

## TRIGONOMETRIC IDENTITIES

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A self-contained Tutorial Module for practising integration of expressions involving products of trigonometric functions such as  $\sin nx \sin mx$

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## 1. Theory

Integrals of the form

$$\int \sin nx \sin mx ,$$

and similar ones with products like  $\sin nx \cos mx$  and  $\cos nx \cos mx$ , can be solved by making use of the following trigonometric identities:

$$\sin A \sin B = -\frac{1}{2} [\cos (A + B) - \cos (A - B)]$$

$$\sin A \cos B = \frac{1}{2} [\sin (A + B) + \sin (A - B)]$$

$$\cos A \cos B = \frac{1}{2} [\cos (A + B) + \cos (A - B)]$$

Using these identities, such **products are expressed as a sum** of trigonometric functions

This sum is generally more straightforward to integrate

## 2. Exercises

Click on [EXERCISE](#) links for full worked solutions (9 exercises in total).

**Perform the following integrations:**

[EXERCISE 1.](#)

$$\int \cos 3x \cos 2x \, dx$$

[EXERCISE 2.](#)

$$\int \sin 5x \cos 3x \, dx$$

[EXERCISE 3.](#)

$$\int \sin 6x \sin 4x \, dx$$

## EXERCISE 4.

$$\int \cos 2\omega t \sin \omega t dt, \text{ where } \omega \text{ is a constant}$$

## EXERCISE 5.

$$\int \cos 4\omega t \cos 2\omega t dt, \text{ where } \omega \text{ is a constant}$$

## EXERCISE 6.

$$\int \sin^2 x dx$$

## EXERCISE 7.

$$\int \sin^2 \omega t dt, \text{ where } \omega \text{ is a constant}$$

## EXERCISE 8.

$$\int \cos^2 t dt$$

## EXERCISE 9.

$\int \cos^2 kx \, dx$ , where  $k$  is a constant

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### 3. Answers

1.  $\frac{1}{10} \sin 5x + \frac{1}{2} \sin x + C,$

2.  $-\frac{1}{16} \cos 8x - \frac{1}{4} \cos 2x + C,$

3.  $-\frac{1}{20} \sin 10x + \frac{1}{4} \sin 2x + C,$

4.  $-\frac{1}{6\omega} \cos 3\omega t + \frac{1}{2\omega} \cos \omega t + C,$

5.  $\frac{1}{12\omega} \sin 6\omega t + \frac{1}{4\omega} \sin 2\omega t + C,$

6.  $-\frac{1}{4} \sin 2x + \frac{1}{2} x + C,$

7.  $-\frac{1}{4\omega} \sin 2\omega t + \frac{1}{2} t + C,$

8.  $\frac{1}{4} \sin 2t + \frac{1}{2} t + C,$

9.  $\frac{1}{4k} \sin 2kx + \frac{1}{2} x + C.$

## 4. Standard integrals

$f(x)$	$\int f(x)dx$	$f(x)$	$\int f(x)dx$
$x^n$	$\frac{x^{n+1}}{n+1} \quad (n \neq -1)$	$[g(x)]^n g'(x)$	$\frac{[g(x)]^{n+1}}{n+1} \quad (n \neq -1)$
$\frac{1}{x}$	$\ln x $	$\frac{g'(x)}{g(x)}$	$\ln g(x) $
$e^x$	$e^x$	$a^x$	$\frac{a^x}{\ln a} \quad (a > 0)$
$\sin x$	$-\cos x$	$\sinh x$	$\cosh x$
$\cos x$	$\sin x$	$\cosh x$	$\sinh x$
$\tan x$	$-\ln \cos x $	$\tanh x$	$\ln \cosh x$
$\operatorname{cosec} x$	$\ln\left \tan \frac{x}{2}\right $	$\operatorname{cosech} x$	$\ln\left \tanh \frac{x}{2}\right $
$\sec x$	$\ln \sec x + \tan x $	$\operatorname{sech} x$	$2 \tan^{-1} e^x$
$\sec^2 x$	$\tan x$	$\operatorname{sech}^2 x$	$\tanh x$
$\cot x$	$\ln \sin x $	$\operatorname{coth} x$	$\ln \sinh x $
$\sin^2 x$	$\frac{x}{2} - \frac{\sin 2x}{4}$	$\sinh^2 x$	$\frac{\sinh 2x}{4} - \frac{x}{2}$
$\cos^2 x$	$\frac{x}{2} + \frac{\sin 2x}{4}$	$\cosh^2 x$	$\frac{\sinh 2x}{4} + \frac{x}{2}$



