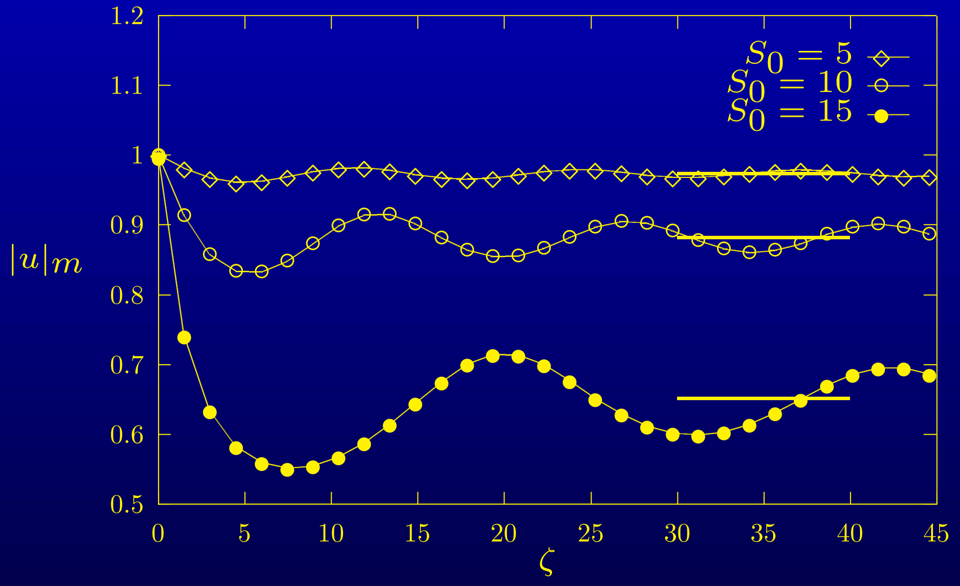
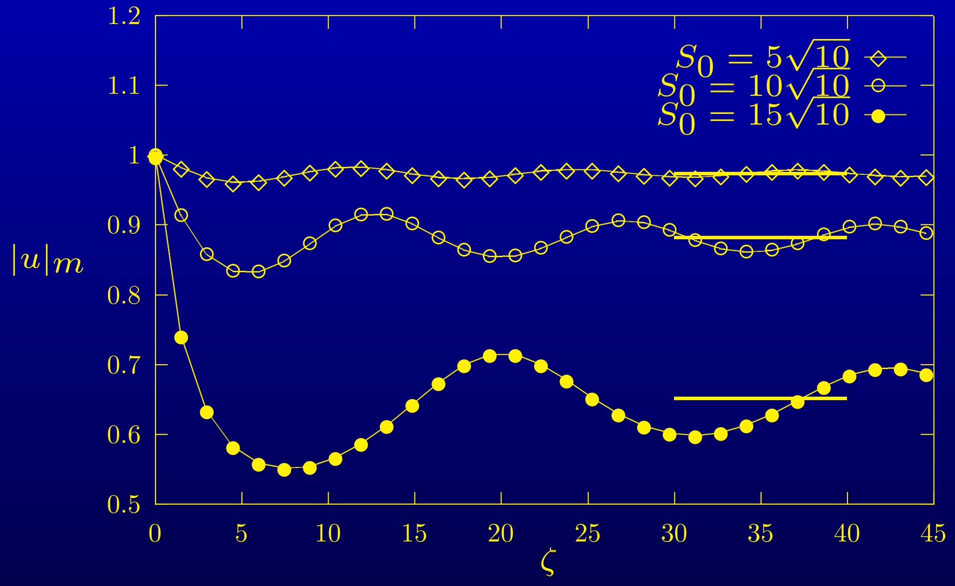



