Helmholtz-Manakov Solitons

James M Christian

Graham S McDonald

Joule Physics Laboratory, Institute of Materials Research,
School of Computing, Science and Engineering,
University of Salford, Salford M5 4WT, U.K.

and

Pedro Chamorro-Posada

Departmento de Teoría de la Señal y Comunicaciones e Ingeniería Telemática,
Universidad de Valladolid,ETSI Telecomunicación,
Campus Miguel Delibes s/n, 47011 Valladolid, Spain.

Talk Outline

- Non-paraxial optical beams
- Non-Linear Helmholtz equations
- Advantages of NLH-type models
- Helmholtz-type of non-paraxiality
- Angular limitations of paraxial models
- New vector solitons
- Solitons as robust attractors
- Conclusions and closing remarks

Non-Paraxial Optical Beams

- Working definition:

> "Non-paraxiality" refers to any optical beam that cannot be adequately described by the paraxial approximation.

- 3 regimes to consider:

Ultra-narrow beams, ✗
Intense self-focusing, ✗
Oblique propagation. ✓

Non-Linear Helmholtz Equations

- Do not invoke the slowly-varying envelope approximation and consider broad beams …

NLH equation → drop $\kappa\partial_{\zeta\zeta}$ → NLS equation …

$$\kappa \frac{\partial^2 u}{\partial \zeta^2} + i\frac{\partial u}{\partial \zeta} + \frac{1}{2}\frac{\partial^2 u}{\partial \xi^2} \pm |u|^2 u = 0$$

Helmholtz term

Helmholtz-Manakov equation → drop $\kappa\partial_{\zeta\zeta}$ → Manakov equation …

$$\kappa \frac{\partial^2 \mathbf{U}}{\partial \zeta^2} + i\frac{\partial \mathbf{U}}{\partial \zeta} + \frac{1}{2}\frac{\partial^2 \mathbf{U}}{\partial \xi^2} \pm \left(\mathbf{U}^\dagger \mathbf{U}\right)\mathbf{U} = \mathbf{0}$$

$$\mathbf{U}(\xi,\zeta) = \begin{bmatrix} A(\xi,\zeta) \\ B(\xi,\zeta) \end{bmatrix}$$

$$\zeta \propto z \qquad \xi \propto x \qquad \kappa = \frac{1}{4\pi^2 n_0^2}\left(\frac{\lambda}{w_0}\right)^2$$

Advantages of NLH-type Models

- Rotationally-invariant model – respects inherent (x,z) symmetry of the uniform medium

  x* and *z* are *PHYSICALLY EQUIVALENT

- Describes accurately waves propagating at **ARBITRARILY LARGE PROPAGATION ANGLES** w.r.t. the z axis

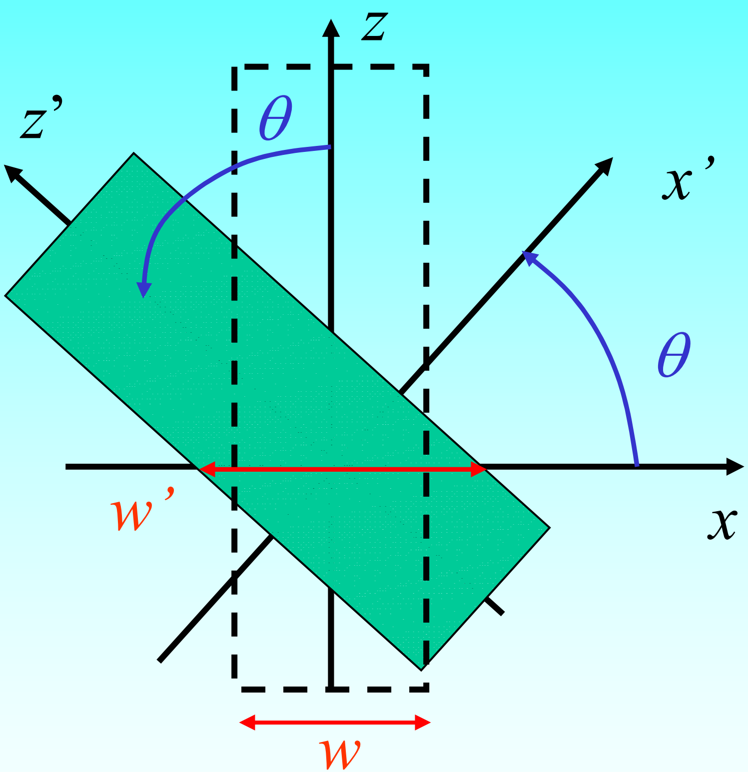
- Can describe multiple beams interacting at arbitrary angles

