Optical fractals: characterization, cavities, coherence and connections

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Abstract: Uniform and regular systems can generate optical fractals. After characterizing different fractal-generating systems, emphasis is placed on the roles of boundary conditions and cavity feedback. New aspects of linear and nonlinear optical fractals are presented, along with considerations of system coherence and novel connections to some classic systems and configurations.

I. INTRODUCTION

Universal instabilities in Nature tend to transcend the particular details of individual physical systems, and their effect is to drive the emergence of familiar universal patterns. In an essential way, these patterns may often be classified as either *simple* (have structure governed by a single dominant scale length) or *fractal* (possessing proportional levels of detail across many orders of scale). Both types of pattern are fundamentally important to research in the field of complexity, which exploits universal features in trans-disciplinary applications. Our interest here lies primarily with optical fractals and in the candidate systems that are potentially capable of generating them. A focus will be placed on transverse and longitudinal boundary conditions that may dictate, or modify, fractal-generating capacity. In this Invited talk, we consider details of two simple cavities (one linear, one nonlinear) that can generate optical fractal patterns. Introduction of incoherence effects is also examined and potential connections to classic systems and configurations, that are not normally associated with fractal generation, are outlined.

II. OPTICAL FRACTAL CHARACTERIZATION

We characterize optical fractals as falling broadly into four distinct categories, each of which has its own particular nature of generating multi-scale structure. These categories are:

(i) Linear fractals. One of the earliest reports of linear optical fractals is *diffractals* (plane waves scattered by non-regular/fractal objects) [1]. Although it is intuitive that light diffracted by complex gratings might acquire complex structure, even a simple (regular) square-wave grating can also produce fractal light patterns through repeated self-imaging of the grating itself (the Talbot effect) [2]. Self-imaging is a property of many linear optical systems. For instance, the transverse empty-cavity modes of classic unstable strip resonators have fractal character [3], where the eigenvalue problem (a criterion for self-reproducing mode profiles) involves an interplay between small-scale diffraction effects at the mirror edges and successive round-trip magnifications [4]. Mode fractality was later confirmed in so-called *kaleidoscope lasers*, that include non-trivial transverse boundary conditions [5,6]. Alternative schemes for optical self-imaging, such as multiple-reduction copiers and pixellated video feedback setups [7], have provided further (potentially linear) contexts for spatial fractal formation;

(ii) Soliton fractals. A range of fractal patterns in soliton-supporting systems has been identified over the last two decades. The existence of these patterns is directly related to nonlinear light–material coupling. Self-similarity has been predicted during the amplification of parabolic pulses in optical fibres [8] and also in the distributions of soliton profiles in systems with series of abrupt material discontinuities (that can induce individual new scale lengths through splitting phenomena) [9];

(iii) Nonlinear phase-space fractals. Fractals can appear in the parameter characterization of nonlinear optical phenomena (while their real-space and time representations remain non-fractal). Examples include bifurcations in the phase-space of chaotic pixel-pixel mappings in optical memory applications [10] and in the properties of interacting vector solitons [11];

(iv) Spontaneous nonlinear spatial fractals. Finite-amplitude simple universal patterns (e.g., stripes, squares, hexagons, honeycombs, etc.) may grow spontaneously from the homogeneous states of a reaction-diffusion system that is sufficiently stressed. Turing showed that the origin of simple-pattern emergence is the existence of a *single* threshold instability minimum whose characteristics dictate the dominant scale length of the pattern [12]. More recently, we proposed that any system whose threshold instability spectrum comprises a hierarchy of comparable Turing minima may be susceptible to truly spontaneous fractal pattern formation. The first prediction of such patterns was made for a simple system: the Kerr slice with a single feedback mirror [13,14]. Our subsequent analyses of dispersive and absorptive ring cavities [15] have offered further evidence that multi-Turing threshold minima can be a generic signature of a system's innate fractal-generating capacity.

III. FRACTALS IN LINEAR CAVITIES

Kaleidoscope lasers are generalizations of classic unstable resonators to two non-trivial transverse dimensions (2D), and where the system aperture has the shape of a regular polygon [6]. The non-orthogonal edges of this element have a profound

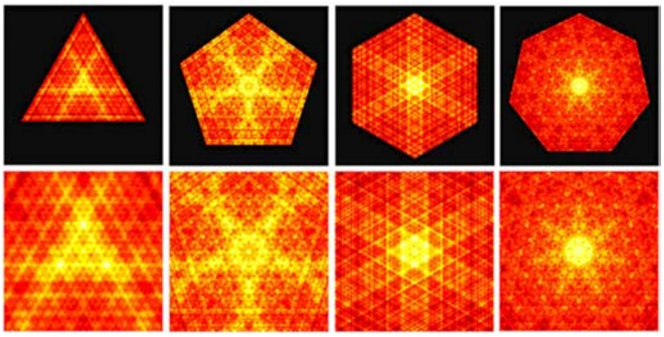

Fig. 1. Lowest-loss kaleidoscope laser modes for a range of transverse geometries when N_{eq} = 30 and M = 1.5. The lower panes are a magnification of the central portion of the pattern.

