Helmholtz Solitons in Non-Kerr Media

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The generalized Non-Linear Helmholtz (gNLH) equation can be used to model broad spatial beams propagating at arbitrarily large angles, relative to the reference direction, in planar-waveguide geometry. The physical equivalence of transverse and longitudinal dimensions is preserved, and new effects are predicted which have no counterpart in paraxial theory. Exact analytical soliton solutions and conserved quantities of the gNLH equation are presented. Well-tested numerical perturbative techniques examine the role of the new solitons as robust attractors in the system dynamics.

gNLH Equation & Soliton Solutions

The propagation of spatial beams in planar waveguides is routinely described by models based on the Non-Linear Schrödinger (NLS) equation. Such equations are constrained by the paraxial approximation, making them inappropriate for studying some regimes of physical interest. When the dominant type of non-paraxiality in the system arises solely from the oblique propagation of broad beams, an accurate model requires the restoration of (x,z) symmetry in the governing wave equation, rather than higher-order ultranarrow-beam corrections [1]. The gNLH equation has been derived to describe the evolution of beams propagating at arbitrary angles (with respect to the reference direction) in media whose non-linear refractive-index deviates from the archetypal Kerr (proportional to intensity) response [2-4].

$$\kappa\frac{\partial^2 u}{\partial\zeta^2}+i\frac{\partial u}{\partial\zeta}+\frac{1}{2}\frac{\partial^2 u}{\partial\xi^2}+\alpha|u|^{\sigma}u+\gamma|u|^{2\sigma}u=0$$

This equation admits exact analytical soliton solutions that exhibit non-trivial corrections to their paraxial counterparts [5]:

$$u(\xi,\zeta)=\left[\frac{\eta}{\cosh\Theta+\Gamma}\right]^{\frac{1}{\sigma}}\exp\left[i\sqrt{\frac{1+2\kappa(\mu/\sigma)^2}{1+2\kappa V^2}}\left(-V\xi+\frac{\zeta}{2\kappa}\right)\right]\exp\left(-i\frac{\zeta}{2\kappa}\right),$$

$$\Theta(\xi,\zeta)=\frac{\mu(\xi+V\zeta)}{\sqrt{1+2\kappa V^2}},\qquad \Gamma\equiv\Gamma_{\pm}(\mu)=\pm\left[1+\left(\frac{\mu^2}{2\sigma^2}\right)\frac{(2+\sigma)^2}{1+\sigma}\frac{\gamma}{\alpha^2}\right]^{-\frac{1}{2}},\qquad \eta=\eta(\mu)=\left(\frac{\mu^2}{2\sigma^2}\right)\frac{2+\sigma}{\alpha}\Gamma(\mu).$$

This solution represents a spatial beam propagating at a non-trivial angle $\theta=\tan^{-1}\left(\sqrt{2\kappa}V\right)$ with respect to the reference direction.

The Helmholtz correction factor $2\kappa V^2$ can be arbitrarily large, even for broad beams. This regime defines a Helmholtz-type of non-paraxiality.

Soliton Stability

Pure Focusing ($\alpha>0,\gamma>0$)	**Pure defocusing** ($\alpha<0,\gamma<0$)
$\Gamma=\Gamma_+$ $\sigma\le 2$ stable $2<\sigma<4$ stable $\mu<\mu_{th}$ $\sigma>4$ unstable	NO STABLE LOCALIZED SOLITON SOLUTION
Type-I Competing ($\alpha<0,\ \gamma>0$)	**Type-II Competing** ($\alpha>0,\ \gamma<0$)
$\Gamma=\Gamma_-$ $\sigma\le 1$ stable $1<\sigma<2$ stable $\mu>\mu_{th}$ $\sigma\ge 4$ unstable	$\Gamma=\Gamma_+$ Stable for: $\mu\le\mu_{cr}=\frac{\alpha\sigma}{2+\sigma}\sqrt{\frac{2(1+\sigma)}{\lvert\gamma\rvert}}$

The stability of generalized solitons against small perturbations in the corresponding paraxial model [5] can be studied by using the well-known Vakhitov-Kolokolov criterion. A solution $u(\xi,\zeta,;\beta)$ is stable if

$$\frac{d}{d\beta}\int_{-\infty}^{+\infty}d\xi\left|u(\xi,\zeta;\beta)\right|^2>0,$$

where $u(\xi,\zeta;\beta)=\left|u(\xi;\beta)\right|\exp(i\beta\zeta)$.