$f(x)$	$\int f(x) dx$	$f(x)$	$\int f(x) dx$
$\frac{1}{a^2+x^2}$	$\frac{1}{a} \tan^{-1} \frac{x}{a}$ $(a > 0)$	$\frac{1}{a^2-x^2}$	$\frac{1}{2a} \ln \left  \frac{a+x}{a-x} \right $ ( $0 <  x  < a$ ) $\frac{1}{2a} \ln \left  \frac{x-a}{x+a} \right $ ( $ x  > a > 0$ )
$\frac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1} \frac{x}{a}$ $(-a < x < a)$	$\frac{1}{\sqrt{a^2+x^2}}$	$\ln \left  \frac{x+\sqrt{a^2+x^2}}{a} \right $ ( $a > 0$ ) $\ln \left  \frac{x+\sqrt{x^2-a^2}}{a} \right $ ( $x > a > 0$ )
$\sqrt{a^2-x^2}$	$\frac{a^2}{2} \left[ \sin^{-1} \left( \frac{x}{a} \right) + \frac{x\sqrt{a^2-x^2}}{a^2} \right]$	$\sqrt{a^2+x^2}$	$\frac{a^2}{2} \left[ \sinh^{-1} \left( \frac{x}{a} \right) + \frac{x\sqrt{a^2+x^2}}{a^2} \right]$ $\frac{a^2}{2} \left[ -\cosh^{-1} \left( \frac{x}{a} \right) + \frac{x\sqrt{x^2-a^2}}{a^2} \right]$

## 5. Tips

- STANDARD INTEGRALS are provided. Do not forget to use these tables when you need to
- When looking at the THEORY, STANDARD INTEGRALS, ANSWERS or TIPS pages, use the [Back](#) button (at the bottom of the page) to return to the exercises
- Use the solutions intelligently. For example, they can help you get started on an exercise, or they can allow you to check whether your intermediate results are correct
- Try to make less use of the full solutions as you work your way through the Tutorial

## Full worked solutions

### Exercise 1.

$$\int \cos 3x \cos 2x \, dx: \quad \text{Use } \cos A \cos B = \frac{1}{2} [\cos (A + B) + \cos (A - B)]$$

i.e. taking  $A = 3x$  and  $B = 2x$ :

$$\begin{aligned} \int \cos 3x \cos 2x \, dx &= \frac{1}{2} \int [\cos (3x + 2x) + \cos (3x - 2x)] \\ &= \frac{1}{2} \int [\cos 5x + \cos x] \, dx \end{aligned}$$

Each term under the integration sign is a function of a linear function of  $x$ , i.e.

$$\int f(ax+b) \, dx = \frac{1}{a} \int f(u) \, du, \text{ where } u = ax+b, \, du = a \, dx, \text{ i.e. } dx = \frac{du}{a}.$$

$$\text{i.e. } \int \cos 3x \cos 2x \, dx = \frac{1}{2} \frac{1}{5} \sin 5x + \frac{1}{2} \sin x + C = \frac{1}{10} \sin 5x + \frac{1}{2} \sin x + C.$$

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**Exercise 2.**

$$\int \sin 5x \cos 3x \, dx: \quad \text{Use } \sin A \cos B = \frac{1}{2} [\sin(A+B) + \sin(A-B)]$$

i.e. taking  $A = 5x$  and  $B = 3x$ :

$$\begin{aligned} \int \sin 5x \cos 3x \, dx &= \frac{1}{2} \int [\sin(5x+3x) + \sin(5x-3x)] \\ &= \frac{1}{2} \int [\sin 8x + \sin 2x] \, dx \end{aligned}$$

i.e.

$$\begin{aligned} \int \sin 5x \cos 3x \, dx &= -\frac{1}{2} \frac{1}{8} \cos 8x - \frac{1}{2} \frac{1}{2} \cos 2x + C \\ &= -\frac{1}{16} \cos 8x - \frac{1}{4} \cos 2x + C. \end{aligned}$$

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**Exercise 3.**

$$\int \sin 6x \sin 4x \, dx: \quad \text{Use } \sin A \sin B = -\frac{1}{2} [\cos (A + B) - \cos (A - B)]$$

i.e. taking  $A = 6x$  and  $B = 4x$ :

$$\begin{aligned} \int \sin 6x \sin 4x \, dx &= -\frac{1}{2} \int [\cos (6x + 4x) - \cos (6x - 4x)] \\ &= -\frac{1}{2} \int [\cos 10x - \cos 2x] \, dx \end{aligned}$$

i.e.

$$\begin{aligned} \int \sin 6x \sin 4x \, dx &= -\frac{1}{2} \frac{1}{10} \sin 10x + \frac{1}{2} \frac{1}{2} \sin 2x + C \\ &= -\frac{1}{20} \sin 10x + \frac{1}{4} \sin 2x + C. \end{aligned}$$

