Helmholtz Solitons of Polynomial Non-Linearity

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Abstract

The non-paraxial non-linear Schrödinger (NNLS) equation describing spatial beam propagation in a Kerr medium can be generalized to allow for a polynomial-type non-linearity in the field modulus. Soliton solutions to this novel scalar equation, the generalized NNLS (gNNLS), are presented and their stability in different parameter regimes examined through the use of a well-tested numerical perturbative approach. Novel Helmholtz algebraic solitons, with power-law asymptotics, are also presented.

Introduction

Models of spatial beam propagation in Kerr planar waveguide are usually based upon the paraxial non-linear Schrödinger (NLS) equation. Despite possessing many desirable mathematical features, the NLS equation can be unsuitable for describing wave propagation in some circumstances. Of particular significance here is its inability to describe accurately spatial beams which travel at non-trivial angles with respect to the longitudinal z direction. To overcome this the NNLS equation was derived [1], in which the slowly-varying envelope approximation was omitted. The resultant NNLS framework is rotationally-symmetric and respects certain symmetry aspects of the propagation problem ignored by NLS-type approaches. This equation can also be generalized to allow for beam propagation in a dielectric medium whose response deviates from the archetypal Kerr (cubic) non-linearity [2-4]. The generalized NNLS (gNNLS) equation is:

$$\kappa\frac{\partial^2 u}{\partial\zeta^2}+i\frac{\partial u}{\partial\zeta}+\frac{1}{2}\frac{\partial^2 u}{\partial\xi^2}+\alpha|u|^{\sigma}u+\gamma|u|^{2\sigma}u=0$$

Retain Helmholtz term → **Non-Trivial Helmholtz-Type Corrections…**

- Well-defined angles, e.g. $\tan^2\theta=2\kappa V^2$,
- Angular beam broadening,
- Longitudinal and transverse dimensions physically identical,
- Corrections to phase shifts.

This equation is a Helmholtz-type extension of the gNLS equation in [5]. It admits exact analytical soliton solutions which exhibit non-trivial correction to their paraxial counterpart. For the parametric choice $\alpha<0$ and $\gamma<0$, the solution is unstable, since a purely *defocusing* medium cannot support such a localized beam.

$$u(\xi,\zeta)=\left[\frac{\eta}{\cosh\Theta+\Gamma}\right]^{\frac{1}{\sigma}}\exp\left[i\sqrt{\frac{1+2\kappa(\mu/\sigma)^2}{1+2\kappa V^2}}\left(-V\xi+\frac{\zeta}{2\kappa}\right)\right]\exp\left[-i\frac{\zeta}{2\kappa}\right],$$

where $\Theta(\xi,\zeta)=\frac{\mu(\xi+V\zeta)}{\sqrt{1+2\kappa V^2}}$, $\quad\Gamma\equiv\Gamma_{\pm}(\mu)=\pm\left[1+\left(\frac{\mu^2}{2\sigma^2}\right)\frac{(2+\sigma)^2}{1+\sigma}\frac{\gamma}{\alpha^2}\right]^{-\frac{1}{2}}$, $\quad\eta\equiv\eta(\mu)=\left(\frac{\mu^2}{2\sigma^2}\right)\frac{2+\sigma}{\alpha}\Gamma(\mu)$.

This solution has vanishing asymptotics (strictly, exponential tails). It is most easily sought derived using Ansatz methods combined with the rotational transformation laws of the gNNLS equation (which are identical to those of the NNLS equation [6]).

HELMHOLTZ SOLUTION REDUCES TO ITS PARAXIAL COUNTERPART IN THE MULTIPLE LIMIT

$\kappa\to 0\quad \kappa(\mu/\sigma)^2\to 0\quad \kappa V^2\to 0$

Soliton Stability

The increased number of degrees of freedom in the problem and the non-integrability of the gNNLS equation make predictions about soliton stability very difficult. It has been found by numerical *trial-and-error* that the regimes of stability for the exact analytical generalized Helmholtz solitons are:

