## Differential Equations

PPLATO PROMOTING PHYSICS LEARNING & TEACHING OPPORTUNITIES

# BERNOULLI EQUATIONS

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A Tutorial Module for learning how to solve Bernoulli differential equations



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# 1. Theory

A **Bernoulli differential equation** can be written in the following standard form:

$$\boxed{\frac{dy}{dx} + P(x)y = Q(x)y^n} \quad ,$$

where $n \neq 1$ (the equation is thus **nonlinear**).

To find the solution, change the dependent variable from $y$ to $z$, where $z = y^{1-n}$. This gives a differential equation in $x$ and $z$ that is **linear**, and can be solved using the integrating factor method.

Note: Dividing the above standard form by $y^n$ gives:

$$\begin{aligned} \frac{1}{y^n}\frac{dy}{dx} + P(x)y^{1-n} &= Q(x) \\ \text{i.e. } \frac{1}{(1-n)}\frac{dz}{dx} + P(x)z &= Q(x) \end{aligned}$$

$\left(\text{where we have used } \frac{dz}{dx} = (1-n)y^{-n}\frac{dy}{dx}\right)$.

## 2. Exercises

Click on Exercise links for full worked solutions (there are 9 exercises in total)

Exercise 1.
The general form of a Bernoulli equation is

$$\frac{dy}{dx} + P(x)y = Q(x)\,y^n\,,$$

where $P$ and $Q$ are functions of $x$, and $n$ is a constant. Show that the transformation to a new dependent variable $z = y^{1-n}$ reduces the equation to one that is linear in $z$ (and hence solvable using the integrating factor method).

**Solve the following Bernoulli differential equations:**

Exercise 2.
$\frac{dy}{dx} - \frac{1}{x}y = xy^2$

**Exercise 3.**

$$\frac{dy}{dx} + \frac{y}{x} = y^2$$

**Exercise 4.**

$$\frac{dy}{dx} + \frac{1}{3}y = e^x y^4$$

**Exercise 5.**

$$x\frac{dy}{dx} + y = xy^3$$

**Exercise 6.**

$$\frac{dy}{dx} + \frac{2}{x}y = -x^2 \cos x \cdot y^2$$

Exercise 7.

$$2\frac{dy}{dx} + \tan x \cdot y = \frac{(4x+5)^2}{\cos x}y^3$$

Exercise 8.

$$x\frac{dy}{dx} + y = y^2x^2 \ln\, x$$

Exercise 9.

$$\frac{dy}{dx} = y\cot x + y^3\mathrm{cosec}x$$

# 3. Answers

1. HINT: Firstly, divide each term by $y^n$. Then, differentiate $z$ with respect to $x$ to show that $\frac{1}{(1-n)}\frac{dz}{dx} = \frac{1}{y^n}\frac{dy}{dx}$ ,

2. $\frac{1}{y} = -\frac{x^2}{3} + \frac{C}{x}$ ,

3. $\frac{1}{y} = x(C - \ln\, x)$ ,

4. $\frac{1}{y^3} = e^x(C - 3x)$ ,

5. $y^2 = \frac{1}{2x+Cx^2}$ ,

6. $\frac{1}{y} = x^2(\sin x + C)$ ,

7. $\frac{1}{y^2} = \frac{1}{12\cos x}(4x+5)^3 + \frac{C}{\cos x}$ ,

8. $\frac{1}{xy} = C + x(1 - \ln\, x)$ ,

9. $y^2 = \frac{\sin^2 x}{2\cos x + C}$ .

# 4. Integrating factor method

Consider an ordinary differential equation (o.d.e.) that we wish to solve to find out how the variable $z$ depends on the variable $x$.

If the equation is **first order** then the highest derivative involved is a first derivative.

If it is also a **linear** equation then this means that each term can involve $z$ either as the derivative $\frac{dz}{dx}$ OR through a single factor of $z$.

Any such linear first order o.d.e. can be re-arranged to give the following standard form:

$$\boxed{\frac{dz}{dx} + P_1(x)z = Q_1(x)}$$

where $P_1(x)$ and $Q_1(x)$ are functions of $x$, and in some cases may be constants.

A linear first order o.d.e. can be solved using the **integrating factor method**.