- Supports non-linear standing-wave solutions

- Offers well-defined connection between transverse velocities in (ξ, ζ) and propagation angles in the (x,z) laboratory frame

Helmholtz-Type Non-Paraxiality

- Invariance laws of the NLH and H-M equations show that …



$$\tan\theta = \sqrt{2\kappa V}$$

Helmholtz (angular) correction can assume any positive value!

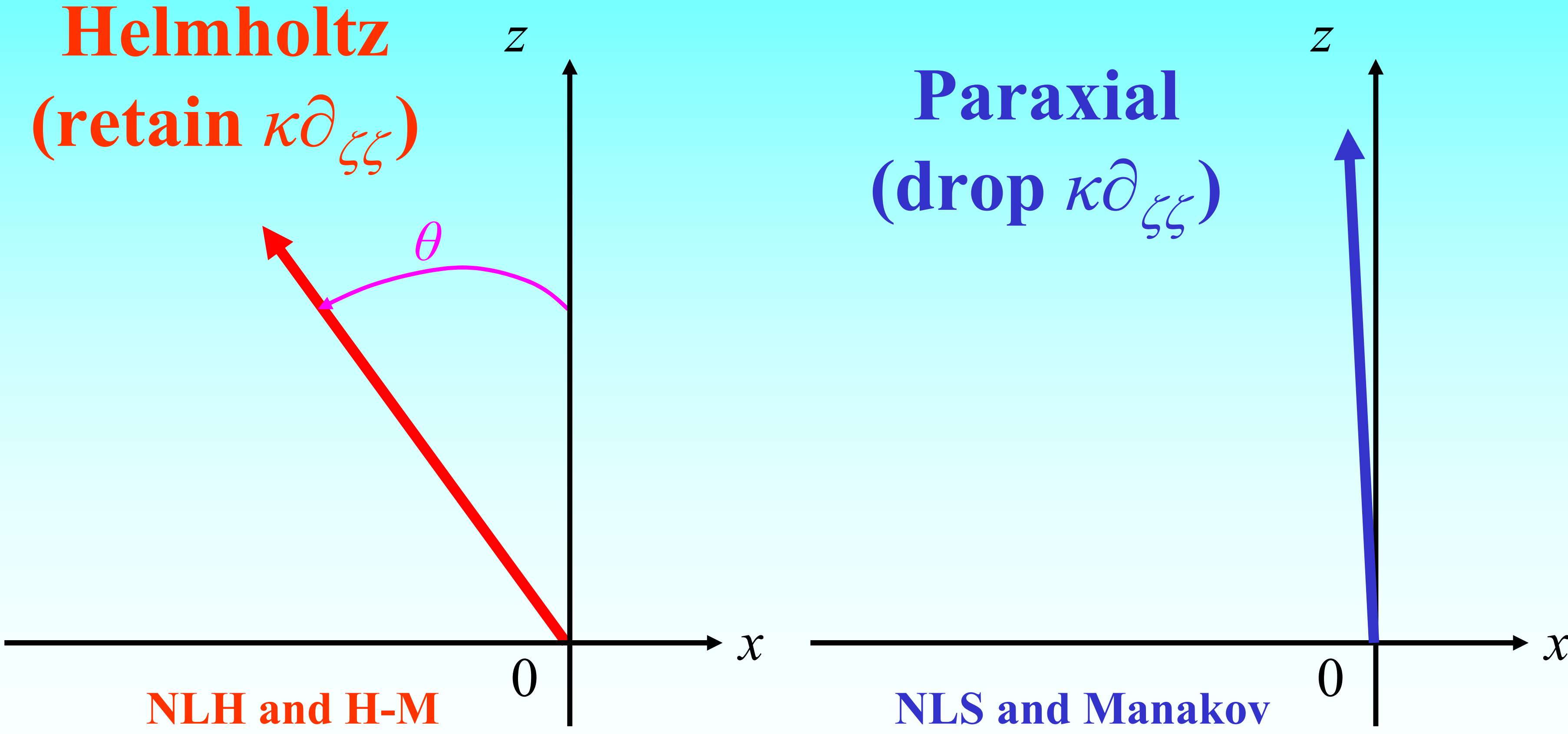
Defines

“Helmholtz-type non-paraxial beams”

Paraxiality: $\theta << 1$ (in radians) so $\kappa V^2 \rightarrow 0$

Angular Limitations of Paraxial Models

- NLS and Manakov equations accurate only for
  VANISHINGLY SMALL PROPAGATION ANGLES
  (arrows denote propagation directions)



REQUIRES THE RESTORATION OF (x-z) SYMMETRY

Solitons of the H-M Equation

- H-M equation admits four new exact analytical soliton solutions ...

2 solutions in a focusing medium ...

BRIGHT-BRIGHT SOLITON

$$\mathbf{U}(\xi,\zeta)=\mathbf{C}\,\eta\,\mathrm{sech}\left(\eta\frac{\xi+V\zeta}{\sqrt{1+2\kappa V^2}}\right)\exp\left[i\sqrt{\frac{1+2\kappa\eta^2}{1+2\kappa V^2}}\left(-V\xi+\frac{\zeta}{2\kappa}\right)\right]\exp\left(-i\frac{\zeta}{2\kappa}\right)$$

$$\mathbf{C}=\begin{bmatrix}\cos(\alpha)\exp(i\delta_A)\\ \sin(\alpha)\exp(i\delta_B)\end{bmatrix}$$

- Generalization of the Manakov and NLH scalar bright solitons

More exotic solutions also exist...