Pure Focusing ($\alpha>0$ $\gamma>0$) $\Gamma=\Gamma_+$ $\sigma\le 2$ stable $2<\sigma<4$ stable $\mu<\mu_{th}$ $\sigma>4$ unstable	Pure defocusing ($\alpha<0$ $\gamma<0$) **NO STABLE LOCALIZED SOLITON SOLUTION**
Type-I Competing ($\alpha<0$ $\gamma>0$) $\Gamma=\Gamma_-$ $\sigma\le 1$ stable $1<\sigma<2$ stable $\mu>\mu_{th}$ $\sigma\ge 4$ unstable	Type-II Competing ($\alpha>0$ $\gamma<0$) $\Gamma=\Gamma_+$ Stable for: $\mu\le\mu_{cr}=\frac{\alpha\sigma}{2+\sigma}\sqrt{\frac{2(1+\sigma)}{\lvert\gamma\rvert}}$

These results are almost identical to those of the gNLS equation. In parallel with the conventional linear stability criterion for waves of NLS-type models [5], the Helmholtz wave is proposed to be linearly stable if,

$$\frac{dW(\mu)}{d\mu}>0$$

where $W(\mu)=\int_{-\infty}^{\infty}d\xi\left[|u|^2-i\kappa\left(u^*\frac{\partial u}{\partial\zeta}-u\frac{\partial u^*}{\partial\zeta}\right)\right]$ is the energy-flow invariant of the gNNLS equation [7].

For example, see figure 1. In the pure-focusing regime, with $\alpha=1$, $\gamma=1$, these curves allow the identification of a critical μ value for each σ. Above the critical value there are no stable solutions, and the launched beams diffract. For $\sigma>4$ *(not shown)*, there are no stable solitons.

Although there exists no proof of this hypothesis, its quantitative predictions match those found by solving the gNNLS equation numerically.

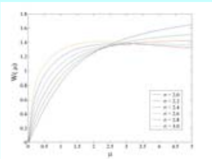

Figure 1. Energy-flow curves of gNNLS solitons.

Non-Linear Beam Reshaping

The numerical perturbative approach uses exact solitons of the gNLS [5] as initial conditions: $u(\xi,0)=\left[\frac{\eta}{\cosh(\mu\xi)+\Gamma_{\pm}}\right]^{\frac{1}{\sigma}}\exp(-iS_0\xi)$. This yields insight into the dynamical properties of the novel gNNLS solitons. In all simulations presented here, $|\alpha|=|\gamma|=1$, with $\kappa=10^{-3}$, $\mu/\sigma=O(1)$ *and* $\kappa(\mu/\sigma)^2\ll 1$.

$S_0=5,10,15$ corresponds to the non-trivial propagation angles of $\theta=12.9°$, $26.6°$ and $42.1°$, respectively, for the value $\kappa=10^{-3}$.

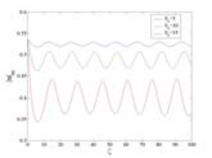

Figure 2. Pure Focusing ($\mu=1$, $\sigma=1$)

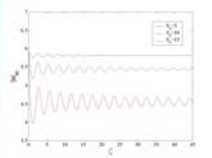

Figure 3. Type-I Competing ($\mu=1$, $\sigma=0.5$)

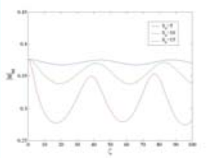

Figure 4. Type-II Competing ($\mu=0.6$, $\sigma=1$)

- **Pure Focusing:** In the middle of the *unconditionally stable* region, a Helmholtz soliton does not emerge from the input paraxial beam. Instead, long-lived self-persistent oscillations result and this is associated with *limit cycle*-type behaviour (figure 2). Paraxial beams launched with μ in the *conditionally stable* regime also display similar characteristics in their long term dynamics.
- **Type-I Competing** In the *unconditionally stable* regime the beam undergoes a slow reshaping effect toward an asymptotic soliton state with a well-defined amplitude (figure 3). The rate at which the steady-state is approached depends on S_0, that is, on the strength of the perturbation. At threshold ($\sigma=1$) and beyond, weakly perturbed beams can give rise to Helmholtz soliton formation; increasing the strength of the perturbation leads to diffractive spreading and collapse (no soliton state).
- **Type-II Competing** Close to threshold (in figure 4, $\mu_{th}=0.666$) the dynamical evolution resembles that in the pure-focusing regime. The paraxial input beam undergoes long-lived self-persistent oscillation, characteristic of *limit cycle*-type behaviour. A immediately noticeable difference is the much lower oscillation frequency in this parametric regime. Far away from threshold, where the beam is more stable, it may be expected that the transient oscillations vanish and an asymptotic Helmholtz soliton emerges. In fact, this is not the case and the oscillations still occur.