After writing the equation in standard form, $P_1(x)$ can be identified. One then multiplies the equation by the following “integrating factor”:

$$\text{IF}= e^{\int P_1(x)dx}$$

This factor is defined so that the equation becomes equivalent to:

$$\frac{d}{dx}(\text{IF}\, z) = \text{IF}\, Q_1(x),$$

whereby integrating both sides with respect to $x$, gives:

$$\text{IF}\, z = \int \text{IF}\, Q_1(x)\, dx$$

Finally, division by the integrating factor (IF) gives $z$ explicitly in terms of $x$, i.e. gives the solution to the equation.

# 5. Standard integrals

| $f(x)$ | $\int f(x)dx$ | $f(x)$ | $\int f(x)dx$ |
|---|---|---|---|
| $x^n$ | $\frac{x^{n+1}}{n+1} \quad (n \neq -1)$ | $[g(x)]^n g'(x)$ | $\frac{[g(x)]^{n+1}}{n+1} \quad (n \neq -1)$ |
| $\frac{1}{x}$ | $\ln\lvert x\rvert$ | $\frac{g'(x)}{g(x)}$ | $\ln\lvert g(x)\rvert$ |
| $e^x$ | $e^x$ | $a^x$ | $\frac{a^x}{\ln a} \quad (a > 0)$ |
| $\sin x$ | $-\cos x$ | $\sinh x$ | $\cosh x$ |
| $\cos x$ | $\sin x$ | $\cosh x$ | $\sinh x$ |
| $\tan x$ | $-\ln\lvert\cos x\rvert$ | $\tanh x$ | $\ln\cosh x$ |
| $\operatorname{cosec} x$ | $\ln\left\lvert\tan\frac{x}{2}\right\rvert$ | $\operatorname{cosech} x$ | $\ln\left\lvert\tanh\frac{x}{2}\right\rvert$ |
| $\sec x$ | $\ln\lvert\sec x + \tan x\rvert$ | $\operatorname{sech} x$ | $2\tan^{-1} e^x$ |
| $\sec^2 x$ | $\tan x$ | $\operatorname{sech}^2 x$ | $\tanh x$ |
| $\cot x$ | $\ln\lvert\sin x\rvert$ | $\coth x$ | $\ln\lvert\sinh x\rvert$ |
| $\sin^2 x$ | $\frac{x}{2} - \frac{\sin 2x}{4}$ | $\sinh^2 x$ | $\frac{\sinh 2x}{4} - \frac{x}{2}$ |
| $\cos^2 x$ | $\frac{x}{2} + \frac{\sin 2x}{4}$ | $\cosh^2 x$ | $\frac{\sinh 2x}{4} + \frac{x}{2}$ |

| $f(x)$ | $\int f(x)\,dx$ | $f(x)$ | $\int f(x)\,dx$ |
|---|---|---|---|
| $\frac{1}{a^2+x^2}$ | $\frac{1}{a}\tan^{-1}\frac{x}{a}$ | $\frac{1}{a^2-x^2}$ | $\frac{1}{2a}\ln\left\lvert\frac{a+x}{a-x}\right\rvert \quad (0<\lvert x\rvert<a)$ |
| | $(a>0)$ | $\frac{1}{x^2-a^2}$ | $\frac{1}{2a}\ln\left\lvert\frac{x-a}{x+a}\right\rvert \quad (\lvert x\rvert>a>0)$ |
| $\frac{1}{\sqrt{a^2-x^2}}$ | $\sin^{-1}\frac{x}{a}$ | $\frac{1}{\sqrt{a^2+x^2}}$ | $\ln\left\lvert\frac{x+\sqrt{a^2+x^2}}{a}\right\rvert \quad (a>0)$ |
| | $(-a<x<a)$ | $\frac{1}{\sqrt{x^2-a^2}}$ | $\ln\left\lvert\frac{x+\sqrt{x^2-a^2}}{a}\right\rvert \quad (x>a>0)$ |
| $\sqrt{a^2-x^2}$ | $\frac{a^2}{2}\left[\sin^{-1}\left(\frac{x}{a}\right) + \frac{x\sqrt{a^2-x^2}}{a^2}\right]$ | $\sqrt{a^2+x^2}$ | $\frac{a^2}{2}\left[\sinh^{-1}\left(\frac{x}{a}\right)+\frac{x\sqrt{a^2+x^2}}{a^2}\right]$ |
| | | $\sqrt{x^2-a^2}$ | $\frac{a^2}{2}\left[-\cosh^{-1}\left(\frac{x}{a}\right)+\frac{x\sqrt{x^2-a^2}}{a^2}\right]$ |

# 6. Tips on using solutions

● When looking at the THEORY, ANSWERS, IF METHOD, INTEGRALS or TIPS pages, use the Back button (at the bottom of the page) to return to the exercises.