Solitons of the H-M Equation

$$A(\xi,\zeta)=\eta\,\mathrm{sech}\left(a\frac{\xi+V\zeta}{\sqrt{1+2\kappa V^2}}\right)\exp\left[i\sqrt{\frac{1+2\kappa\eta^2}{1+2\kappa V^2}}\left(-V\xi+\frac{\zeta}{2\kappa}\right)\right]\exp\left(-i\frac{\zeta}{2\kappa}\right)$$

$$B(\xi,\zeta)=\sqrt{\eta^2-a^2}\,\tanh\left(a\frac{\xi+V\zeta}{\sqrt{1+2\kappa V^2}}\right)\exp\left[i\sqrt{\frac{1+4\kappa\left(\eta^2-a^2\right)}{1+2\kappa V^2}}\left(-V\xi+\frac{\zeta}{2\kappa}\right)\right]\exp\left(-i\frac{\zeta}{2\kappa}\right)$$

***B* tends to weaken the overall guiding effect**
(e.g. anti-guiding soliton)

Bright-Dark H-M Soliton

Solitons of the H-M Equation

2 solutions in a defocusing medium …

$$A(\xi,\zeta)=A_0\left[\cos\phi\tanh\left(a\frac{\xi+W\zeta}{\sqrt{1+2\kappa W^2}}\right)+i\sin\phi\right]\exp\left[i\sqrt{\frac{1-4\kappa A_0^2}{1+2\kappa V^2}}\left(-V\xi+\frac{\zeta}{2\kappa}\right)\right]\exp\left(-i\frac{\zeta}{2\kappa}\right)$$

$$B(\xi,\zeta)=\sqrt{A_0^2\cos^2\phi-a^2}\,\mathrm{sech}\left(a\frac{\xi+W\zeta}{\sqrt{1+2\kappa W^2}}\right)\exp\left[i\sqrt{\frac{1+2\kappa\left(a^2-2A_0^2\right)}{1+2\kappa W^2}}\left(-W\xi+\frac{\zeta}{2\kappa}\right)\right]\exp\left(-i\frac{\zeta}{2\kappa}\right)$$

$$W=\frac{V-V_0}{1+2\kappa VV_0}\qquad V_0=\frac{a\tan\phi}{\sqrt{1-2\kappa\left(2A_0^2+a^2\tan^2\phi\right)}}$$

