Fresnel Diffraction from Polygonal Apertures

Jungang. Huang, J. M. Christian, and G. S. McDonald
Joule Physics Laboratory, School of Computing, Science and Engineering,
Institute of Materials Research, University of Salford, Salford M5 4WT, UK

Two new general analytical descriptions of Fresnel diffraction patterns from polygon apertures are reported, for the first time. Our results have fundamental importance as well as specific applications, promising new physical insights into diffraction-related phenomena. Fractal laser mode patterns, calculated to prescribed accuracy, illustrate the unique power of our new results.

Introduction

We present, for the first time, two complementary analytical techniques for calculating the Fresnel diffraction patterns from a polygonal aperture illuminated by a plane wave. These **frameworks are exact**, in that they do not involve any further approximation beyond the (paraxial) Fresnel integral. Here, we consider regular polygonal apertures, but our results are readily extended to describe near-field diffraction from **apertures of arbitrary shape**. Our results are of fundamental importance and have specific applications where standard methods, such as Fast Fourier Transform (FFT) techniques, fail. For example, in unstable-resonator mode calculations, both (paraxial beam) ABCD matrix modelling and existing semi-analytical methods can only give accurate results in limited parameter regimes. Consequently, **a complete and detailed study of optical fractal laser modes [1] has not previously been possible**. A specific advantage of our formulisms is the ability to calculate and store the fine details of only a small portion of one, or many, complex diffraction patterns.

Moreover, the explicit mathematical form of our results may also lend **physical insight into a wide range of diffraction-related phenomena** in physics. For example, insight into the physical origin of excess quantum noise in lasers, where the transverse symmetry of an aperturing element has been shown to play a central role in the observed phenomena [2]. While Fraunhoffer (far-field) diffraction patterns have been known for many years, there has been relatively little equivalent published work in the Fresnel (near-field) regime. The far-field approximation allows the expression of diffraction patterns and descriptions of derivative concepts (eg in holography, filtering, convolution and coherence) as simple Fourier integrals and transform theorems, respectively. Our new results permit the mathematical and physical expression of near-field diffraction patterns in terms of their elemental spatial structures (edge-waves). It is plausible that our results may also open future doors in the **development of derivative concepts in Fresnel Optics**.

(i): *S*-Function Method

With the aperture-plane and image-plane axes denoted by (ξ, η) and (x, y), respectively, we define relative coordinates as $u=\sqrt{2/\lambda L}\,(x-\xi)$ and $v=\sqrt{2/\lambda L}\,(y-\eta)$, where L is the distance between the aperture and image planes and λ is the wavelength of the incident plane wave. The functions $V_1(u)$ and $V_2(u)$ describe the shape of the aperture (see ***FIG. 1***). The Fresnel diffraction pattern can then be written as [3-5]:

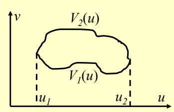

FIG. 1. *Schematic of the u and v parametric description of an aperture with arbitrary shape.*

$$U(P)=U_0\{\varepsilon+B(u_2,u_1)+\frac{1}{1+i}\int_{u_1}^{u_2}\exp(i\frac{\pi}{2}u^2)B[V_2(u),V_1(u)]du\}$$

where ε is equal to unity if the observation point P lies within the aperture, and is zero if P is outside this region. U_0 is the complex amplitude of the illuminating plane wave; and

$$B(\chi_1,\chi_2)=\frac{C(\chi_1)-C(\chi_2)+i[S(\chi_1)-S(\chi_2)]}{1+i}-1$$

$C(\chi)$ and $S(\chi)$ denote the Fresnel cosine and sine integrals, respectively. These integrals can be evaluated efficiently, and to a prescribed accuracy, using well-known rational approximations.