$\kappa = 0.001$

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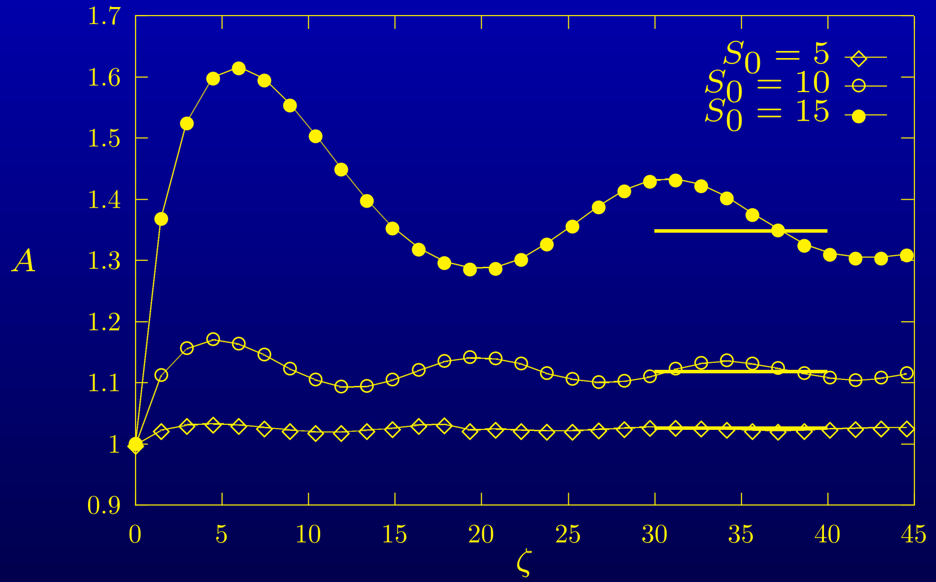


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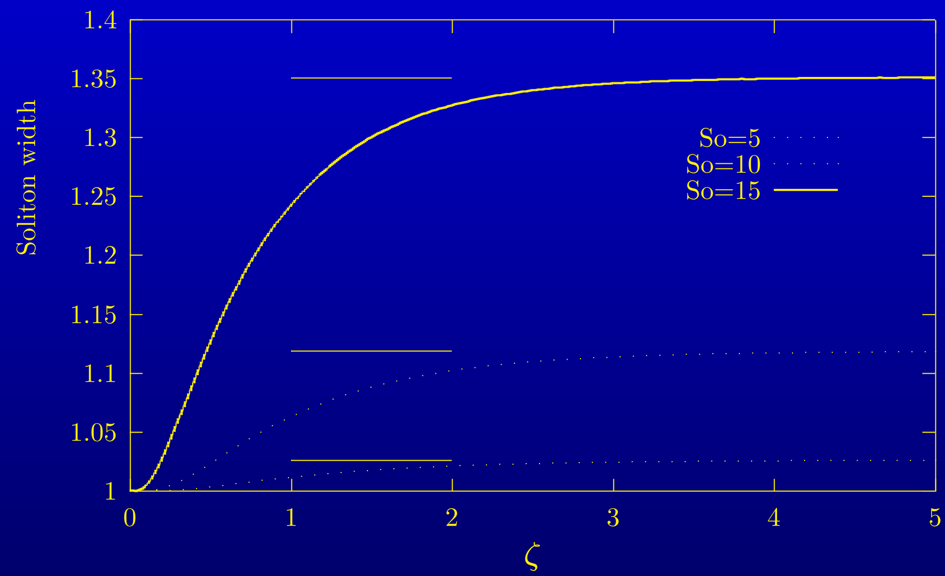


- Asymptotic values of the Helmholtz beam width are $\sqrt{1+2\kappa V^2}$, where $V = S_0/\sqrt{1+2\kappa V^2}$.

- Fast convergence to the asymptotic solutions.