***B* tends to weaken the overall guiding effect**
(e.g. anti-guiding soliton)

Dark-Bright H-M Soliton

Solitons of the H-M Equation

$$A(\xi,\zeta) = A_0\left[\cos\phi_1 \tanh\left(a\frac{\xi + W\zeta}{\sqrt{1+2\kappa W^2}}\right) + i\sin\phi_1\right]\exp\left[i\sqrt{\frac{1-4\kappa\chi^2}{1+2\kappa V_1^2}}\left(-V_1\xi + \frac{\zeta}{2\kappa}\right)\right]\exp\left(-i\frac{\zeta}{2\kappa}\right)$$

$$B(\xi,\zeta) = B_0\left[\cos\phi_2 \tanh\left(a\frac{\xi + W\zeta}{\sqrt{1+2\kappa W^2}}\right) + i\sin\phi_2\right]\exp\left[i\sqrt{\frac{1-4\kappa\chi^2}{1+2\kappa V_2^2}}\left(-V_2\xi + \frac{\zeta}{2\kappa}\right)\right]\exp\left(-i\frac{\zeta}{2\kappa}\right)$$

$$\chi^2 \equiv A_0^2 + B_0^2$$

$$A_0^2\cos^2\phi_1 + B_0^2\cos^2\phi_2 = a^2$$

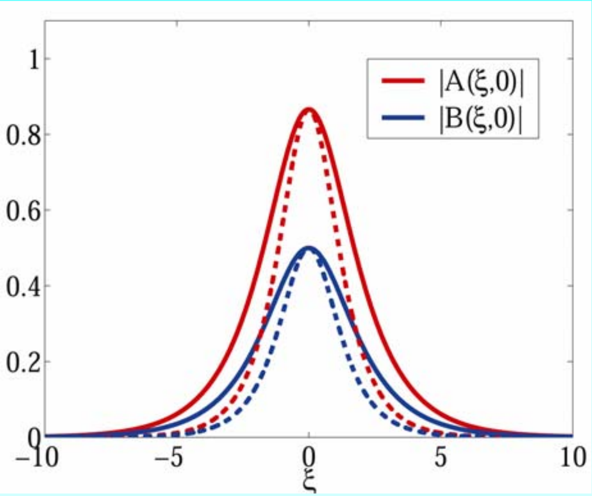
$$W = \frac{V - V_0}{1 + 2\kappa V V_0} \qquad V_{0j} = \frac{a\tan\phi_j}{\sqrt{1-2\kappa\left(2\chi^2 + a^2\tan^2\phi_j\right)}}$$

Dark-Dark H-M Soliton

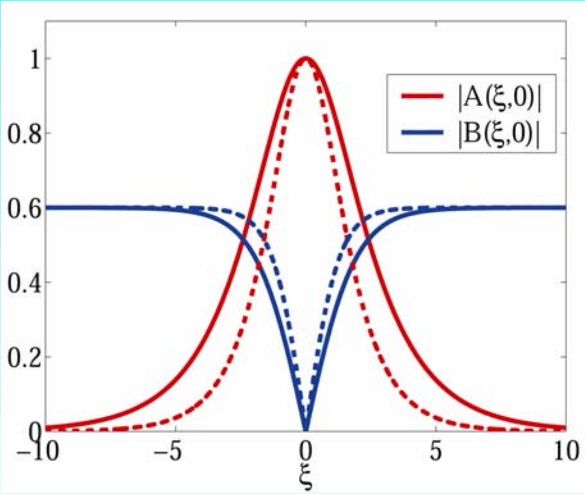
Features of the New Solitons

- Angular beam broadening present in all solutions, e.g. $\theta \approx 48.2^o$,