(ii): Line-Integral Method

Here, we define two complex vectors: $\mathbf{p}=x+iy$ and $\mathbf{q}=\xi+i\eta$. For a polygonal aperture with N sides (***FIG. 2***), the position of the vertex A_j is given by the position vector $\mathbf{q}_j$. By defining vectors $\mathbf{L}_j=\mathbf{q}_{j+1}-\mathbf{q}_j$ and $\mathbf{t}_j=\mathbf{L}_j/L_j$ (where L_j is the length of $\mathbf{L}_j$), the displacement vector of a general point on side L_j can be written as $\mathbf{q}=\mathbf{q}_j+\mathbf{t}_j l$, where l is the distance from vertex A_j. Denoting the unit outward normal of side L_j as $\mathbf{n}_j$, and the illuminating spatial wavenumber as k, the Fresnel diffraction pattern is found to be [6]:

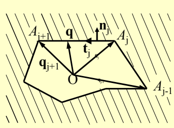

FIG. 2. *Aperture description in the Line-Integral Method, .*

$$U(\mathbf{p})=U_0\Big[\varepsilon-\frac{1}{2\pi}\sum_{j=1}^{N}(\mathbf{q}_j-\mathbf{p})\bullet\mathbf{n}_j\exp\{i\frac{k}{2L}[(\mathbf{q}_j-\mathbf{p})\bullet\mathbf{n}_j]^2\}\times\int_{(\mathbf{q}_j-\mathbf{p})\cdot\mathbf{t}_j}^{L_j+(\mathbf{q}_j-\mathbf{p})\cdot\mathbf{t}_j}\exp(i\frac{k}{2L}l^2)/\{l^2+[(\mathbf{q}_j-\mathbf{p})\bullet\mathbf{n}_j]^2\}dl\Big]$$

Results: Fresnel Diffraction Patterns

The *S*-Function and Line-Integral approaches have been used to calculate the diffraction patterns for *N*-sided polygonal apertures. Extensive computational investigations have verified that our **two approaches** are completely equivalent, and **produce identical results**.

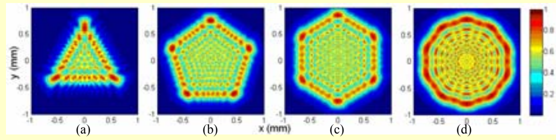

FIG. 3. *Diffraction patterns for: (a) triangular, (b) pentagonal, (c) hexagonal, and (d) decahedronal apertures. Parameters are* λ = 0.5 μm, a = 1 mm, L = 100a. *In these cases, the number of sides is:* n = 3, 5, 6 *and* 10 , *respectively.*

The amount of fine detail present in each Fresnel diffraction pattern can be quantified by the effective Fresnel Number $F_{eff}=a^2/\lambda L+f(n)$ [7], where a is the radius of a circle inscribing an n-sided polygon, L is the distance from the aperture to the observation plane ($L^2 \gg a^2$ is a paraxial approximation inherent to this near-field description), λ is the wavelength of the illuminating light, and $f(n) = 0.30618n^2 - 0.19533n - 0.68095$ is an expression accounting for the geometrical structure of the aperture. Our results verify that the detail of the diffraction patterns increases with an increase of this Fresnel number. ***FIG. 3*** and ***FIG. 4*** illustrate the change in diffraction pattern with the increase of n and decrease of L, respectively.

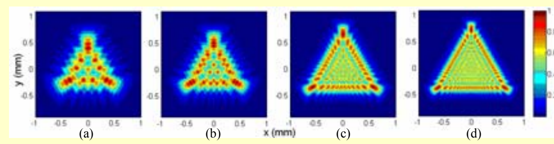

FIG. 4. *Diffraction patterns of a triangular aperture at different Fresnel numbers* F_{eff}. *(a)* F_{eff} = 3.06, L = 200 mm, *(b)* F_{eff} = 3.89, L = 150 mm, *(c)* F_{eff} = 7.23, L = 75 mm, *(d)* F_{eff} = 10.56, L = 50 mm. *Other parameters are* λ = 0.5 μm *and* a = 1 mm.