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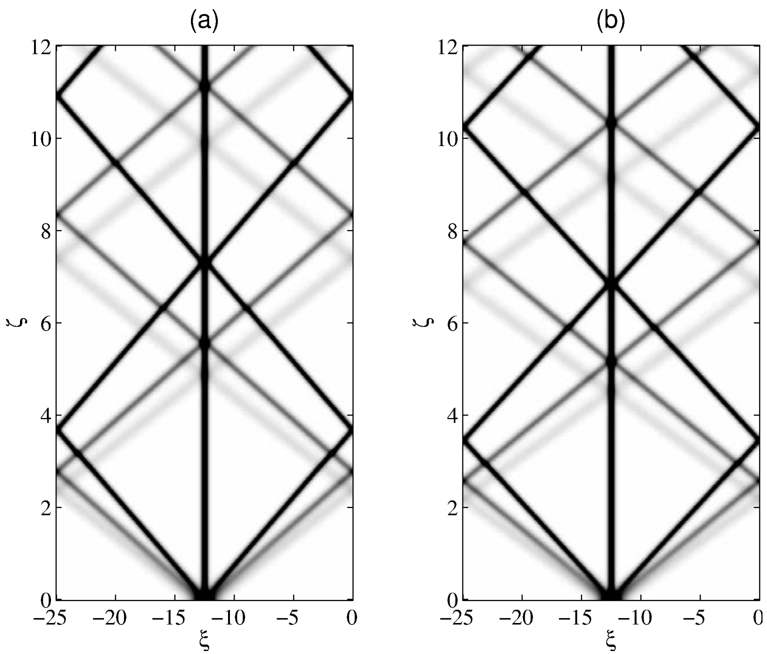
- The figure shows the paraxial (a) and Helmholtz (b) results for $a = 0.26$ and $u_0 = 5$.



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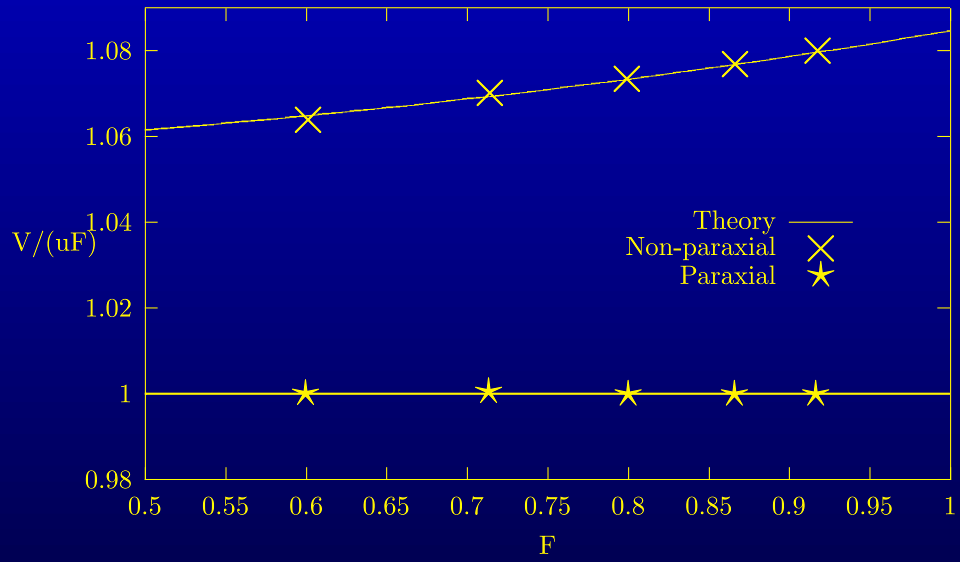


Normalised transverse velocities of simulated grey solitons (symbols) and the corresponding analytical predictions (curves).

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  - ⋆ Perturbed bright and dark solitons evolve asymptotically to exact Helmholtz solitons.
  - ⋆ The spontaneously generated Helmholtz solitons fit the exact solutions presented.

