POWER SERIES

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A Tutorial Module for learning the usage of power series representations

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1. Theory

This tutorial deals with the approximation of functions of x, $f(x)$, using power series expansions:

$$f(x) = \sum_{i=0}^{\infty} c_i x^i = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \ldots \ ,$$

where c_i are constant coefficients.

Power series open the door to the rapid calculation, manipulation and interpretation of analytical expressions that are, otherwise, difficult to deal with.

In each exercise, an appropriate power series can be derived by using the Standard series (accessed from the "toolbar" at the bottom of each of the Exercises pages).

It will greatly simplify each calculation if, at an early stage, you manage to deduce how many terms are required from each standard series to give the final result to the specified accuracy (e.g. to x^3).

Note also that writing the given function in terms of partial fractions, where appropriate, can result in a simpler derivation.

2. Exercises

Use Standard series, to expand the following functions in power series, as far as the terms shown. Also state the range of values of x for which the power series converges:

Click on Exercise links for full worked solutions (there are 10 exercises in total).

Exercise 1.

$e^{-3x}\cos 2x$, up to x^3

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Exercise 4.

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$$(\cos 4x)\ln(1+2x), \text{ up to } x^4$$

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$$\frac{\sqrt{1+2x}}{1-3x}, \text{ up to } x^3$$

Exercise 10.

$$\sqrt{\frac{1+x}{1-2x}}, \text{ up to } x^3$$

3. Answers

1. $1 - 3x + \frac{5}{2}x^2 + \frac{3}{2}x^3 + \ldots$ (converges for all x),
2. $-2x^2 - 2x^3 - \frac{7}{3}x^4 + \ldots$ (converges for $-\frac{1}{2} \le x < \frac{1}{2}$),
3. $1 - \frac{3}{2}x + \frac{7}{8}x^2 - \frac{1}{48}x^3 + \ldots$ (converges for $-1 < x < 1$),
4. $x - x^2 + \frac{4}{3}x^3 - \frac{7}{3}x^4 + \ldots$ (converges for $-\frac{1}{2} < x < \frac{1}{2}$),
5. $2x - 2x^2 - \frac{40}{3}x^3 + 12x^4 + \ldots$ (converges for $-\frac{1}{2} < x \le \frac{1}{2}$),
6. $5 - x + 11x^2 - 13x^3 + \ldots$ (converges for $-\frac{1}{2} < x < \frac{1}{2}$),
7. $\frac{3}{2} - \frac{7}{4}x + \frac{15}{8}x^2 - \frac{31}{16}x^3 + \ldots$ (converges for $-1 < x < 1$),
8. $2 + \frac{7}{2}x + \frac{11}{4}x^2 + \frac{25}{8}x^3 + \ldots$ (converges for $-1 < x < 1$),
9. $1 + 4x + \frac{23}{2}x^2 + 35x^3 + \ldots$ (converges for $-\frac{1}{3} < x < \frac{1}{3}$),
10. $1 + \frac{3}{2}x + \frac{15}{8}x^2 + \frac{51}{16}x^3 + \ldots$ (converges for $-\frac{1}{2} < x < \frac{1}{2}$).

4. Standard series

$f(x)$	Power Series	Convergence
$\frac{1}{1-x}$	$1 + x + x^2 + x^3 + x^4 + \ldots$	$(-1 < x < 1)$
$(1+x)^n$	$1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots$	$(-1 < x < 1,$ $n \neq 1, 2, 3, \ldots)$
e^x	$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$	(for all x)
$\ln(1+x)$	$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$	$(-1 < x \leq 1)$
$\sin x$	$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$	(for all x)
$\cos x$	$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$	(for all x)
$\tan x$	$x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \ldots$	$(-\frac{\pi}{2} < x < \frac{\pi}{2})$
$\sinh x$	$x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \ldots$	(for all x)
$\cosh x$	$1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \ldots$	(for all x)

Note: $2! = 2 \cdot 1 = 2$, $3! = 3 \cdot 2 \cdot 1 = 6$, $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$, etc.