This is a solvability condition preventing exponential growth in an associated linearized eigenvalue problem. In the Helmholtz case, there is no known (mathematically rigorous) analogue of this expression due to the analytical complexities introduced by the $\kappa\,\partial_{\zeta\zeta}$ operator. The analytical determination of stability remains open, and is currently a purely numerical pursuit. The results of this analysis are presented

Conserved Quantities

Using Lagrangian field-theoretic techniques, three conserved quantities of the gNLH equation can be identified. These are the energy-flow W, the momentum M and the Hamiltonian H of the system. The gNLH model is regarded as the Euler-Lagrange equation of motion for a Lagrangian density, and a pair of canonically-conjugate momentum variables are defined. Noether's theorem can then be exploited to derive the system invariants.

$$W=\int_{-\infty}^{+\infty}d\xi\left[|u|^2-i\kappa\left(u^*\frac{\partial u}{\partial\zeta}-u\frac{\partial u^*}{\partial\zeta}\right)\right],$$

$$M=\int_{-\infty}^{+\infty}d\xi\left[\frac{i}{2}\left(u^*\frac{\partial u}{\partial\xi}-u\frac{\partial u^*}{\partial\xi}\right)-\kappa\left(\frac{\partial u^*}{\partial\zeta}\frac{\partial u}{\partial\xi}+\frac{\partial u^*}{\partial\xi}\frac{\partial u}{\partial\zeta}\right)\right],$$

$$H=\int_{-\infty}^{+\infty}d\xi\left[\frac{1}{2}\left|\frac{\partial u}{\partial\xi}\right|^2-\kappa\left|\frac{\partial u}{\partial\zeta}\right|^2-\alpha\frac{|u|^{2+\sigma}}{1+\frac{1}{2}\sigma}-\gamma\frac{|u|^{2(1+\sigma)}}{1+\sigma}\right].$$

The integral representation of the invariants is vital for monitoring the integrity of any numerical scheme used to solve the gNLH equation [6].

Helmholtz Algebraic Solitons

A weakly-localized non-linear wave of the gNLH equation is the **algebraic soliton**. This solution has much slower *power-law* (e.g. Lorentzian) asymptotics as opposed to the strongly-localized *sech-type* (e.g. exponential) solutions. The algebraic soliton is supported only in the *Type-I competing* regime where, in the limit $\mu\to 0$, there remains a *non-zero energy-flow*, $W(\mu\to 0)>0$.

$$u_a(\xi,\zeta)=\left[\frac{(1+\sigma)(2+\sigma)/|\alpha|}{\sigma^2(1+\sigma)\frac{(\xi+V\zeta)^2}{1+2\kappa V^2}+(2+\sigma)^2\frac{\gamma}{2\alpha^2}}\right]^{\frac{1}{\sigma}}\exp\left[i\frac{1}{\sqrt{1+2\kappa V^2}}\left(-V\xi+\frac{\zeta}{2\kappa}\right)\right]\exp\left(-i\frac{\zeta}{2\kappa}\right).$$

The Lorentzian asymptotics are seen from $\lim_{\xi\to\pm\infty}\left|u_a(\xi,\zeta)\right|\to|\xi|^{-\frac{2}{\sigma}}$.

Algebraic solitary waves are a common feature of non-linear systems, occurring, for example, in fluid mechanics. It could be predicted *a-priori* that paraxial wave optics, governed by NLS-type equations, must also support algebraic solitons from the *fluid mechanics–nonlinear optics* analogue. Such waves do exist and have been reported by several authors, such as in [6]. However, this is the first known reporting of algebraic solitons in fully 2[nd]-order Helmholtz-type systems.

Recovery of Paraxial Solutions

Known solutions of the paraxial model [5] corresponding to the gNLH equation can be recovered from the full Helmholtz solutions when an appropriate multiple limit is enforced:

PARAXIAL LIMIT

- $\kappa\to 0$ (broad beams)
- $\kappa\left(\frac{\mu}{\sigma}\right)^2\to 0$ (low intensity or slow phase variations)
- $\kappa V^2\to 0$ (vanishingly-small propagation angle)

$$u(\xi,\zeta)\approx\left[\frac{\eta}{\cosh\left[\mu(\xi+V\zeta)\right]+\Gamma}\right]^{\frac{1}{\sigma}}\exp\left[-iV\xi+\frac{i}{2}\left(\frac{\mu^2}{\sigma^2}-V^2\right)\zeta\right],$$

$$u_a(\xi,\zeta)\approx\left[\frac{(1+\sigma)(2+\sigma)/|\alpha|}{\sigma^2(1+\sigma)(\xi+V\zeta)^2+(2+\sigma)^2\frac{\gamma}{2\alpha^2}}\right]^{\frac{1}{\sigma}}\exp\left(-iV\xi-i\frac{V^2}{2}\zeta\right).$$