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- Almost exactly counterpropagating solitons.
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# Helmholtz soliton collisions

Three interaction geometries can be identified:

- Almost exactly copropagating solitons.
  (Paraxial theory)
- Almost exactly counterpropagating solitons.
  (Paraxial theory)
- Collisions at intermediate angles.
  (Helmholtz type of nonparaxiality)

# Helmholtz soliton collisions

- Considering two beams $u_1(\xi, \zeta)$ and $u_2(\xi, \zeta)$ propagating at angles $\theta$ and $-\theta$ to the $\zeta$-axis and setting $u(\xi, \zeta) = u_1(\xi, \zeta) + u_2(\xi, \zeta)$ in the NNLS.

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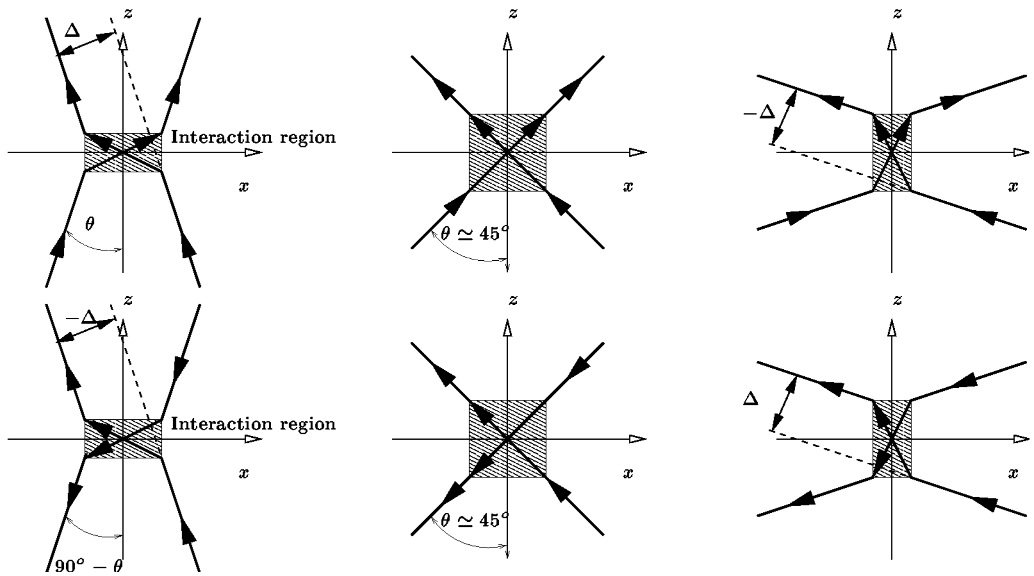
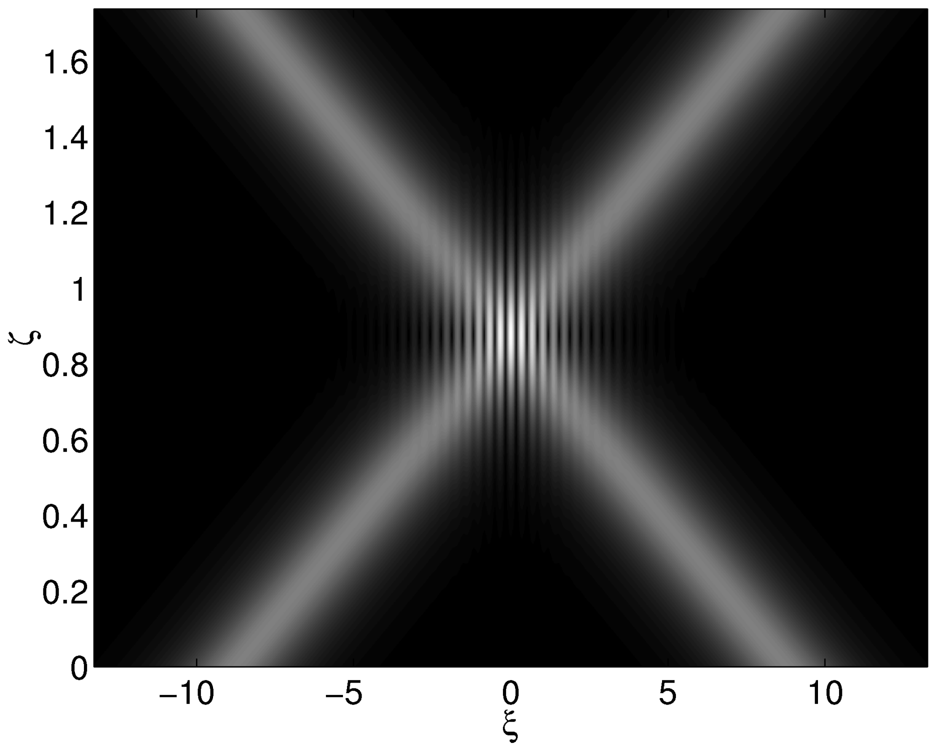
$$\kappa \frac{\partial^2 u_j}{\partial \zeta} + i \frac{\partial u_j}{\partial \zeta} + \frac{1}{2} \frac{\partial^2 u_j}{\partial \xi^2} + \left[ |u_j|^2 + (1+h) |u_{3-j}|^2 \right] u_j = 0$$