5. Tips on using solutions

● When looking at the THEORY, ANSWERS, STANDARD SERIES, TIPS or NOTATION pages, use the Back button (at the bottom of the page) to return to the exercises

● Use the solutions intelligently. For example, they can help you get started on an exercise, or they can allow you to check whether your intermediate results are correct

● Try to make less use of the full solutions as you work your way through the Tutorial

6. Alternative notation

The power series for $(1+x)^n$ is an example of a binomial series.

When n is not a whole number (i.e. $n \neq 0, 1, 2, 3, ...$) then the series is an infinite series and it is only true for $-1 < x < 1$.

On the other hand, when n is a whole number (i.e. $n = 0, 1, 2, 3, ...$) then the series no longer has an infinite number of terms and it is valid for all values of x.

Some books use notation to distinguish the above two cases, where:

$(1+x)^n$ is written when the power is a whole number, and
$(1+x)^r$ is written otherwise.

In this Tutorial, we use the symbol n to denote the power in either case.

Full worked solutions

Exercise 1.

$\boxed{e^{-3x}\cos 2x, \text{ up to } x^3}$

$$\begin{array}{llcl} \bullet & e^x & = & 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \ \boxed{\text{true for all } x} \\ \therefore & e^{-3x} & = & 1 + (-3x) + \frac{(-3x)^2}{2!} + \frac{(-3x)^3}{3!} + \ldots \\ \text{i.e.} & e^{-3x} & = & 1 - 3x + \frac{9x^2}{2} - \frac{9}{2}x^3 + \ldots \end{array}$$

$$\begin{array}{llcl} \bullet & \cos x & = & 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots \ \boxed{\text{true for all } x} \\ \therefore & \cos 2x & = & 1 - \frac{(2x)^2}{2} + \ldots \quad (\text{we only need up to } x^3) \\ \text{i.e.} & \cos 2x & = & 1 - 2x^2 + \ldots \end{array}$$

$$
\begin{aligned}
\therefore \quad e^{-3x}\cos 2x &= \left(1-3x+\tfrac{9}{2}x^2-\tfrac{9}{2}x^3\right)\left(1-2x^2\right)+\ldots \\
&= 1-3x+\tfrac{9}{2}x^2-\tfrac{9}{2}x^3 \\
&\quad -2x^2(1-3x)+\ldots \text{ (again, we only need up to } x^3\text{)} \\
&= 1-3x+\tfrac{9}{2}x^2-\tfrac{9}{2}x^3-2x^2+6x^3+\ldots \\
&= 1-3x+\left(\tfrac{9-4}{2}\right)x^2+\left(\tfrac{12-9}{2}\right)x^3+\ldots
\end{aligned}
$$

$$
\therefore \quad e^{-3x}\cos 2x = 1-3x+\tfrac{5}{2}x^2+\tfrac{3}{2}x^3+\ldots
$$

(true for all x, $\therefore$ converges for all x values).

Return to Exercise 1

Exercise 2.

$\boxed{(\sin x)\ln(1-2x),\text{ up to } x^4}$

- $\sin x \;=\; x - \frac{x^3}{3!} + \ldots = x - \frac{x^3}{6} + \ldots$ $\boxed{\text{true for all } x}$
- $\ln(1+x) \;=\; x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$ true for $-1 < x \leq 1$

$$\therefore \quad \ln(1-2x) = (-2x) - \frac{(-2x)^2}{2} + \frac{(-2x)^3}{3} - \frac{(-2x)^4}{4} + \ldots$$

true for $-1 < (-2x) \leq 1$

$$= -2x - \frac{4x^2}{2} - \frac{8x^3}{3} - \frac{16x^4}{4} - \ldots \quad \boxed{\text{true for } -\tfrac{1}{2} \leq x < \tfrac{1}{2}}$$

(note the switching of "$<$" and "$\leq$"

because of the minus sign of $-2x$)

$$\text{i.e.} \quad \ln(1-2x) = -2x - 2x^2 - \frac{8}{3}x^3 - 4x^4 - \ldots$$

$$\therefore\ (\sin x)\ln(1-2x) = \left(x - \frac{x^3}{6}\right)\left(-2x - 2x^2 - \frac{8}{3}x^3 - 4x^4\right) + \ldots$$

$$= x\left(-2x - 2x^2 - \frac{8}{3}x^3 - 4x^4\right) - \frac{x^3}{6}(-2x - \ldots) + \ldots$$

$$= -2x^2 - 2x^3 - \frac{8}{3}x^4 + \frac{x^4}{3} + \ldots$$

$$\therefore\ (\sin x)\ln(1-2x) = -2x^2 - 2x^3 - \frac{7}{3}x^4 + \ldots$$

(converges for $-\frac{1}{2} \le x < \frac{1}{2}$).

