PPLATO PROMOTING PHYSICS LEARNING & TEACHING OPPORTUNITIES

Basic Mathematics

Inequalities

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The aim of this document is to provide a short, self-assessment programme for students who wish to acquire a basic competence in the use of inequalities.

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1. Inequalities (Introduction)

A number a is *greater than* a number b if $a - b$ is *positive*. In symbols this is written as $a > b$.

Thus	$2 > 1$	because	$2 - 1 = 1$ is positive,
	$3 > -1$	because	$3 - (-1) = 4$ is positive,
BUT	$-1 > 2$ is *false*	because	$-1 - 2 = -3$ is negative.

Example 1 Prove or disprove the following inequalities.

$$\textbf{(a)}\ 0.4 > \frac{1}{4}\,, \qquad \textbf{(b)}\ (0.7)^2 > \frac{1}{2}\,.$$

Solution

(a) As a decimal, $1/4 = 0.25$ and so $0.4 - 1/4 = 0.4 - 0.25 = 0.15$, which is positive. Thus $0.4 > 1/4$ is true.

(b) Here $(0.7)^2 = 0.7 \times 0.7 = 0.49$. As a fraction $1/2$ is 0.5. In this case, $(0.7)^2 - 1/2 = 0.49 - 0.5 = -0.01$, which is *negative*. This means that the inequality $(0.7)^2 > 1/2$ is false.

For this latter example we would write $(0.7)^2 < 1/2$, or in words, $(0.7)^2$ *is less than* $1/2$. In general we say:

A number a is *less than* a number b if $a - b$ is *negative*. In symbols this is written as $a < b$.

If $a < b$ then $b > a$ and vice versa.

Example 2 In each of the following pairs of numbers, use one of the symbols $>$ or $<$ to give the correct ordering of the numbers *for the order in which they appear*.

$$\textbf{(a)}\ -1\,,2\,, \qquad\qquad \textbf{(b)}\ \frac{1}{4}, \frac{1}{5}\,.$$

Solution

(a) Taking $a = -1$ and $b = 2$ the difference $a - b$, becomes $a - b = (-1) - 2 = -3$, which is negative. The correct inequality is $-1 < 2$.

(b) In decimal form $1/4 = 0.25$ and $1/5 = 0.2$. Since $0.25 - 0.2 = 0.05$, and this is positive, the correct inequality is $1/4 > 1/5$.

In addition to these two inequalities there are two further symbols, $\geq$ and $\leq$. The first of these is read as *greater than or equal to* and the second as *less than or equal to*.

EXERCISE 1. For each of the following pairs of numbers use one of the symbols $>$, $<$, $\geq$, $\leq$ to give the correct ordering *for the order in which they appear*.

(a) 10^2, 2^{10}.

(b) $(-1)^2$, $(-1/2)^2$.

(c) 0.2, 1/5.

(d) $-\frac{1}{3}$, $-\frac{1}{2}$.

Here is a short quiz.

Quiz Determine which of the following inequalities is correct.

(a) $3^2 > 2^3$, (b) $2^4 < 4^2$, (c) $2^5 < 5^2$, (d) $3^5 > 4^4$.

2. Rules for Inequalities

The section following this one will deal with the solution of inequalities. As with the solution of equations, there are certain rules that may be used. In the case of inequalities these are

Rule 1 An equal quantity may be added to, (or subtracted from) both sides of an inequality without changing the inequality.

Rule 2 An equal positive quantity may multiply (or divide) both sides of an inequality without changing the inequality.

Rule 3 If both sides of an inequality are multiplied (or divided) by a *negative* quantity then the inequality is *reversed*.

N.B. It is very important to be careful with the last rule.

Before looking at the solution of inequalities it is useful to see why the above rules hold. This is done in the following example.

Example 3 Given the (true) inequality $4 > -1$, verify each of the rules

(a) by adding 3 to both sides,
(b) by subtracting 3 from both sides,
(c) by multiplying both sides by 3,
(d) by multiplying both sides by -3.

Solution

(a) Adding 3 to both sides gives $7 > 2$, which is also true.

(b) Subtracting 3 from both sides gives $4 - 3 > -1 - 3$ or $1 > -4$, which is also true.

(c) Multiplying both sides by 3 gives $12 > -3$, which is also true.

(d) Multiplying the inequality $4 > -1$ by -3 gives, according to **rule 3**, $4 \times -3 < -1 \times -3$, or $-12 < 3$, which is correct.

3. Solving Inequalities

Example 4 Solve the following inequalities.

(a) $x-3>5$, **(b)** $2x-1>7$, **(c)** $3-2x>-5$.

Solution

(a)
$$\begin{aligned} x-3 &> 5, && \text{add 3 to both sides} \\ x-3+3 &> 5+3, \\ x &> 8. \end{aligned}$$

(b)
$$\begin{aligned} 2x-1 &> 7, && \text{add 1 to both sides} \\ 2x &> 8, && \text{divide both sides by 2,} \\ x &> 4. \end{aligned}$$

(c)
$$\begin{aligned} 3-2x &> -5, && \text{subtract 3 from both sides} \\ -2x &> -8, && \text{divide both sides by } -2 \\ x &< 4, && \textbf{rule 3} \text{ has reversed the inequality !} \end{aligned}$$

Here are some exercises on solving inequalities.