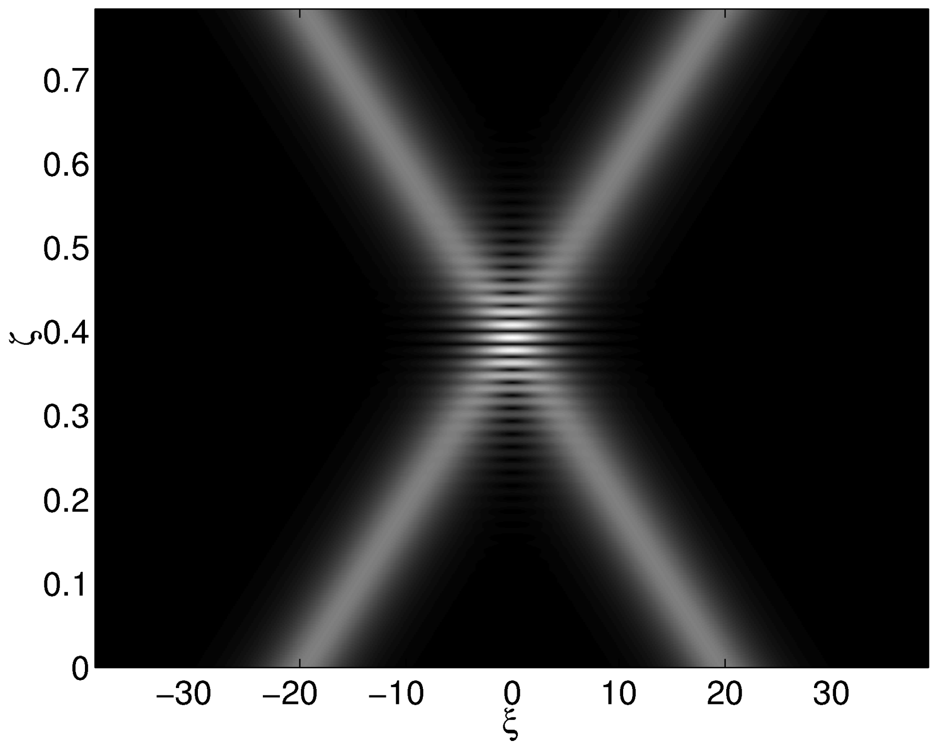
# Helmholtz soliton collisions: geometry

Geometry of Helmholtz soliton collisions in which interactions are dominated by the individual beam intensities. Top panel: copropagating solitons; bottom panel: counterpropagating solitons



# Helmholtz soliton collisions: numerical results




Intensity profiles of two interacting solitons with equal amplitudes. Left panel: two co-propagating solitons; right panel: two counterpropagating solitons.