Note: We will assume convergence for the values of x that both the $\sin x$ and the $\ln(1-2x)$ power series are true.

Return to Exercise 2

Exercise 3.

$\boxed{\sqrt{1+x}\cdot e^{-2x}, \text{ up to } x^3}$

- $e^{x} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$ $\boxed{\text{true for all } x}$

$\therefore \quad e^{-2x} = 1 + (-2x) + \frac{(-2x)^2}{2} + \frac{(-2x)^3}{6} + \ldots$

i.e. $\quad e^{-2x} = 1 - 2x + 2x^2 - \frac{4}{3}x^3 + \ldots$

- $\sqrt{1+x} = (1+x)^{\frac{1}{2}}$

$$\equiv (1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots,$$

where $n = \frac{1}{2}$

$\therefore \quad \sqrt{1+x} = 1 + \frac{1}{2}x + \frac{1}{2}\cdot\left(-\frac{1}{2}\right)\frac{x^2}{2} + \left(\frac{1}{2}\right)\cdot\left(-\frac{1}{2}\right)\cdot\left(-\frac{3}{2}\right)\frac{x^3}{6} + \ldots$

$\boxed{\text{true for } -1 < x < 1}$

i.e. $$\sqrt{1+x} = 1+\tfrac{1}{2}x-\tfrac{1}{8}x^2+\tfrac{1}{16}x^3+\ldots$$

$$\begin{aligned}
\therefore \quad \sqrt{1+x}\,e^{-2x} &= \left(1+\tfrac{1}{2}x-\tfrac{1}{8}x^2+\tfrac{1}{16}x^3\right)\left(1-2x+2x^2-\tfrac{4}{3}x^3\right)+\ldots \\
&= 1\cdot\left(1-2x+2x^2-\tfrac{4}{3}x^3\right) \\
&\qquad +\tfrac{1}{2}x(1-2x+2x^2) \\
&\qquad\qquad -\tfrac{1}{8}x^2(1-2x) \\
&\qquad\qquad\qquad +\tfrac{1}{16}x^3(1)+\ldots \\
&= 1-2x+2x^2-\tfrac{4}{3}x^3 \\
&\qquad +\tfrac{1}{2}x-x^2+x^3 \\
&\qquad\qquad -\tfrac{1}{8}x^2+\tfrac{1}{4}x^3 \\
&\qquad\qquad\qquad +\tfrac{1}{16}x^3+\ldots
\end{aligned}$$

$$\therefore \quad \sqrt{1+x}\,e^{-2x} = 1-\tfrac{3}{2}x+\tfrac{7}{8}x^2-\tfrac{1}{48}x^3+\ldots \text{ (converges for } -1<x<1\text{).}$$

Return to Exercise 3

Exercise 4.

$$\boxed{\frac{\sin x}{\sqrt{1+2x}}, \text{ up to } x^4}$$

- $$\sin x = x - \frac{x^3}{3!} + \ldots = x - \frac{x^3}{6} + \ldots \quad \boxed{\text{true for all } x}$$

- $$(1+X)^{-\frac{1}{2}} = 1 + \left(-\tfrac{1}{2}\right) X + \left(-\tfrac{1}{2}\right)\cdot\left(-\tfrac{3}{2}\right)\frac{X^2}{2} + \left(-\tfrac{1}{2}\right)\cdot\left(-\tfrac{3}{2}\right)\cdot\left(-\tfrac{5}{2}\right)\frac{X^3}{6} + \ldots$$