● Use the solutions intelligently. For example, they can help you get started on an exercise, or they can allow you to check whether your intermediate results are correct.

● Try to make less use of the full solutions as you work your way through the Tutorial.

## Full worked solutions

**Exercise 1.**

$$\frac{dy}{dx} + P(x)y = Q(x)y^n$$

$\underline{\text{DIVIDE by } y^n}$: $\quad \frac{1}{y^n}\frac{dy}{dx} + P(x)y^{1-n} = Q(x)$

$\underline{\text{SET } z = y^{1-n}}$: i.e. $\quad \frac{dz}{dx} = (1-n)y^{(1-n-1)}\frac{dy}{dx}$

i.e. $\quad \frac{1}{(1-n)}\frac{dz}{dx} = \frac{1}{y^n}\frac{dy}{dx}$

SUBSTITUTE $\quad \frac{1}{(1-n)}\frac{dz}{dx} + P(x)z = Q(x)$

i.e. $\quad \boxed{\frac{dz}{dx} + P_1(x)z = Q_1(x)}$ linear in $z$

where $\quad P_1(x) = (1-n)P(x)$
$\quad Q_1(x) = (1-n)Q(x)$ .

Return to Exercise 1

**Exercise 2.**

This is of the form $\frac{dy}{dx} + P(x)y = Q(x)y^n$ where

$$\begin{array}{rrcl} \text{where} & P(x) & = & -\frac{1}{x} \\ & Q(x) & = & x \\ \text{and} & n & = & 2 \end{array}$$

<u>DIVIDE by $y^n$</u>: i.e. $\frac{1}{y^2}\frac{dy}{dx} - \frac{1}{x}y^{-1} = x$

<u>SET $z = y^{1-n} = y^{-1}$</u>: i.e. $\frac{dz}{dx} = -y^{-2}\frac{dy}{dx} = -\frac{1}{y^2}\frac{dy}{dx}$

$$\therefore \quad -\frac{dz}{dx} - \frac{1}{x}z = x$$

$$\text{i.e.} \quad \frac{dz}{dx} + \frac{1}{x}z = -x$$

Integrating factor, $\quad \text{IF} = e^{\int \frac{1}{x}\,dx} = e^{\ln\,x} = x$

$$
\begin{aligned}
\therefore \quad & x\frac{dz}{dx} + z = -x^2 \\
\text{i.e.} \quad & \frac{d}{dx}[x \cdot z] = -x^2 \\
\text{i.e.} \quad & xz = -\int x^2 dx \\
\text{i.e.} \quad & xz = -\frac{x^3}{3} + C
\end{aligned}
$$

Use $z = \frac{1}{y}$: $\quad \frac{x}{y} = -\frac{x^3}{3} + C$

$$
\text{i.e.} \quad \frac{1}{y} = -\frac{x^2}{3} + \frac{C}{x}\ .
$$

Return to Exercise 2

**Exercise 3.**

This is of the form $\dfrac{dy}{dx} + P(x)y = Q(x)y^n$

where $P(x) = \frac{1}{x}$,

$$Q(x) = 1,$$

and $\quad n = 2$

DIVIDE by $y^n$: i.e. $\frac{1}{y^2}\frac{dy}{dx} + \frac{1}{x}y^{-1} = 1$

SET $z = y^{1-n} = y^{-1}$: i.e. $\frac{dz}{dx} = -1 \cdot y^{-2}\frac{dy}{dx} = -\frac{1}{y^2}\frac{dy}{dx}$

