Helmholtz Dark Solitons

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Abstract

- The general dark soliton solution of the nonlinear Helmholtz equation $\partial_{zz}A+\partial_{xx}A+f(A)=0$, where $f(A)$ introduces defocusing Kerr nonlinearity, is reported.
- Modifications to soliton transverse velocity, width, phase period and existence conditions are explicitly derived an explained in geometrical terms.
- Full numerical simulations verify analytical predictions, along with demonstrating the spontaneous formation of Helmholtz solitons and their transparency in mutual interactions.

Introduction

The effects of non-paraxiality on soliton propagation in self-focusing media have been previously addressed using a scalar approach based on the non-paraxial nonlinear Schrödinger equation (NNSE) [1,2,3] which is fully equivalent to the Helmholtz equation[4]

$$\kappa\frac{\partial^2 u}{\partial\zeta^2}+j\frac{\partial u}{\partial\zeta}+\frac{1}{2}\frac{\partial^2 u}{\partial\xi^2}\pm|u|^2u=0, \qquad (1)$$

where

$$\zeta=\frac{z}{L_D},\quad \xi=\frac{\sqrt{2}x}{w_0},\quad u(\xi,\zeta)=\sqrt{\frac{kn_2L_D}{n_0}}A(\xi,\zeta),$$

$$\kappa=\frac{1}{kw_0^2}\quad\text{and}\quad E(x,z)=A(x,z)e^{jkz} \qquad (2)$$

Earlier studies uncovered the exact non-paraxial bright soliton solution and its detailed physical interpretation[1] as well as general analytical properties of the solution of the NNSE[2]. This proved fundamental in the development and testing of new non-paraxial numerical techniques[3].

The solutions of the NNSE have been shown to posses invariance under rotational transformation[1,2] given by

$$\xi=\frac{\xi'+V\zeta'}{\sqrt{1+2\kappa V^2}}\qquad \zeta=\frac{-2\kappa V\xi'+\zeta'}{\sqrt{1+2\kappa V^2}} \qquad (3)$$

$$u(\xi,\zeta)=\exp\left[j\left(\frac{V\zeta'}{\sqrt{1+2\kappa V^2}}+\frac{1}{2\kappa}\left(1-\frac{1}{\sqrt{1+2\kappa V^2}}\right)\zeta'\right)\right]u'(\xi'\zeta')$$

The rotation angle in the original (unscaled) coordinate system is given by $\sec\theta=\sqrt{1+2\kappa V^2}$.

In the paraxial limit, $\kappa V^2\to 0$, the galilean transformation of the NSE is recovered.

Exact dark solitons

We report here the exact dark soliton solution of the NNSE:

$$u(\xi,\zeta)=u_0\left\{\frac{(\lambda+i\nu)^2+e^Z}{1+e^Z}\right\}\exp\left[i\sqrt{\frac{1-4\kappa u_0^2}{1+2\kappa V^2}}\left(-V\xi+\frac{\zeta}{2\kappa}\right)\right]\exp\left[-i\frac{\zeta}{2\kappa}\right] \qquad (4)$$

where

$$Z=\frac{2u_0\nu(\xi+W\zeta)}{\sqrt{1+2\kappa W^2}} \qquad (5)$$

and

$$W=\frac{V-V_0}{1+2\kappa VV_0} \qquad (6)$$

is a net transverse velocity involving V (arising from the choice of reference frame) and V_0 (an amplitude-dependent component asociated with grey solitons), given by

$$V_0=\frac{u_0\lambda}{\sqrt{1-(2+\lambda^2)2\kappa u_0^2}}. \qquad (7)$$

The dark soliton parameter is $\lambda=\pm(1-\nu^2)^{1/2}$. For $\lambda=0$, the black Helmhltz soliton is obtained:

$$u(\xi,\zeta)=u_0A\tanh\left[\frac{u_0(\xi+V\zeta)}{\sqrt{1+2\kappa V^2}}\right]\exp\left[i\sqrt{\frac{1-4\kappa u_0^2}{1+2\kappa V^2}}\left(-V\xi+\frac{\zeta}{2\kappa}\right)\right]\exp\left[-i\frac{\zeta}{2\kappa}\right] \qquad (8)$$

$0<\lambda<1$ defines grey solitons with varying degree of greyness.
$\lambda=1$ ($\nu=0$) gives a nonlinear plane wave.