(true for $-1 < X < 1$)

$$= 1 - \tfrac{1}{2}X + \tfrac{3}{2^2}\frac{X^2}{2} - \tfrac{15}{2^3}\frac{X^3}{6} + \ldots$$

$$\therefore \quad (1+2x)^{-\frac{1}{2}} = 1 - \tfrac{1}{2}(2x) + \tfrac{3}{2^2}\frac{(2x)^2}{2} - \tfrac{15}{2^3}\frac{(2x)^3}{6} + \ldots \text{ (true for } -1 < 2x < 1)$$

$$= 1 - x + \frac{3x^2}{2} - \frac{15}{6}x^3 + \ldots \quad \boxed{\text{true for } -\tfrac{1}{2} < x < \tfrac{1}{2}}$$

Note: In the above, it has been recognised that, to find $\sin(x)(1+2x)^{-\frac{1}{2}}$ one only needs $(1+2x)^{-\frac{1}{2}}$ up to x^3 since $\sin x = x - \frac{x^3}{6}$ will multiply this by at least x.

$$
\begin{aligned}
\therefore \quad \frac{\sin x}{\sqrt{1+2x}} &= \left(x - \frac{x^3}{6}\right)\left(1 - x + \tfrac{3}{2}x^2 - \tfrac{15}{6}x^3\right) + \ldots \\
&= x\left(1 - x + \tfrac{3}{2}x^2 - \tfrac{5}{2}x^3\right) - \tfrac{x^3}{6}(1-x) + \ldots \\
&= x - x^2 + \tfrac{3}{2}x^3 - \tfrac{5}{2}x^4 - \tfrac{1}{6}x^3 + \tfrac{1}{6}x^4 + \ldots \\
&= x - x^2 + \left(\tfrac{9-1}{6}\right)x^3 + \left(\tfrac{1-15}{6}\right)x^4 + \ldots \\
\therefore \quad \frac{\sin x}{\sqrt{1+2x}} &= x - x^2 + \tfrac{4}{3}x^3 - \tfrac{7}{3}x^4 + \ldots
\end{aligned}
$$

(converges for $-\frac{1}{2} < x < \frac{1}{2}$).

Return to Exercise 4

Exercise 5.

$\boxed{(\cos 4x)\ln(1+2x), \text{ up to } x^4}$

- $$\cos X = 1 - \frac{X^2}{2!} + \frac{X^4}{4!} - \ldots \quad \boxed{\text{true for all } X}$$
$$= 1 - \frac{X^2}{2} + \frac{X^4}{24} - \ldots$$

$$\therefore \quad \cos 4x = 1 - \frac{(4x)^2}{2} + \frac{(4x)^4}{24} - \ldots$$
$$= 1 - 8x^2 + \frac{64}{6}x^4 - \ldots = 1 - 8x^2 + \frac{32}{3}x^4 - \ldots$$

- $$\ln(1+X) = X - \frac{X^2}{2} + \frac{X^3}{3} - \frac{X^4}{4} + \ldots \text{ (true for } -1 < X \leq 1)$$

$$\therefore \quad \ln(1+2x) = (2x) - \frac{(2x)^2}{2} + \frac{(2x)^3}{3} - \frac{(2x)^4}{4} + \ldots \text{ (true for } -1 < 2x \leq 1)$$

$$\text{i.e.} \quad \ln(1+2x) = 2x - 2x^2 + \frac{8}{3}x^3 - 4x^4 + \ldots \quad \boxed{\text{true for } -\tfrac{1}{2} < x \leq \tfrac{1}{2}}$$

$$\begin{aligned}\therefore \quad (\cos 4x)\ln(1+2x) &= (1-8x^2)\left(2x-2x^2+\tfrac{8}{3}x^3-4x^4\right)+\ldots\\ &= 2x-2x^2+\tfrac{8}{3}x^3-4x^4-8x^2(2x-2x^2)+\ldots\\ &= 2x-2x^2-\tfrac{40}{3}x^3+12x^4+\ldots\\ &\quad \text{(converges for } -\tfrac{1}{2}<x\leq\tfrac{1}{2}).\end{aligned}$$

Return to Exercise 5

Exercise 6.

$$\boxed{\frac{5+4x}{(1+2x)(1-x)},\ \text{up to } x^3}$$

Firstly, express in terms of partial fractions:

$$\frac{5+4x}{(1+2x)(1-x)} = \frac{A}{1+2x} + \frac{B}{1-x} = \frac{A(1-x)+B(1+2x)}{(1+2x)(1-x)}$$