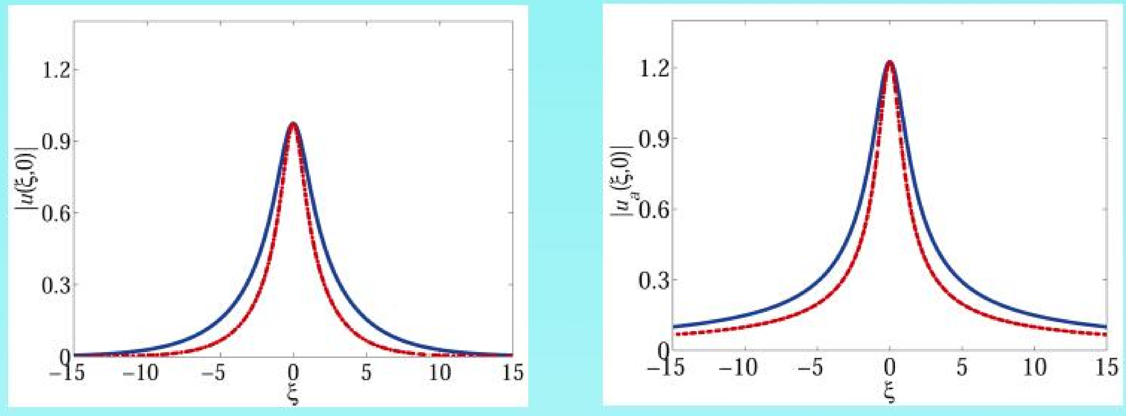

FIG. 1. Angular beam broadening effect of the Helmholtz solitons (solid curves) compared with their corresponding paraxial solutions (dotted curves) [5]. Left: *sech*-type and; right: algebraic.

Parameters: $\kappa=10^{-3}$ and $V=25$, so that the propagation angle is $\theta\approx 48.2°$. Here, $\sigma=0.9$ and $\mu=1$ for the *sech*-type solution. In all simulations, $|\alpha|=|\gamma|=1$.

Solitons as Robust Attractors

The numerical perturbative approach involves using initial conditions that correspond to exact solutions of the paraxial equation with transverse velocity S_0. For quasi-paraxial beams (where the first two conditions in the *paraxial limit* are met), rotational symmetry establishes a connection between V and S_0. Examination of the beam along its propagation axis shows that the evolution is equivalent to that of an on-axis paraxial soliton whose width has been reduced by $(1+2\kappa V^2)^{1/2}$, where $V=S_0/(1-2\kappa S_0^2)^{1/2}$ [7].

The figures below depict beam self-reshaping situations. Depending upon the parameters involved (S_0, μ, σ) a propagation-invariant (stationary) soliton may emerge asymptotically from the initial condition. The corresponding Helmholtz soliton is then classified as a *stable fixed point*. Alternatively, the oscillations may persist in the long term. In this case, the solution is a new type of *stable limit cycle* soliton.

S_0 = 5 (—), 10(—), 15(—) correspond to non-trivial propagation angles of θ = 12.9°, 26.6° and 42.1