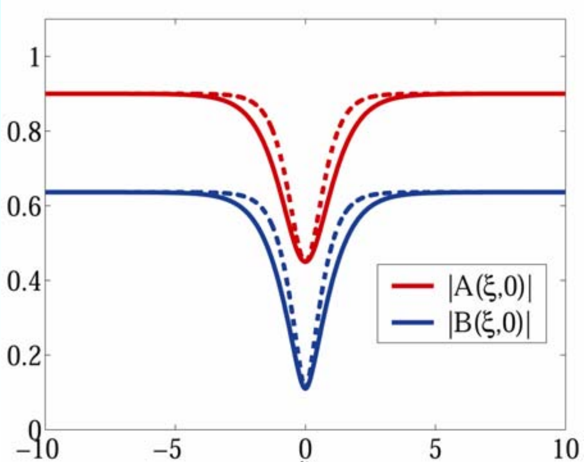
Bright-Bright



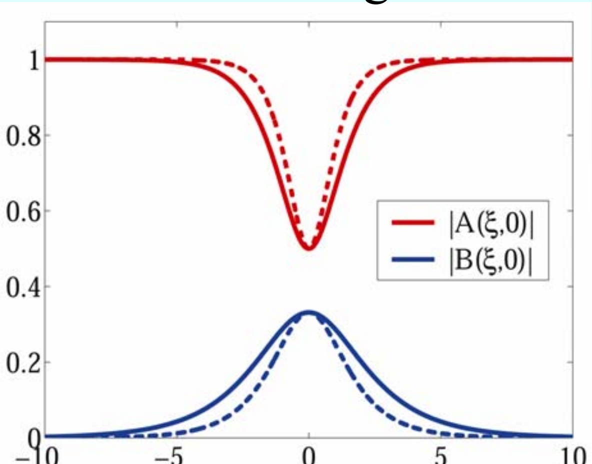
Bright-Dark



Dark-Dark



Dark-Bright



Features of the New Solitons

- Non-trivial corrections to the soliton velocities …

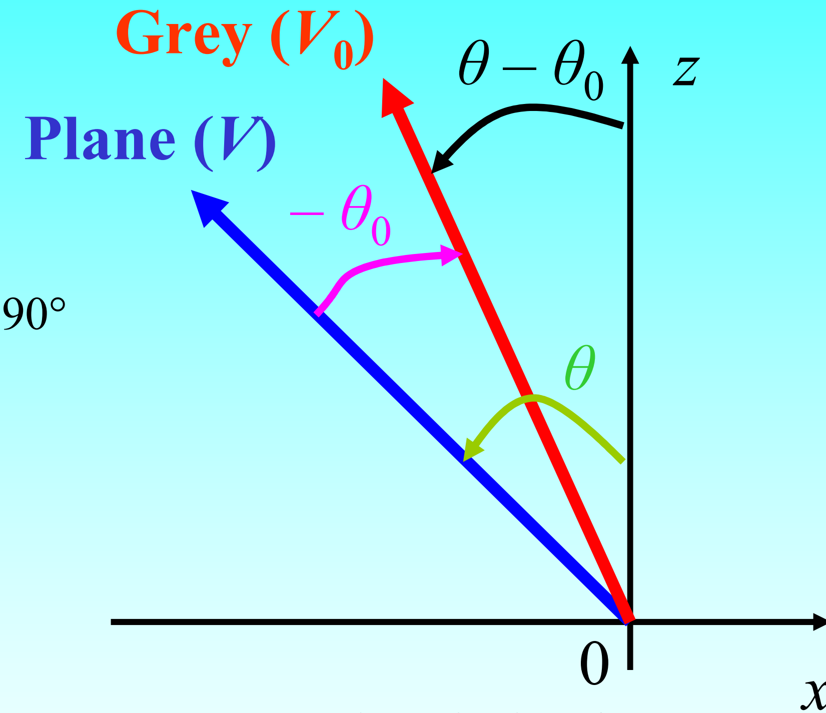
Maximum value of the "greyness" parameter

$$\phi_{max} = \tan^{-1}\left(\frac{1-2\kappa A_0^2}{2\kappa a^2}\right),$$

when the grey component propagates at $\theta_0 = 90°$ (w.r.t. the plane background) and $V_0 \to \infty$.