$\therefore \quad -\frac{dz}{dx} + \frac{1}{x}z = 1$

i.e. $\frac{dz}{dx} - \frac{1}{x}z = -1$

Integrating factor, $\quad \text{IF} = e^{-\int \frac{dx}{x}} = e^{-\ln\, x} = e^{\ln\, x^{-1}} = \dfrac{1}{x}$

$$\therefore \quad \tfrac{1}{x}\tfrac{dz}{dx} - \tfrac{1}{x^2}z = -\tfrac{1}{x}$$

$$\text{i.e.} \quad \tfrac{d}{dx}\left[\tfrac{1}{x}\cdot z\right] = -\tfrac{1}{x}$$

$$\text{i.e.} \quad \tfrac{1}{x}\cdot z = -\int \tfrac{dx}{x}$$

$$\text{i.e.} \quad \tfrac{z}{x} = -\ln\, x + C$$

Use $z = \frac{1}{y}$: $\quad \frac{1}{yx} = C - \ln\, x$

$$\text{i.e.} \quad \tfrac{1}{y} = x(C - \ln\, x)\ .$$

Return to Exercise 3

**Exercise 4.**

This of the form $\dfrac{dy}{dx} + P(x)y = Q(x)y^n$

$$\begin{array}{rrcl} \text{where} & P(x) & = & \dfrac{1}{3} \\ & Q(x) & = & e^x \\ \text{and} & n & = & 4 \end{array}$$

<u>DIVIDE by $y^n$</u>: i.e. $\dfrac{1}{y^4}\dfrac{dy}{dx} + \dfrac{1}{3}y^{-3} = e^x$

<u>SET $z = y^{1-n} = y^{-3}$</u>: i.e. $\dfrac{dz}{dx} = -3y^{-4}\dfrac{dy}{dx} = -\dfrac{3}{y^4}\dfrac{dy}{dx}$

$$\therefore \quad -\frac{1}{3}\frac{dz}{dx} + \frac{1}{3}z = e^x$$

$$\text{i.e.} \quad \frac{dz}{dx} - z = -3e^x$$

Integrating factor, $\quad \mathrm{IF} = e^{-\int dx} = e^{-x}$

$$\therefore \quad e^{-x}\frac{dz}{dx} - e^{-x}z = -3e^{-x}\cdot e^{x}$$

$$\text{i.e.} \quad \frac{d}{dx}[e^{-x}\cdot z] = -3$$

$$\text{i.e.} \quad e^{-x}\cdot z = \int -3\,dx$$

$$\text{i.e.} \quad e^{-x}\cdot z = -3x + C$$

Use $z = \frac{1}{y^3}$: $\quad e^{-x}\cdot\frac{1}{y^3} = -3x + C$

$$\text{i.e.} \quad \frac{1}{y^3} = e^{x}(C - 3x)\ .$$

Return to Exercise 4

**Exercise 5.**

Bernoulli equation: $\dfrac{dy}{dx} + \dfrac{y}{x} = y^3$ with $P(x) = \dfrac{1}{x}, Q(x) = 1, n = 3$

$\underline{\text{DIVIDE by } y^n \quad \text{i.e. } y^3}$: $\qquad \frac{1}{y^3}\frac{dy}{dx} + \frac{1}{x}y^{-2} = 1$

$\underline{\text{SET } z = y^{1-n} \quad \text{i.e.} \quad z = y^{-2}}$: $\qquad \frac{dz}{dx} = -2y^{-3}\frac{dy}{dx}$

i.e. $\quad -\frac{1}{2}\frac{dz}{dx} = \frac{1}{y^3}\frac{dy}{dx}$

$\therefore \quad -\frac{1}{2}\frac{dz}{dx} + \frac{1}{x}z = 1$

i.e. $\quad \frac{dz}{dx} - \frac{2}{x}z = -2$

Integrating factor, $\quad \text{IF} = e^{-2\int \frac{dx}{x}} = e^{-2\ln x} = e^{\ln x^{-2}} = \dfrac{1}{x^2}$

$$\therefore \quad \tfrac{1}{x^2}\tfrac{dz}{dx} - \tfrac{2}{x^3}z = -\tfrac{2}{x^2}$$

$$\text{i.e.} \quad \tfrac{d}{dx}\left[\tfrac{1}{x^2}z\right] = -\tfrac{2}{x^2}$$

$$\text{i.e.} \quad \tfrac{1}{x^2}z = (-2)\cdot(-1)\tfrac{1}{x} + C$$

$$\text{i.e.} \quad z = 2x + Cx^2$$

Use $z = \frac{1}{y^2}$: $\quad y^2 = \frac{1}{2x+Cx^2}$ .