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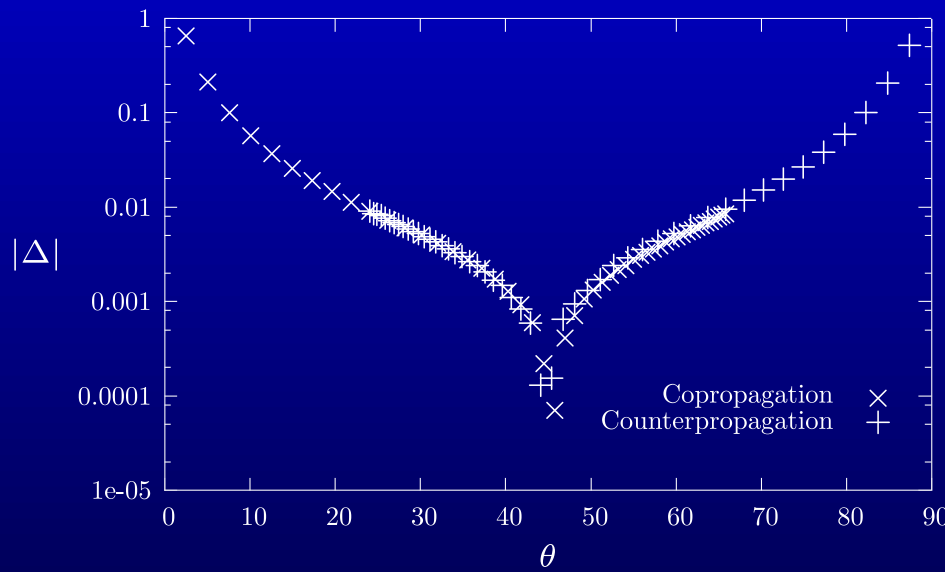


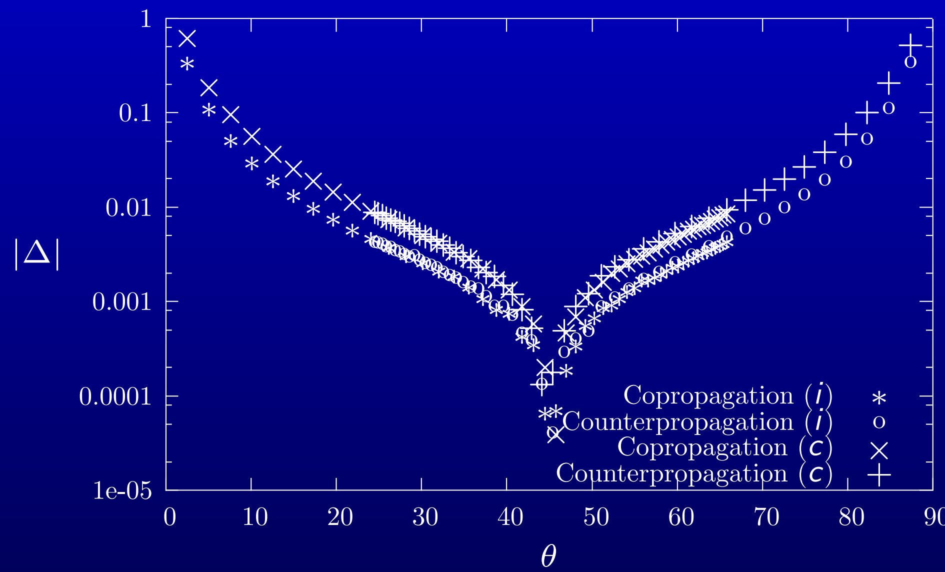


$$\kappa\frac{\partial^2 u}{\partial \zeta} + i\frac{\partial u}{\partial \zeta} + \frac{1}{2}\frac{\partial^2 u}{\partial \xi^2} + |u|^2 u = 0$$

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Magnitude of the trajectory phase shift as a function of the interaction angle $\theta$ for both copropagation and counterpropagation configurations ($\kappa = 10^{-3}$).

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- **Non-integrability of the NHE**.
- Helmholtz solitons are robust modes (individually and during interactions).

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