Advantage and Application

Compared with the standard alternative (FFT) method, the advantage of the *S*-Function and Line-Integral Methods is that they allow just a small portion of the diffraction pattern to be calculated and examined in detail. This is obviously important when localized fine structure is an essential feature of the phenomenon examined. This is precisely our motivation within a new generalization of Southwell's virtual source theory [8-10] that allows us to calculate the field of **fractal laser modes of arbitrary order and of unprecedented accuracy**. Calculating these fractal eigenmodes entails unfolding the resonator into a sequence of apertures, but **only specific regions of the resulting diffraction patterns are needed**. In comparison, the FFT method requires one to calculate the entire field pattern arising from each aperture. There can be very many such apertures and these are typically of widely-varying size. This renders the FFT method inappropriate and ineffective in important parameter regimes of this phenomenon. ***FIG. 5*** shows detailed eigenmode patterns calculated from our new formulations.

Fractal Laser Modes of Prescribed Accuracy

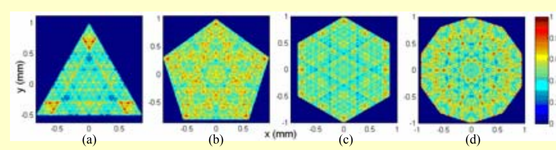

FIG. 5 *Detailed two-dimensional fractal modes of unstable resonators with (a) triangular, (b) pentagonal, (c) hexagonal, and (d) decahedronal output mirrors. The cavities have a magnification of* M = 1.5 , *and an equivalent Fresnel number, defined in [1-4], of* F_{eq} = 50.

References

[1] G. P. Karman, G. S. McDonald, G. H. C. New, and J. P. Woerdman, Nature **402**, 138 (1999); G. S. McDonald, G. P. Karman, G. H. C. New and J. P. Woerdman, J. Opt Soc Am B **17**, 524 (2000); G. H. C. New, M. A. Yates, J. P. Woerdman and G. S. McDonald, Opt. Commun. **193**, 261 (2001).

[2] G. P. Karman, G. S. McDonald, J. P. Woerdman and G. H. C. New, App. Opt. **38**, 6874 (1999).

[3] M. Born and E. Wolf, *Principles of Optics*, 6th edition (Pergamon, Oxford, 1980).

[4] A. E. Siegman, *Lasers* (University Science Books, Oxford, 1986).

[5] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions* (Wiley-Interscience, New York, 1972); K. D. Mielenz, J. Res. Natl. Inst. Stand. Technol. **102**, 363 (1997); K. D. Mielenz, J. Res. Natl. Inst. Stand. Technol. **105**, 589 (2000).

[6] J. H. Hannay, J. Mod Opt. **47** 121(2000); J. S. Asvestas, J. Opt. Soc. Am. A **2**, 879 (1985).

[7] S. Wang, Q. Lin, L. Yu, and X. Xu, Appl. Opt. **39**, 3453 (2000).

[8] W. H. Southwell, Opt. Lett. **6**, 487 (1981); W. H. Southwell, J. Opt. Soc. Am. A **3**, 1885 (1986); M. Berry, C. Storm and W. van Saarloos, Opt. Commun. **197**, 393 (2001); M Berry, Opt. Commun. **200,** 321 (2001); M. A. Yates, G. H. C. New and T. Albaho, J. Mod. Opt. **51**, 657 (2004).

[9] K. Petermann, IEEE J. Quantum Electron. **QE-15**, 566 (1979); G. H. C. New, J. Mod. Opt. **42**, 799 (1995).

[10] S. Ganci, J. Opt. Soc. Am. A **1**, 559 (1984); R. C. Smith and J. S. Marsh, J. Opt. Soc. Am **64**, 798 (1974); Jirí Komrska, J. Opt. Soc. Am. **72**, 1382 (1982).