Return to Exercise 5

**Exercise 6.**

This is of the form $\frac{dy}{dx} + P(x)y = Q(x)y^n$ where

$$\begin{array}{rrcl} \text{where} & P(x) & = & \frac{2}{x} \\ & Q(x) & = & -x^2 \cos x \\ \text{and} & n & = & 2 \end{array}$$

<u>DIVIDE by $y^n$</u>: i.e. $\frac{1}{y^2}\frac{dy}{dx} + \frac{2}{x}y^{-1} = -x^2 \cos x$

<u>SET $z = y^{1-n} = y^{-1}$</u>: i.e. $\frac{dz}{dx} = -1 \cdot y^{-2}\frac{dy}{dx} = -\frac{1}{y^2}\frac{dy}{dx}$

$$\therefore \quad -\frac{dz}{dx} + \frac{2}{x}z = -x^2 \cos x$$

$$\text{i.e.} \quad \frac{dz}{dx} - \frac{2}{x}z = x^2 \cos x$$

Integrating factor, $\quad \text{IF} = e^{\int -\frac{2}{x}\,dx} = e^{-2\int \frac{dx}{x}} = e^{-2\ln x} = e^{\ln x^{-2}} = \dfrac{1}{x^2}$

$$
\begin{aligned}
\therefore \quad & \frac{1}{x^2}\frac{dz}{dx} - \frac{2}{x^3}z = \frac{x^2}{x^2}\cos x \\
\text{i.e.} \quad & \frac{d}{dx}\left[\frac{1}{x^2}\cdot z\right] = \cos x \\
\text{i.e.} \quad & \frac{1}{x^2}\cdot z = \int \cos x\,dx \\
\text{i.e.} \quad & \frac{1}{x^2}\cdot z = \sin x + C
\end{aligned}
$$

Use $z = \frac{1}{y}$: $\quad \frac{1}{x^2 y} = \sin x + C$

$$
\text{i.e.} \quad \frac{1}{y} = x^2(\sin x + C)\,.
$$

Return to Exercise 6

**Exercise 7.**

Divide by 2 to get standard form:

$$\frac{dy}{dx} + \frac{1}{2}\tan x \cdot y = \frac{(4x+5)^2}{2\cos x} y^3$$

This is of the form $\dfrac{dy}{dx} + P(x)y = Q(x)y^n$

$$\begin{aligned} \text{where} \qquad P(x) &= \frac{1}{2}\tan x \\ Q(x) &= \frac{(4x+5)^2}{2\cos x} \\ \text{and} \qquad n &= 3 \end{aligned}$$

DIVIDE by $y^n$: i.e. $\frac{1}{y^3}\frac{dy}{dx} + \frac{1}{2}\tan x \cdot y^{-2} = \frac{(4x+5)^2}{2\cos x}$

SET $z = y^{1-n} = y^{-2}$: i.e. $\frac{dz}{dx} = -2y^{-3}\frac{dy}{dx} = -\frac{2}{y^3}\frac{dy}{dx}$

$$\therefore \quad -\frac{1}{2}\frac{dz}{dx} + \frac{1}{2}\tan x \cdot z = \frac{(4x+5)^2}{2\cos x}$$

i.e. $\frac{dz}{dx} - \tan x \cdot z = \frac{(4x+5)^2}{\cos x}$

Integrating factor, $\quad \text{IF} = e^{\int -\tan x \cdot dx} = e^{\int -\frac{\sin x}{\cos x}\,dx} \quad \left[\equiv e^{\int \frac{f'(x)}{f(x)}\,dx}\right]$

$$= e^{\ln\cos x} = \cos x$$

$$\therefore \quad \cos x\frac{dz}{dx} - \cos x \tan x \cdot z = \cos x \frac{(4x+5)^2}{\cos x}$$

$$\text{i.e.} \quad \cos x\frac{dz}{dx} - \sin x \cdot z = (4x+5)^2$$

$$\text{i.e.} \quad \frac{d}{dx}[\cos x \cdot z] = (4x+5)^2$$

$$\text{i.e.} \quad \cos x \cdot z = \int (4x+5)^2 dx$$

$$\text{i.e.} \quad \cos x \cdot z = \left(\frac{1}{4}\right) \cdot \frac{1}{3}(4x+5)^3 + C$$