$\therefore \quad 5+4x = A(1-x)+B(1+2x) \qquad$ [identity, true for all x]

True for $x=1$: $\quad 5+4 = 0+B(3) \quad$ i.e. $B=3$

True for $x=-\frac{1}{2}$: $\quad 5-2 = A\left(\frac{3}{2}\right)+0 \quad$ i.e. $A=2$

So, $\dfrac{5+4x}{(1+2x)(1-x)} = \dfrac{2}{1+2x} + \dfrac{3}{1-x}$

[This form allows addition of series rather than multiplication, i.e. it makes the calculation easier]

- $$\frac{1}{1-X} = 1 + X + X^2 + X^3 + \ldots \quad \boxed{\text{true for } -1 < X < 1}$$

$$\therefore \quad \frac{1}{1+2x} = 1 + (-2x) + (-2x)^2 + (-2x)^3 + \ldots$$

$$(\text{true for } -1 < (-2x) < 1)$$

$$\text{i.e.} \quad \frac{1}{1+2x} = 1 - 2x + 4x^2 - 8x^3 + \ldots \quad \boxed{\text{true for } -\tfrac{1}{2} < x < \tfrac{1}{2}}$$

$$\therefore \quad \frac{2}{1+2x} + \frac{3}{1-x} = 2(1 - 2x + 4x^2 - 8x^3) + \ldots$$

$$+3(1 + x + x^2 + x^3) = 5 - x + 11x^2 - 13x^3 + \ldots$$

$$(\text{converges for } -\tfrac{1}{2} < x < \tfrac{1}{2}).$$

Return to Exercise 6

Exercise 7.

$$\boxed{\frac{3+x}{(2+x)(1+x)}, \text{ up to } x^3}$$

Express as partial fractions:

$$\frac{3+x}{(2+x)(1+x)} = \frac{A}{2+x} + \frac{B}{1+x} = \frac{A(1+x)+B(2+x)}{(2+x)(1+x)}$$

$\therefore \quad 3+x = A(1+x)+B(2+x) \qquad$ [identity, true for all x]

True for $x=-1$: $\quad 3-1 = 0+B(2-1) \quad$ i.e. $B=2$

True for $x=-2$: $\quad 3-2 = A(1-2)+0 \quad$ i.e. $A=-1$

So, $\dfrac{3+x}{(2+x)(1+x)} = \dfrac{-1}{2+x} + \dfrac{2}{1+x}$

- $$\frac{1}{1-X} = 1 + X + X^2 + X^3 + \ldots \quad \boxed{\text{true for } -1 < X < 1}$$

$$\therefore \quad \frac{1}{1+x} = 1 + (-x) + (-x)^2 + (-x)^3 + \ldots$$

$$= 1 - x + x^2 - x^3 + \ldots \quad \boxed{\text{true for } -1 < x < 1}$$

- $$\frac{1}{2+x} = \left(\frac{1}{2}\right) \frac{1}{1+\left(\frac{x}{2}\right)} = \left(\frac{1}{2}\right)\left[1 - \left(\frac{x}{2}\right) + \left(\frac{x}{2}\right)^2 - \left(\frac{x}{2}\right)^3 + \ldots\right]$$

$$\left(\text{true for } -1 < \frac{x}{2} < 1\right)$$

i.e. $$-\frac{1}{2+x} = -\frac{1}{2} + \frac{x}{4} - \frac{x^2}{8} + \frac{x^3}{16} + \ldots \quad \boxed{\text{true for } -2 < x < 2}$$

$$\therefore \quad \frac{3+x}{(2+x)(1+x)} = -\frac{1}{2+x} + \frac{2}{1+x} = -\frac{1}{2} + \frac{x}{4} - \frac{x^2}{8} + \frac{x^3}{16} + 2(1 - x + x^2 - x^3) + \ldots$$

$$= \frac{3}{2} - \frac{7}{4}x + \frac{15}{8}x^2 - \frac{31}{16}x^3 + \ldots$$

(converges for $-1 < x < 1$).