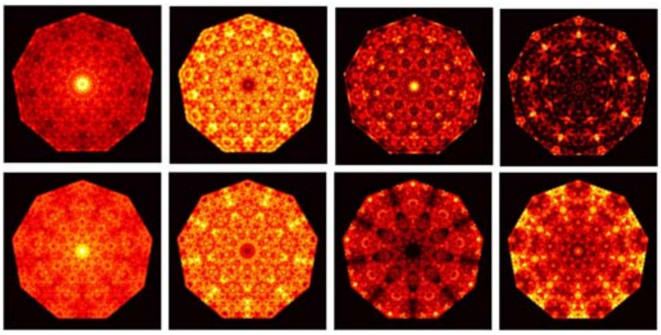

Fig. 2. Lowest-loss mode and first three higher-order modes (left to right) for a kaleidoscope laser with N = 9 and M = 1.5. Top row: N_{eq} = 20. Bottom row: N_{eq} = 30.

impact on the structure of the cavity eigenmodes, which exhibit a striking level of beauty and complexity. Unstable cavity lasers are typically described by two free parameters: the equivalent Fresnel number N_{eq} (which quantifies the cavity aspect ratio) and round-trip magnification M. Previous analyses of kaleidoscope lasers have been restricted to regimes where $N_{eq} = O(1)$ (then fully-numerical ABCD paraxial matrix modelling can be deployed [5,6]) or $N_{eq} >> O(1)$ (in which case asymptotic approximations may be used [16]). Our approach, which is based on a fully-2D generalization of Southwell's Virtual Source (VS) method [17], exploits exact mathematical descriptions of the fundamental spatial structures – Fresnel edge-waves [18].

Virtual source theory unfolds a geometrically unstable cavity into an equivalent sequence of $N_S = \log(250N_{eq})/\log(M)$ virtual apertures. Any desired eigenmode can then be constructed from a *linear superposition* of the edge-waves diffracted by each aperture, plus a plane-wave component. In scaled units, the mode pattern $V(\mathbf{X})$ is given by

$$V(\mathbf{X}) \propto \frac{E_{N_S+1}(\mathbf{X}_C)}{\alpha^{N_S}(\alpha-1)} + \sum_{m=1}^{N_S} \alpha^{-m} E_m(\mathbf{X}), \qquad (1)$$

where $\mathbf{X}$ denotes an appropriate set of transverse coordinates, $\mathbf{X}_C$ is any point on the boundary of the feedback mirror, and $E_m(\mathbf{X})$ is the composite edge-wave pattern arising from the m^{th} virtual aperture [18]. The weighting factor α plays the role of the mode eigenvalue; it is obtained by finding the roots of an

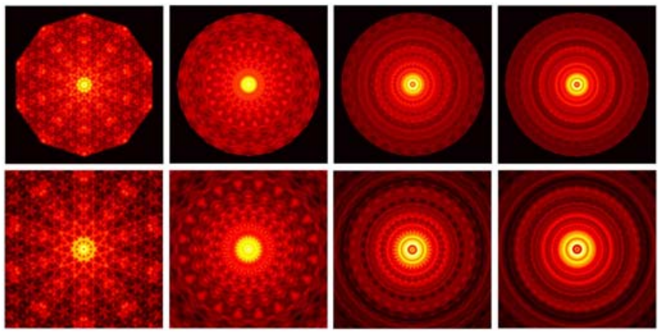

Fig. 3. Transition to circularity for the lowest-loss mode of a kaleidoscope laser with N_{eq} = 30 and M = 1.5. The lower panes show a magnification of the central portion of the pattern. The number of aperture edges in each case is (left to right): N = 10, 20, 30, and 40.

associated polynomial equation.

The eigenmodes of arbitrary-N cavities have been calculated for the first time, and with unprecedented accuracy, with any desired N_{eq} and M (see Fig. 1). A particularly useful facet of our 2D-VS modeling is that a single application can determine, simultaneously, the lowest-loss mode *and* all higher-order modes (see Fig. 2). As $N \to \infty$, the feedback mirror becomes circular and the cavity essentially has only a single transverse dimension. This limit has been investigated by Berry under the assumption $N_{eq} >> O(1)$, and only for the lowest-loss mode [16]. For cavities with *arbitrary* N_{eq} and M, this type of fully-2D convergence does not lend itself to asymptotic analysis; it can only truly be addressed via numerical computation (see Fig. 3). We will present, what is to the best of our knowledge, the first in-depth treatment of the quite subtle circular limit of families of kaleidoscope-laser modes.