Use $z = \frac{1}{y^2}$: $\quad \frac{\cos x}{y^2} = \frac{1}{12}(4x+5)^3 + C$

$$\text{i.e.} \quad \frac{1}{y^2} = \frac{1}{12\cos x}(4x+5)^3 + \frac{C}{\cos x}\ .$$

Return to Exercise 7

**Exercise 8.**

Standard form: $\frac{dy}{dx} + \left(\frac{1}{x}\right) y = (x \ln x) y^2$

i.e. $P(x) = \frac{1}{x},\ Q(x) = x \ln x\ ,\ n = 2$

DIVIDE by $y^2$: $\frac{1}{y^2}\frac{dy}{dx} + \left(\frac{1}{x}\right) y^{-1} = x \ln x$

SET $z = y^{-1}$: $\frac{dz}{dx} = -y^{-2}\frac{dy}{dx} = -\frac{1}{y^2}\frac{dy}{dx}$

$\therefore$ $-\frac{dz}{dx} + \left(\frac{1}{x}\right) z = x \ln x$

i.e. $\frac{dz}{dx} - \frac{1}{x} \cdot z = -x \ln x$

Integrating factor: $\quad \text{IF} = e^{-\int \frac{dx}{x}} = e^{-\ln\, x} = e^{\ln\, x^{-1}} = \dfrac{1}{x}$

$$\therefore \quad \tfrac{1}{x}\tfrac{dz}{dx} - \tfrac{1}{x^2}z = -\ln\, x$$

$$\text{i.e.} \quad \tfrac{d}{dx}\left[\tfrac{1}{x}z\right] = -\ln\, x$$

$$\text{i.e.} \quad \tfrac{1}{x}z = -\int \ln\, x\, dx + C'$$

[Use integration by parts: $\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx,$

with $u = \ln\, x\,,\ \frac{dv}{dx} = 1$ ]

$$\text{i.e.} \quad \tfrac{1}{x}z = -\left[x\ \ln\, x - \int x\cdot\tfrac{1}{x}dx\right] + C$$

Use $z = \frac{1}{y}$: $\quad \dfrac{1}{xy} = x(1 - \ln\, x) + C\ .$

Return to Exercise 8

**Exercise 9.**

Standard form: $\frac{dy}{dx} - (\cot x) \cdot y = (\operatorname{cosec} x)\, y^3$

DIVIDE by $y^3$: $\frac{1}{y^3}\frac{dy}{dx} - (\cot x) \cdot y^{-2} = \operatorname{cosec} x$

SET $z = y^{-2}$: $\frac{dz}{dx} = -2y^{-3}\frac{dy}{dx} = -2 \cdot \frac{1}{y^3}\frac{dy}{dx}$

$$\therefore \quad -\frac{1}{2}\frac{dz}{dx} - \cot x \cdot z = \operatorname{cosec} x$$

$$\text{i.e.} \quad \frac{dz}{dx} + 2\cot x \cdot z = -2 \operatorname{cosec} x$$

Integrating factor: $\text{IF} = e^{2\int \frac{\cos x}{\sin x}\,dx} \equiv e^{2\int \frac{f'(x)}{f(x)}\,dx} = e^{2\,\ln(\sin x)} = \sin^2 x.$

$$\therefore \quad \sin^2 x \cdot \tfrac{dz}{dx} + 2\sin x\ \cdot \cos x\ \cdot z = -2\sin x$$

$$\text{i.e.} \quad \tfrac{d}{dx}\left[\sin^2 x\ \cdot z\right] = -2\sin x$$

$$\text{i.e.} \quad z\sin^2 x = (-2)\cdot(-\cos x) + C$$

Use $z = \frac{1}{y^2}$: $\quad \frac{\sin^2 x}{y^2} = 2\cos x + C$

$$\text{i.e.} \quad y^2 = \tfrac{\sin^2 x}{2\cos x + C}\ .$$

Return to Exercise 9