Return to Exercise 7

Exercise 8.

$$\boxed{\frac{4+5x}{(2+x)(1-x)}\text{, up to } x^3}$$

Partial fractions:

$$\frac{4+5x}{(2+x)(1-x)} = \frac{A}{2+x} + \frac{B}{1-x} = \frac{A(1-x)+B(2+x)}{(2+x)(1-x)}$$

$\therefore \quad 4+5x = A(1-x)+B(2+x) \qquad$ [identity, true for all x]

True for $x=+1$: $\quad 4+5 = 0 + B\cdot(3) \quad$ i.e. $B=3$

True for $x=-2$: $\quad 4-10 = A\cdot(3)+0 \quad$ i.e. $A=-2$

So, $\dfrac{4+5x}{(2+x)(1-x)} = -\dfrac{2}{2+x} + \dfrac{3}{1-x}$

- $$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots \boxed{\text{true for } -1 < x < 1}$$
- $$\frac{1}{2+x} = \left(\frac{1}{2}\right)\frac{1}{\left(1+\frac{x}{2}\right)} = \frac{1}{2}\left[1 - \left(\frac{x}{2}\right) + \left(\frac{x}{2}\right)^2 - \left(\frac{x}{2}\right)^3 + \ldots\right]$$

(true for $-1 < \frac{x}{2} < 1$)

$$= \frac{1}{2} - \frac{x}{4} + \frac{x^2}{8} - \frac{x^3}{16} + \ldots \boxed{\text{true for } -2 < x < 2}$$

$$\therefore \quad \frac{4+5x}{(2+x)(1-x)} = -2\left(\frac{1}{2} - \frac{x}{4} + \frac{x^2}{8} - \frac{x^3}{16}\right) + 3(1 + x + x^2 + x^3) + \ldots$$

$$= 2 + \frac{7}{2}x + \frac{11}{4}x^2 + \frac{25}{8}x^3 + \ldots$$

(converges for $-1 < x < 1$).

Return to Exercise 8

Exercise 9.

$$\boxed{\frac{\sqrt{1+2x}}{1-3x}, \text{ up to } x^3}$$

- $$(1+X)^n = 1 + nX + \tfrac{n(n-1)}{2!}X^2 + \tfrac{n(n-1)(n-2)}{3!}X^3 + \ldots$$

(true for $-1 < X < 1$)

$$\sqrt{1+X} = 1+\tfrac{1}{2}X+\left(\tfrac{1}{2}\right)\cdot\left(-\tfrac{1}{2}\right)\tfrac{X^2}{2!}+\left(\tfrac{1}{2}\right)\cdot\left(-\tfrac{1}{2}\right)\cdot\left(-\tfrac{3}{2}\right)\tfrac{X^3}{3!}+\ldots$$

$$\therefore \quad \sqrt{1+2x} = 1+\tfrac{1}{2}(2x)-\left(\tfrac{1}{2}\right)\cdot\left(\tfrac{1}{2}\right)\tfrac{(2x)^2}{2!}+\left(\tfrac{1}{2}\right)\cdot\left(\tfrac{1}{2}\right)\cdot\left(\tfrac{3}{2}\right)\tfrac{(2x)^3}{3!}+\ldots$$

(true for $-1 < 2x < 1$)

$$= 1 + x - \tfrac{x^2}{2} + \tfrac{x^3}{2} - \ldots \quad \boxed{\text{true for } -\tfrac{1}{2} < x < \tfrac{1}{2}}$$

- $$\frac{1}{1-X} = 1 + X + X^2 + X^3 + \ldots \text{ (true for } -1 < X < 1)$$

$$\therefore \quad \frac{1}{1-3x} = 1 + (3x) + (3x)^2 + (3x)^3 + \ldots \text{ (true for } -1 < 3x < 1)$$

$$= 1 + 3x + 9x^2 + 27x^3 + \ldots \quad \boxed{\text{true for } -\tfrac{1}{3} < x < \tfrac{1}{3}}$$

$$\therefore \quad \frac{\sqrt{1+2x}}{1-3x} = \left(1 + x - \frac{x^2}{2} + \frac{x^3}{2}\right)(1 + 3x + 9x^2 + 27x^3) + \ldots$$