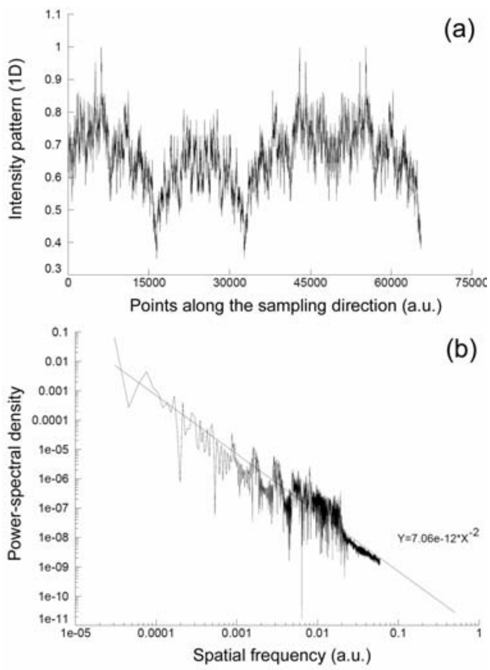

Fig. 4. (a) Slice through the lowest-loss mode of a triangular kaleidoscope laser with N_{eq} = 703.3 and M = 1.9 (sampling line passing through a vertex and perpendicularly through the opposite edge). (b) BENOIT power spectrum calculation and best-fit line [whose slope is directly related to the fractal dimension of the pattern shown in part (a)].

We have been begun looking into the fractal dimension [19] of kaleidoscope laser modes, a topic that is still not well understood. Dimension calculations are computationally non-trivial, and specialist software has been used to help facilitate analysis [20]. Some unexpected results have been uncovered. For example, the power-spectrum fractal dimension of a slice through the lowest-loss mode pattern for a triangular cavity with $N_{eq} = 703.3$ and $M = 1.9$ turns out to be approximately 1.5 (see Fig. 4), which is essentially identical to that for the corresponding mode of a confocal strip resonator with the same parameters [19]. This surprising result, which is in stark contrast to asymptotic analysis [16], suggests something quite profound: *the fractal dimension of kaleidoscope laser modes may be largely independent of N.*

IV. Fractals in Nonlinear Cavities

Earlier analyses and simulations of a simple system with a single Kerr slice and a feedback mirror [13,14] demonstrated spontaneous nonlinear fractal formation. Our later analyses of dispersive and absorptive ring cavities [15] provided further evidence that multi-Turing threshold minima can be a generic signature of a system's innate fractal-generating capacity.

A nonlinear Fabry-Pérot cavity is a deceptively simple system, since it has potential for highly complex behaviours even in the plane-wave limit [21]. Results of analyses of this new geometry will be presented. The full transverse system can be analysed as a direct generalisation of a single feedback mirror system [13,14], in which one of the faces of the thin Kerr slice is partially reflecting. Spontaneous pattern instabilities arise from an interplay that involves: cavity boundary conditions; diffraction of light beams; diffusion of the medium excitation (whose optical response gives rise to nonlinearity); *and* the interaction of counter-propagating light beams.

A systematic approach is taken to quantify the generalisation to the nonlinear Fabry-Pérot system. First, we examine an intermediate configuration in which a lossless and resonant cavity is considered. Analysis predicts, and simulations verify, both simple and fractal pattern formation in this case. Second, we present a full analysis of the spatiotemporal stability of the complete Fabry-Pérot system. The single feedback mirror configuration is found to be recovered as a limit of the more general stability analysis; the weakly-reflecting slice case exhibits only gentle modulations of the instability threshold curves. *Fractal formation is thus also predicted in the full nonlinear Fabry-Pérot system.*

V. Coherence And Connections

Within the process of generalisation to systems of counter-propagating beams in nonlinear cavities, the role of incoherence in spatial pattern formation of systems, with instantaneous nonlinear response, has been examined. We will demonstrate that coherent spatial patterns can indeed emerge from initial states of high incoherence. The case where a system is maintained as incoherent, though noisy complex pump beams, has also been considered. Pattern formation has been found to be surprising stable also in this case.

Finally, having presented further evidence that systems with multi-Turing minima have a strong likelihood to have a fractal-generating capacity, the question of *whether systems presenting a single Turing minimum can be induced to form spatial fractals* is examined. We draw on, what we believe to be is, a novel space-time analogy to propose that inducing spatial fractals may actually be relatively straightforward. This element of generalisation provokes intriguing connections with the behaviour of classic optical, and non-optical, nonlinear systems.

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