For Manakov solitons, $V_0 = a\tan\phi$

NO PHYSICAL LIMITS



- For dark-bright and dark-dark solitons, propagation is in the *net direction* $\theta - \theta_0$, assigned a *net velocity* W, where

$$\cos(\theta-\theta_0) = \frac{1}{\sqrt{1+2\kappa W^2}} \quad \text{leads to} \quad W = \frac{V-V_0}{1+2\kappa V V_0}$$

Features of the New Solitons

- For dark-bright soliton, propagating solutions only exist when

$$4\kappa A_0^2 < 1$$

else the beam is evanescent in ζ

→ plane background has a maximum allowable intensity

- Physical interpretations:

(1) non-linear phase shift < linear phase shift
(2) refractive-index must remain positive
(*implicit* in paraxial models)

Helmholtz solitons have an *explicit* maximum refractive-index change!

Recovery of the Manakov Solitons

- Manakov solitons must be recovered

  in the limit that the system behaves paraxially!

- Recovery by enforcing the *simultaneous* triple limit:

Not too narrow

$$\kappa \to 0$$

Not too intense

$$\kappa \times (\text{amplitude})^2 \to 0 \quad \equiv \quad \kappa \left| \frac{\partial^2}{\partial \zeta^2} \right| \to 0$$

Propagation angle not too large

$$\kappa \times (\text{velocity})^2 \to 0$$

- Helmholtz operator contributes to:
  - Propagation of ultra-narrow optical beams
  - Intense self-focusing (rapid phase variations)
  - *Oblique-propagation effects*

***** NLH-*type* DESCRIPTION ESSENTIAL FOR ANGULAR REGIMES *****

Solitons as Robust Attractors

Initial-Value problem → Choose exact Manakov solitons …

e.g. dark-bright,

$$A(\xi,0) = \tanh(a\xi)\exp(-iS_0\xi),$$

$$B(\xi,0) = \sqrt{1-a^2}\,\mathrm{sech}(a\xi)\exp(-iS_0\xi)$$

↓

Rotational symmetry identifies a relationship $\theta = f(S_0)$

$S_0 = 5, 10, 15$

→ $\theta = 12.9°, 26.6°, 42.1°$.

↓

Beam in rotated frame ≡ perturbed on-axis Manakov soliton with width decreased by $\sqrt{1+2\kappa V^2}$