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**Exercise 4.**

$$\int \cos 2\omega t \sin \omega t dt: \quad \text{Use } \sin A \cos B = \frac{1}{2} [\sin (A + B) + \sin (A - B)]$$

i.e. taking  $A = \omega t$  and  $B = 2\omega t$ :

$$\begin{aligned} \int \cos 2\omega t \sin \omega t dt &= \frac{1}{2} \int [\sin (\omega t + 2\omega t) + \sin (\omega t - 2\omega t)] \\ &= \frac{1}{2} \int [\sin 3\omega t + \sin (-\omega t)] dt \\ &= \frac{1}{2} \int [\sin 3\omega t - \sin \omega t] dt \\ &= -\frac{1}{2} \frac{1}{3\omega} \cos 3\omega t + \frac{1}{2} \frac{1}{\omega} \cos \omega t + C \\ &= -\frac{1}{6\omega} \cos 3\omega t + \frac{1}{2\omega} \cos \omega t + C. \end{aligned}$$

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**Exercise 5.**

$$\int \cos 4\omega t \cos 2\omega t dt: \quad \text{Use } \cos A \cos B = \frac{1}{2} [\cos (A + B) + \cos (A - B)]$$

i.e. taking  $A = 4\omega t$  and  $B = 2\omega t$ :

$$\begin{aligned} \int \cos 4\omega t \cos 2\omega t dt &= \frac{1}{2} \int [\cos (4\omega t + 2\omega t) + \cos (4\omega t - 2\omega t)] \\ &= \frac{1}{2} \int [\cos 6\omega t + \cos 2\omega t] dt \\ &= \frac{1}{2} \frac{1}{6\omega} \sin 6\omega t + \frac{1}{2} \frac{1}{2\omega} \sin 2\omega t + C \\ &= \frac{1}{12\omega} \sin 6\omega t + \frac{1}{4\omega} \sin 2\omega t + C. \end{aligned}$$

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**Exercise 6.**

$\int \sin^2 x dx$ : For the particular case:  $A=B=x$ ,

the formula:  $\sin A \sin B = -\frac{1}{2} [\cos(A+B) - \cos(A-B)]$ ,

reduces to:  $\sin^2 x = -\frac{1}{2}(\cos 2x - 1)$  (a half-angle formula)

i.e.

$$\begin{aligned}\int \sin^2 x dx &= -\frac{1}{2} \int (\cos 2x - 1) dx \\ &= -\frac{1}{2} \frac{1}{2} \sin 2x + \frac{1}{2} x + C \\ &= -\frac{1}{4} \sin 2x + \frac{1}{2} x + C.\end{aligned}$$

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**Exercise 7.**

$\int \sin^2 \omega t dt$ : For the particular case:  $A=B=\omega t$ ,

the formula:  $\sin A \sin B = -\frac{1}{2} [\cos (A + B) - \cos (A - B)]$ ,

reduces to:  $\sin^2 \omega t = -\frac{1}{2}(\cos 2\omega t - 1)$

i.e.

$$\begin{aligned}\int \sin^2 \omega t dt &= -\frac{1}{2} \int (\cos 2\omega t - 1) dt \\ &= -\frac{1}{2} \frac{1}{2\omega} \sin 2\omega t + \frac{1}{2} t + C \\ &= -\frac{1}{4\omega} \sin 2\omega t + \frac{1}{2} t + C.\end{aligned}$$

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**Exercise 8.**

$\int \cos^2 t dt$ : For the particular case:  $A=B=t$ ,

the formula:  $\cos A \cos B = \frac{1}{2} [\cos (A + B) + \cos (A - B)]$  ,

reduces to:  $\cos^2 t = \frac{1}{2}(\cos 2t + 1)$

i.e.

$$\begin{aligned}\int \cos^2 t dt &= \frac{1}{2} \int (\cos 2t + 1) dt \\ &= \frac{1}{2} \frac{1}{2} \sin 2t + \frac{1}{2} t + C \\ &= \frac{1}{4} \sin 2t + \frac{1}{2} t + C.\end{aligned}$$

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**Exercise 9.**

$\int \cos^2 kx dx$ : For the particular case:  $A=B=kx$ ,

the formula:  $\cos A \cos B = \frac{1}{2} [\cos (A + B) + \cos (A - B)]$  ,

reduces to:  $\cos^2 kx = \frac{1}{2}(\cos 2kx + 1)$

i.e.

$$\begin{aligned}\int \cos^2 kx dx &= \frac{1}{2} \int (\cos 2kx + 1) dx \\ &= \frac{1}{2} \frac{1}{2k} \sin 2kx + \frac{1}{2} x + C \\ &= \frac{1}{4k} \sin 2kx + \frac{1}{2} x + C.\end{aligned}$$

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