$$= 1 + 3x + 9x^2 + 27x^3$$

$$+x(1 + 3x + 9x^2)$$

$$-\frac{x^2}{2}(1 + 3x)$$

$$+\frac{x^3}{2}(1) + \ldots$$

$$
\begin{aligned}
&= 1 + 3x + 9x^2 + 27x^3 \\
&\qquad +x + 3x^2 + 9x^3 \\
&\qquad\qquad -\tfrac{1}{2}x^2 - \tfrac{3}{2}x^3 \\
&\qquad\qquad\qquad +\tfrac{1}{2}x^3 + \ldots \\
&= 1 + 4x + \tfrac{23}{2}x^2 + 35x^3 + \ldots \\
&\qquad \text{(converges for } -\tfrac{1}{3} < x < \tfrac{1}{3}\text{)}.
\end{aligned}
$$

Return to Exercise 9

Exercise 10.

$$\boxed{\sqrt{\frac{1+x}{1-2x}},\ \text{up to } x^3}$$

- $$\sqrt{1+x} = (1+x)^{\frac{1}{2}} = 1 + \left(\tfrac{1}{2}\right)x + \left(\tfrac{1}{2}\right)\cdot\left(-\tfrac{1}{2}\right)\tfrac{x^2}{2!} + \left(\tfrac{1}{2}\right)\cdot\left(-\tfrac{1}{2}\right)\cdot\left(-\tfrac{3}{2}\right)\tfrac{x^3}{3!} + \ldots$$
  $$= 1 + \tfrac{1}{2}x - \tfrac{1}{8}x^2 + \tfrac{1}{16}x^3 + \ldots$$
  $$\boxed{\text{true for } -1 < x < 1}$$

- $$\tfrac{1}{\sqrt{1+X}} = (1+X)^{-\frac{1}{2}} = 1 + \left(-\tfrac{1}{2}\right)X + \left(-\tfrac{1}{2}\right)\cdot\left(-\tfrac{3}{2}\right)\tfrac{X^2}{2!} + \left(-\tfrac{1}{2}\right)\cdot\left(-\tfrac{3}{2}\right)\cdot\left(-\tfrac{5}{2}\right)\tfrac{X^3}{3!} + \ldots$$
  $$(\text{true for } -1 < X < 1)$$

$$\therefore \quad (1-2x)^{\frac{1}{2}} = 1 - \tfrac{1}{2}(-2x) + \left(\tfrac{1}{2}\right)\cdot\left(\tfrac{3}{2}\right)\frac{(-2x)^2}{2!} - \left(\tfrac{1}{2}\right)\cdot\left(\tfrac{3}{2}\right)\cdot\left(\tfrac{5}{2}\right)\frac{(-2x)^3}{3!} + \ldots$$

$$(\text{true for } -1 < (-2x) < 1)$$

$$= 1 + x + \tfrac{3}{2}x^2 + \tfrac{5}{2}x^3 + \ldots \quad \boxed{\text{true for } -\tfrac{1}{2} < x < \tfrac{1}{2}}$$

$$
\begin{aligned}
\therefore\ (1+x)^{\frac{1}{2}}(1-2x)^{-\frac{1}{2}} &= \left(1+\tfrac{1}{2}x-\tfrac{1}{8}x^2+\tfrac{1}{16}x^3\right)\left(1+x+\tfrac{3}{2}x^2+\tfrac{5}{2}x^3\right)+\ldots \\
&= 1+x+\tfrac{3}{2}x^2+\tfrac{5}{2}x^3 \\
&\quad +\tfrac{1}{2}x(1+x+\tfrac{3}{2}x^2) \\
&\quad -\tfrac{1}{8}x^2(1+x) \\
&\quad +\tfrac{1}{16}x^3(1)+\ldots \\
&= 1+x+\tfrac{3}{2}x^2+\tfrac{5}{2}x^3 \\
&\quad +\tfrac{1}{2}x+\tfrac{1}{2}x^2+\tfrac{3}{4}x^3 \\
&\quad -\tfrac{1}{8}x^2-\tfrac{1}{8}x^3 \\
&\quad +\tfrac{1}{16}x^3+\ldots \\
&= 1+\tfrac{3}{2}x+\tfrac{15}{8}x^2+\tfrac{51}{16}x^3+\ldots
\end{aligned}
$$

(converges for $-\frac{1}{2} < x < \frac{1}{2}$).

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