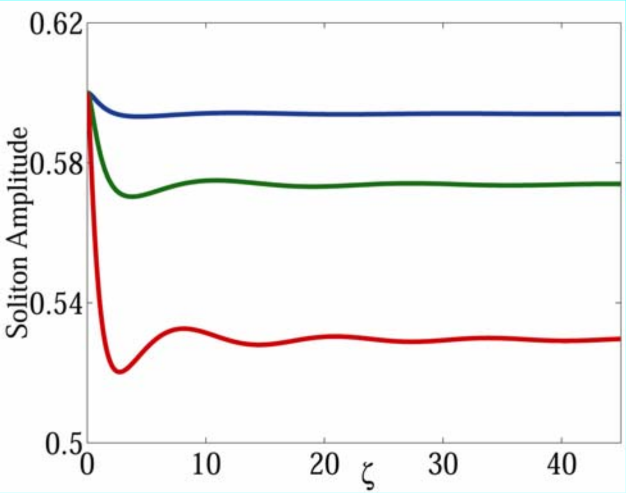
↓

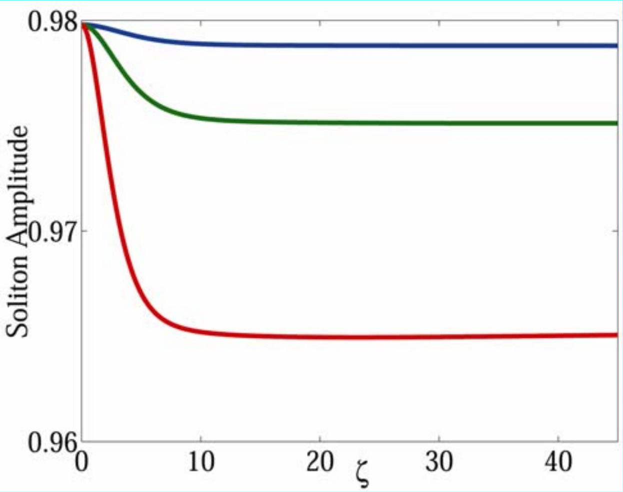
OBSERVE BEAM RESHAPING OSCILLATIONS.

DOES A HELMHOLTZ SOLITON EMERGE ASYMPTOTICALLY?

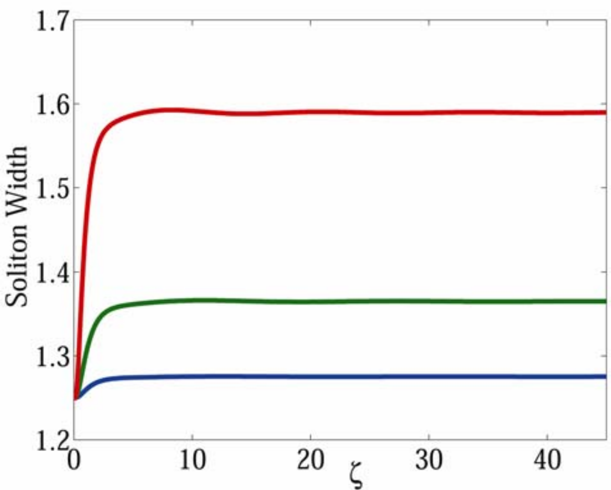
Overview of Solitons as Robust Attractors

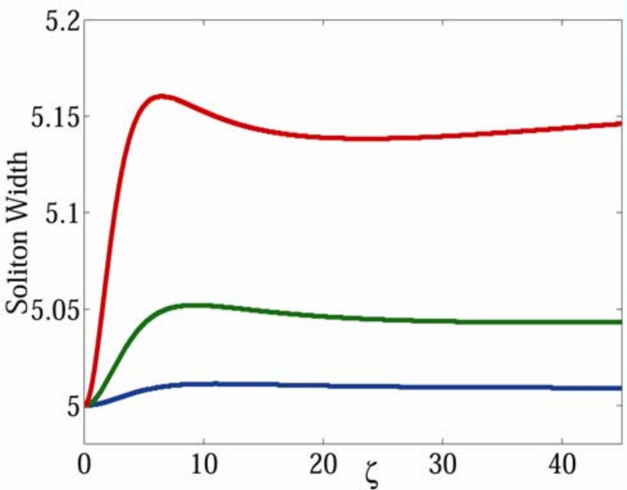
- Initial conditions undergo oscillations in their amplitudes …





… and widths (— $S_0 = 5$, — $S_0 = 10$, — $S_0 = 15$)





Conclusions

- New vector model has been introduced to describe the propagation and interaction of multi-component waves at arbitrarily large angles
- Four new families of exact analytical Helmholtz soliton solution have been derived (Hirota's method)
- New physical properties identified
- Numerical investigations show that H-M solitons are generally robust structures with wide "basins of attraction"