EXERCISE 2. Solve each of the following inequalities using the rules given in **section 2**.

(a) $3x - 4 < 5$, (b) $x + 1 < 0$,
(c) $2x - 6 \geq 10$, (d) $2x \geq x - 3$,
(e) $3x + 1 < 2x + 5$, (f) $3(x - 1) > 2(1 - x)$.

Now try this short quiz.

Quiz Which of the following is the solution to the inequality

$$15 - x > 7 + x\,?$$

(a) $x > 4$, (b) $x > 11$, (c) $x < 11$, (d) $x < 4$.

4. Further Inequalities

Some inequalities contain more information and need further development.

Example 4 Solve the inequality

$$x - 10 < 2x - 2 < x\,.$$

Solution The method is the same as before but now there are **two** inequalities to solve, i.e. $x - 10 < 2x - 2$ and $2x - 2 < x$. The first of these is solved in the left-hand column, the second in the right-hand column.

$x - 10$	$<$	$2x - 2\,,$	$\Vert$	$2x - 2$	$<$	$x\,,$
$x - 10 + 2$	$<$	$2x - 2 + 2\,,$	$\Vert$	$2x - 2 + 2$	$<$	$x + 2\,,$
$x - 8$	$<$	$2x\,,$	$\Vert$	$2x$	$<$	$x + 2\,,$
$x - 8 - x$	$<$	$2x - x\,,$	$\Vert$	$2x - x$	$<$	$x + 2 - x\,,$
-8	$<$	$x\,.$	$\Vert$	x	$<$	$2\,.$

Both of these must hold so the solution is $-8 < x < 2$, i.e. x must be larger than -8 **and** smaller than 2.

Here are some examples for you to practise on.

EXERCISE 3. Find the solution to each of the following inequalities.

(a) $-3 \leq 3x \leq 18$, (b) $10 \leq 2x \leq x + 9$,

(c) $x < 3x - 1 < 2x + 7$, (d) $2x - 7 < 8 < 3x - 11$.

To end this section try the short quiz below.

Quiz Which *prime numbers* satisfy the inequality

$$0 \leq 2w - 3 \leq w + 8\,?$$

(a) 5,7,11,13 (b) 2,5,11,17 (c) 2,3,5,7 (d) 3,7,11,13

5. Quiz on Inequalities

Begin Quiz

1. Which of the following inequalities is correct?

(a) $1/5 \geq 1/4$, (b) $1/4 \geq 0.4$, (c) $0.4 \geq 1/3$, (d) $1/3 \geq 7/25$.

2. Which set of numbers is the solution to the inequality $x + 13 > 2x > x + 1$?

(a) $1 < x < 13$,
(b) $1 < x < 13/2$,
(c) $13/2 > x > 1/2$,
(d) $15 > x > 2$.

3. Which *prime number* satisfies $5 + 2x < 3x < 15 + x$?

(a) 7 , (b) 11, (c) 3, (d) 5.

End Quiz Score: Correct

Solutions to Exercises

Exercise 1(a)

$10^2 = 100$ and $2^{10} = 1024$, so $10^2 < 2^{10}$.

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Exercise 1(b)

$(-1)^2 = 1$ and $(-1/2)^2 = 1/4$ so $(-1)^2 > (-1/2)^2$.

Click on green square to return

Exercise 1(c)

In decimal form $1/5 = 0.2$ so $0.2 \geq 1/5$ and $0.2 \leq 1/5$ are both true.

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Exercise 1(d)

The solution to this can be obtained by converting the fractions to decimals as in previous cases. It may also be obtained using fractions, by writing both with the same denominator 6.

$$-\frac{1}{2} = -\frac{1\times 3}{2\times 3} = -\frac{3}{6}, \quad \text{and} \quad -\frac{1}{3} = -\frac{1\times 2}{3\times 2} = -\frac{2}{6}.$$

Then

$$\begin{aligned} -\frac{1}{2} - \left(-\frac{1}{3}\right) &= -\frac{3}{6} - \left(-\frac{2}{6}\right) = -\frac{3}{6} + \frac{2}{6}, \\ &= \frac{-3+2}{6} = -\frac{1}{6}, \end{aligned}$$

which is negative. The correct inequality is therefore

$$-\frac{1}{2} < -\frac{1}{3}.$$

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Exercise 2(a)

$$\begin{aligned} 3x - 4 &< 5 \\ 3x - 4 + 4 &< 5 + 4 \\ 3x &< 9 \\ x &< 3\,. \end{aligned}$$

Click on green square to return

Exercise 2(b)

$$\begin{aligned} x+1 &< 0 \\ x+1-1 &< 0-1 \\ x &< -1\,. \end{aligned}$$

Click on green square to return

Exercise 2(c)

$$\begin{aligned} 2x - 6 &\geq 10 \\ 2x - 6 + 6 &\geq 10 + 6 \\ 2x &\geq 16 \\ x &\geq 8\,. \end{aligned}$$