In the paraxial limit, $\kappa\to 0$, the appropriate paraxial dark and grey solitons are found as particular solutions

$$u(\xi,\zeta)=u_0\left\{\frac{(\lambda+i\nu)^2+e^Z}{1+e^Z}\right\}\exp(-iV\xi)\exp\left(-iu_0^2\zeta-i\frac{1}{2}V^2\zeta\right) \qquad (9)$$

where

$$Z=2u_0\nu[\xi+(V-\lambda u_0)\zeta] \qquad (10)$$

Both paraxial and Helmholtz soliton solutions represent a dark beam with an intensity minimun of $I_{min}=\lambda^2u_0^2$ on top of background of intensity u_0^2 with a travelling phase moving in the z direction. For $\lambda=0$ the dark beam becomes black $I_{min}=0$ and moves in the z direction (black soliton), whereas for non-zero λ results in a grey beam that has an additional velocity component V_0.

Properties of the Helmholtz dark solitons:

Corrections of unphysical features of paraxial theory

Non-paraxial vs paraxial dark solitons

The V=0 non-paraxial solution reveals modification of:

- The *phase period* of the background.
- The *transverse velocity* parameter of the dark beam. $V_{par}=-u_0\lambda$ whereas $V_{non\text{-}par}=V_0$ defined in (7) which corresponds to $\tan\theta_0=(2\kappa)^{1/2}V_0$, θ_0 being the actual propagation angle of the dark beam in unscaled coordinates.
- The *soliton width*. In the nonparaxial solution this is enlarged by the factor $\sqrt{1+2\kappa V_0^2}=1/\cos\theta_0$ (11)

Physical limits imposed by non-paraxiality

While paraxial soliton solutions exist for arbitrary values of u_0 and $0<\lambda<1$, fundamental limitations come into play once the paraxial approximation is released:

- *Black solitons* are restricted to $4\kappa n_0^2<1$, which is equivalent (in unscaled coordinates) to $2n_2|<n_0$: black solitons can only be found when the *nonlinear phase shift is smaller than the linear phase shift*
- *Grey solutions* only exist if $|F|<|F|_{max}=[(1-4\kappa u_0^2)/(2\kappa u_0^2)]^{1/2}$ which restricts the propagation angle of the dark beam θ_0 *to be real*.

The general solution

The general soliton solution (4) introduces an additional transverse velocity parameter V which describes the background propagating at an angle θ relative to the z axis in unscaled coordinates. The grey beam, makes an angle $\theta-\theta_0$ relative to the z axis. Using the transformation (2) one finds that the correction of the soliton width is

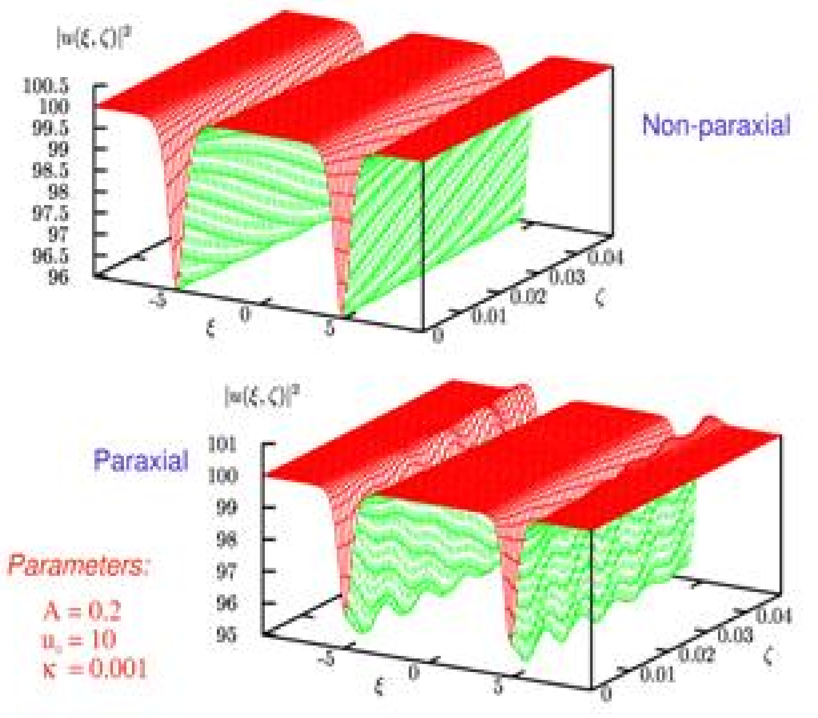
$$\cos(\theta-\theta_0)=\frac{1+2\kappa VV_0}{\sqrt{1+2\kappa V^2}\sqrt{1+2\kappa V_0^2}}. \qquad (12)$$