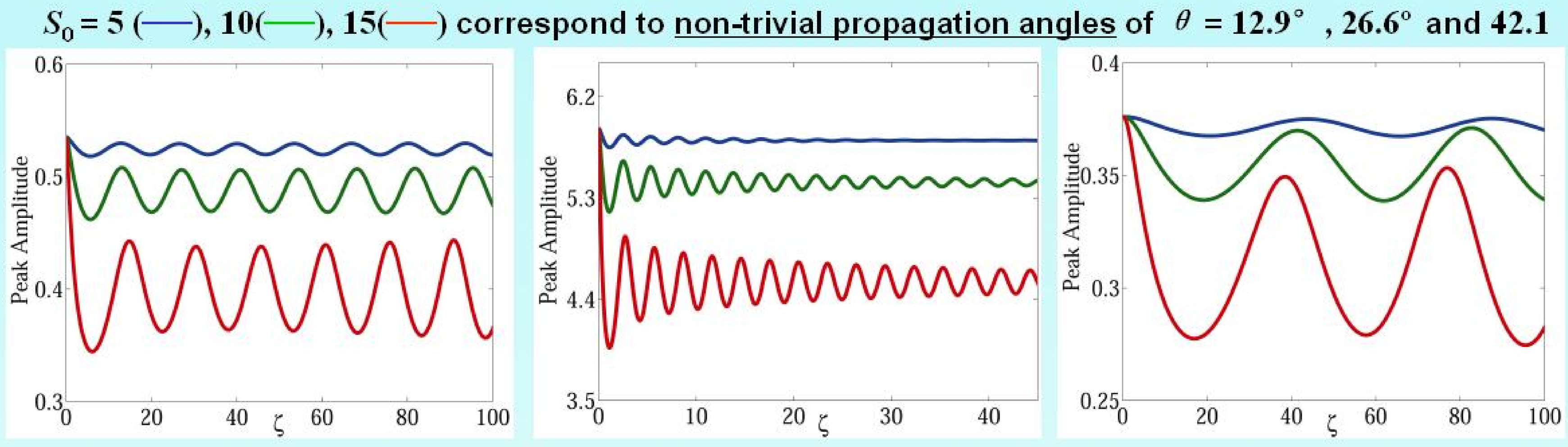

FIG. 2. Reshaping of *sech-type* initial conditions. (a) Pure focusing (σ = 1 and μ = 1) in the middle of the unconditionally-stable regime. Initial conditions with μ in the *conditionally stable* regime can display similar characteristics. (b) Type-I competing (μ = 1 and σ = 0.5). In the *unconditionally stable* regime, the beam evolves asymptotically into a Helmholtz soliton, with a rate of convergence depending on S_0. At threshold (σ = 1) and beyond, weakly perturbed beams can give rise to solitons; increasing the perturbation leads to diffractive spreading and collapse. (c) Type-II competing (μ = 0.6, σ = 1). In all cases below threshold, the initial condition evolves into a limit-cycle solution, resembling the behaviour of the pure-focusing regime. A notable difference is the longitudinal length-scale over which the oscillations occur.

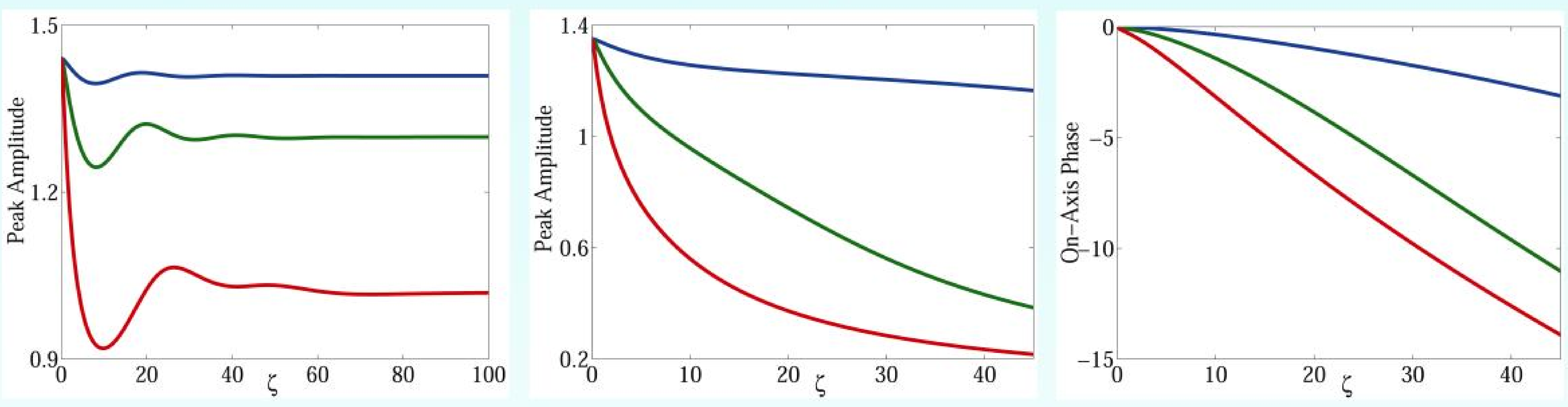

FIG. 3. Reshaping of algebraic initial conditions. (a) σ = 1 and (b) σ = 0.9, where the *sech-type* solitons are unconditionally and conditionally stable, respectively. Far from threshold, the algebraic solitons behave as stable fixed points, with stationary beams emerging asymptotically. As threshold is approached, this stability is lost and the initial condition undergoes diffractive spreading. (c) Paraxial theory predicts this type of instability when the on-axis longitudinal phase gradient is negative.

Conclusions

- The gNLH equation possesses **exact analytical soliton solutions**. Two distinct solution families exhibit strong (exponential) weak (power-law) localization of the beam energy.
- gNLH solitons are of intrinsic mathematical interest. They represent a novel contribution to the knowledge of soliton dynamics in fully 2[nd]-order non-integrable models. **New algebraic solitons have been derived**.
- Experimentally, the gNLH equation **pertains directly to known materials**, such as some semiconductor-doped glasses [2] and non-linear polymers [3,4]. This is an appropriate model for describing **oblique propagation** of spatial optical beams in these media.
- gNLH solitons have been to shown to behave predominantly as **limit cycle attractors** of the system dynamics. In some cases, they can act as **stable fixed points**, but this behaviour is highly dependent on both material parameters and the inverse beam width μ.

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