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Exercise 2(d)

$$\begin{aligned} 2x &\geq x-3 \quad \text{subtract } x \text{ from both sides} \\ 2x-x &\geq x-3-x \\ x &\geq -3\,. \end{aligned}$$

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Exercise 2(e)

$$\begin{aligned} 3x+1 &< 2x+5 \\ 3x+1-1 &< 2x+5-1 \\ 3x &< 2x+4 \\ 3x-2x &< 2x+4-2x \\ x &< 4\,. \end{aligned}$$

Click on green square to return

Exercise 2(f)

In this case the brackets must first be removed using the standard rules - see the package on **brackets**.

$$\begin{aligned} 3(x-1) &> 2(1-x) \\ 3x-3 &> 2-2x \\ 3x-3+3 &> 2-2x+3 \\ 3x &> 5+2x \\ 3x-2x &> 5+2x-2x \\ x &> 5\,. \end{aligned}$$

Click on green square to return

Exercise 3(a)

Here there are two inequalities to be solved, $-3 \leq 3x$ and $3x \leq 18$. The first of these is

$$\begin{aligned} -3 &\leq 3x\,, \quad \text{divide both sides by } 3 \\ -1 &\leq x\,. \end{aligned}$$

The second is

$$\begin{aligned} 3x &\leq 18\,, \quad \text{divide both sides by } 3 \\ x &\leq 6\,. \end{aligned}$$

In both of the above inequalities the divisor is 3, which is positive, so the division does not reverse the inequalities.

The solution to the inequality is thus $-1 \leq x \leq 3$,.

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Exercise 3(b)

There are two inequalities here, $10 \leq 2x$ and $2x \leq x + 9$. The first is

$$\begin{aligned} 10 &\leq 2x\,, \quad \text{divide both sides by } 2 \\ 5 &\leq x\,. \end{aligned}$$

The second is

$$\begin{aligned} 2x &\leq x + 9\,, \quad \text{subtract } x \text{ from both sides} \\ x &\leq 9\,. \end{aligned}$$

The solution to the inequality is $5 \leq x \leq 9$.

Click on green square to return

Exercise 3(c) Here there are two inequalities, $x < 3x - 1$ and $3x - 1 < 2x + 7$. They are solved as follows.

$$\begin{aligned} x &< 3x - 1\,, && \text{adding } 1 \\ x + 1 &< 3x\,, && \text{subtracting } x \\ 1 &< 2x\,, && \text{dividing by } 2 \\ 1/2 &< x\,. \end{aligned}$$

$$\begin{aligned} 3x - 1 &< 2x + 7\,, && \text{adding } 1 \\ 3x &< 2x + 8\,, && \text{subtracting } 2x \\ x &< 8\,. \end{aligned}$$

The solution to the inequality is $1/2 < x < 8$.

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Exercise 3(d)

The two inequalities in this case are $2x - 7 < 8$ and $8 < 3x - 11$. The solution to each is

$$\begin{aligned} 2x - 7 &< 8\,, && \text{add } 7 \\ 2x &< 15\,, && \text{divide by } 2 \\ x &< 15/2\,. \end{aligned}$$

$$\begin{aligned} 8 &< 3x - 11\,, && \text{add } 11 \\ 19 &< 3x\,, && \text{divide by } 3 \\ 19/3 &< x\,. \end{aligned}$$

The solution to this is $19/3 < x < 15/2$.

Click on green square to return

Solutions to Quizzes

Solution to Quiz:

The solution to this is obtained from

$$3^2 = 9 \quad \text{and} \quad 2^3 = 8$$

and $9 > 8$. End Quiz

Solution to Quiz:
The solution is as follows.

$$\begin{aligned} 15 - x &> 7 + x \\ 15 - x + x &> 7 + x + x \\ 15 &> 7 + 2x \\ 15 - 7 &> 7 + 2x - 7 \\ 8 &> 2x \\ 4 &> x \qquad \text{or equivalently} \\ x &< 4\,. \end{aligned}$$

The first step in this solution was adding x to both sides so that (*the positive*) $2x$ appears on the right. This meant that the subsequent division was by the *positive* number 2. Division by positive numbers is always preferable as it generally leads to fewer mistakes.

End Quiz

Solution to Quiz:
As in previous cases there are two inequalities to be solved, $0 \leq 2w-3$ and $2w-3 \leq w+8$. The solution to each of these is

$$\begin{aligned} 0 &\leq 2w-3\,, && \text{add 3} \\ 3 &\leq 2w\,, && \text{divide by 2} \\ 3/2 &\leq w\,. \end{aligned}$$

$$\begin{aligned} 2w-3 &\leq w+8\,, && \text{add 3} \\ 2w &\leq w+11\,, && \text{subtract } w \\ w &\leq 11\,. \end{aligned}$$

so $3/2 \leq w \leq 11$. The *prime numbers* in this range are 2,3,5,7,11 which includes ALL those of part (c) but not all of the other choices on offer.

End Quiz