Identification of the combined transverse velocity parameter, W (6), yields the physical correspondence:

$$\cos(\theta-\theta_0)=\frac{1}{\sqrt{1+2\kappa W^2}} \qquad (13)$$

Numerical Results: Instability of the dark paraxial soliton under the NNSE

Evolution under the NNSE of initial conditions defined as the non-paraxial and parxial exact solitons



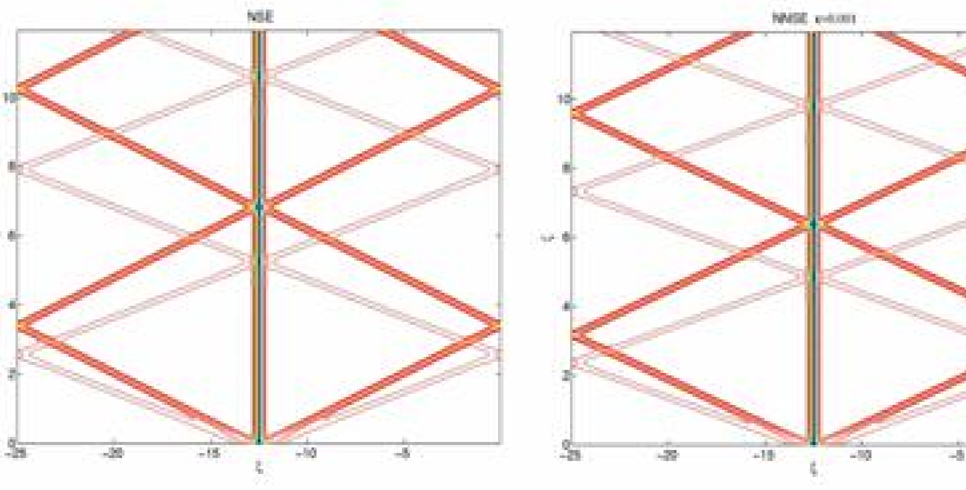
Numerical results: dark soliton splitting

We numerically study[3] the evolution of the initial condition given by

$$u(\xi,0)=u_0\tanh(u_0 a\xi) \qquad (11)$$

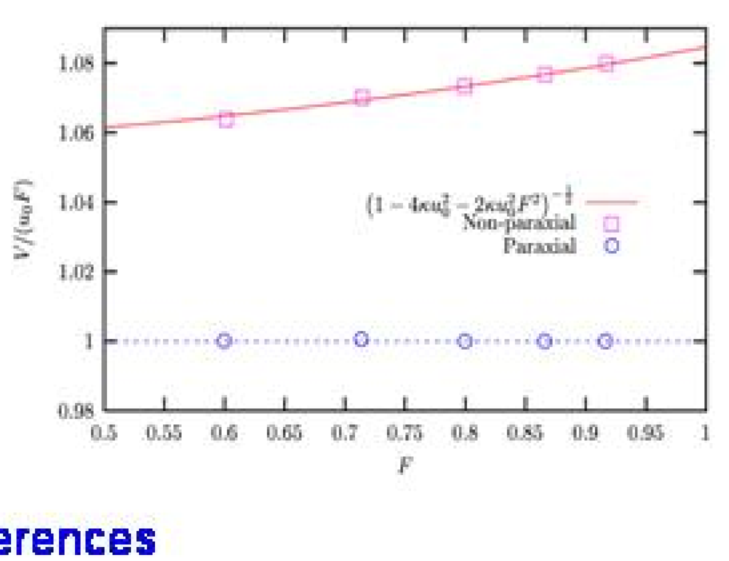
which results in the generation of $2N_0+1$ dark solitons[5], where N_0 is the largest integer satisfying the condition $N_0<1/a$. The following picture shows the splitting in both the paraxial and non-paraxial cases for a=0.3 where one can notice the difference in the transverse velocity of the grey beams after splitting. Helmholtz solitons are also demostrated to be stable and robust attractors that posses the key property of transparent mutual interaction with a trajectory phase shift.

Dark soliton splitting (u_0= 5 and a = 0.3) in the non-paraxial (NNSE) and paraxial (NSE) frameworks



The following figure summarizes the transverse velocities of the various solitons generated both in the paraxial and the non-paraxial cases for u_0=5 and different values of a. The results are in good agreement with the theory.

Transverse velocities of the grey pulses resulting from multisoliton splitting in the non-paraxial (NNSE) and paraxial (NSE) frameworks



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