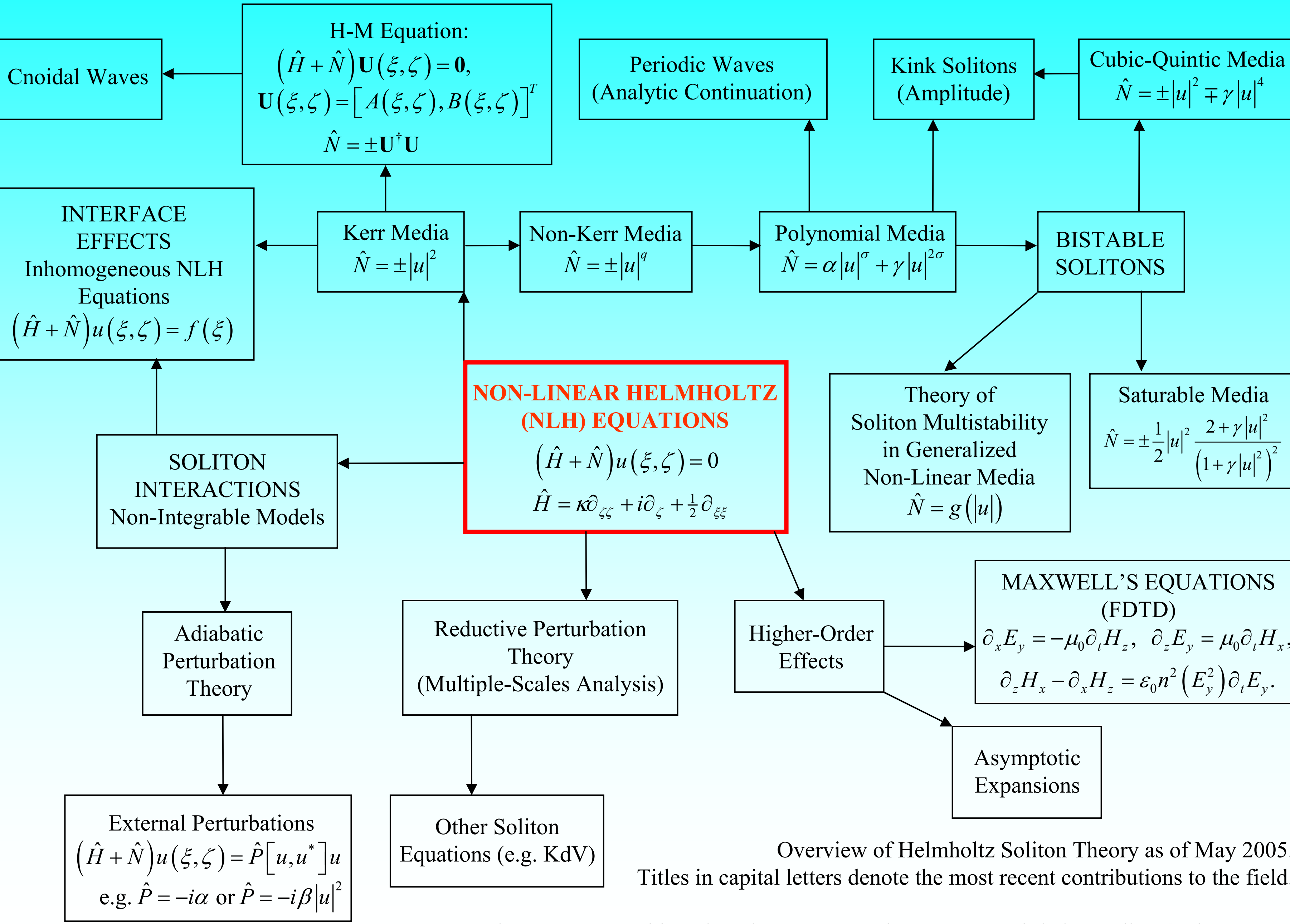
CLOSING REMARKS

It is not necessary to treat the Helmholtz operator $\kappa\partial_{\zeta\zeta}$ as a perturbative term …

… exact analytical solutions to NLH model equations can often be found

For example …

HELMHOLTZ SOLITON MAP



Overview of Helmholtz Soliton Theory as of May 2005.
Titles in capital letters denote the most recent contributions to the field.