Helmholtz Algebraic Solitons

A weakly-localized non-linear wave of the gNNLS equation is the Helmholtz algebraic soliton. This solution has much slower *power-law* asymptotics (Lorentzian) as opposted to the strongly-localized (exponential) sech-type solutions presented earlier. The algebraic soliton is supported only in the *Type-I competing* regime where, in the limit $\mu\to 0$, there remains a *non-zero energy-flow* $W(\mu\to 0)>0$.

$$u_a(\xi,\zeta)=\left[\frac{(1+\sigma)(2+\sigma)/|\alpha|}{\sigma^2(1+\sigma)\frac{(\xi+V\zeta)^2}{1+2\kappa V^2}+(2+\sigma)^2\frac{\gamma}{2\alpha^2}}\right]^{\frac{1}{\sigma}}\exp\left[i\frac{1}{\sqrt{1+2\kappa V^2}}\left(-V\xi+\frac{\zeta}{2\kappa}\right)\right]\exp\left[-i\frac{\zeta}{2\kappa}\right].$$

The Lorentzian asymptotics are seen from:

$$\lim_{\xi\to\pm\infty}|u_a(\xi,\zeta)|\to|\xi|^{-\frac{2}{\sigma}}.$$

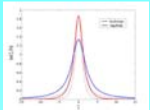

Figure 5. ***Sech-type and Algebraic Soliton Profiles.*** Parameters: $\kappa=10^{-3}$ and $V=25$, so $\theta\approx 48.2°$. Here, $\sigma=0.9$ and $\mu=1$ for the *sech*-type soliton. The non-linear medium is necessarily type-1 competing. Dotted lines correspond to paraxial solutions (no broadening).

Algebraic solitary waves are a common feature of non-linear systems, occurring, for example, in fluid mechanics. It could be predicted *a-priori* that paraxial wave optics, governed by NLS-type equations, must also support algebraic solitons from the *fluid mechanics–nonlinear optics* analogue. Such waves do exist and have been reported by several authors, such as in [6]. However, this is the first known reporting of algebraic solitons in 2nd-order non-paraxial systems.

Stability of Helmholtz Algebraic Solitons

The reshaping of an exact paraxial algebraic soliton launched into the Helmholtz regime is now examined. The initial conditions are taken as: $u_a(\xi,0)=\left[\frac{(1+\sigma)(2+\sigma)/|\alpha|}{\sigma^2(1+\sigma)\xi^2+(2+\sigma)^2\frac{\gamma}{2\alpha^2}}\right]^{\frac{1}{\sigma}}\exp(-iS_0\xi)$.

Two cases of particular interest are where the *sech*-type solutions are unconditionally ($\sigma<1$) and conditionally ($1<\sigma<2$) stable. In the first case (figure 6) where $\sigma=0.5$, a Helmholtz soliton is seen to emerge quite quickly. But as the threshold $\sigma=1$ is approached, such stability is lost and the beam undergoes diffractive spreading (figure 7). By examining its phase in the propagation direction (figure 8), it is found that,

$$\Omega(\zeta)\equiv\frac{d\phi(\zeta)}{d\zeta}<0,$$

where $u(\xi,\zeta)=|u(\xi,\zeta)|\exp[i\phi(\zeta)]$. In the paraxial theory, this is leads to diffractive spreading.

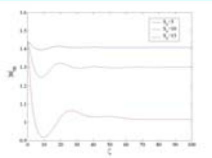

Figure 6. For $\sigma=0.5$ (well below threshold) a Helmholtz algebraic soliton is formed.

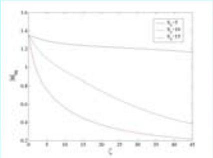

Figure 7. For $\sigma=0.9$ (close to threshold) no Helmholtz algebraic soliton is formed. The Initially-localized beam undergoes diffraction.

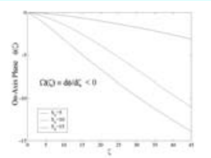

Figure 8. Phase of the field for $\sigma=0.9$ (close to threshold) in figure 7. The slope of the phase is negative, and this leads to diffractive spreading (soliton formation suppressed).

Conclusions

- The gNNLS equation, describing Helmholtz wave propagation in a medium with a polynomial-type non-linear refractive index, allows exact analytical soliton solutions. These solutions may be either strongly- (with exponential tails) or weakly- (Lorentzian profile) localized.
- Solutions to the gNNLS equation are of intrinsic mathematical interest. They represent a *novel contribution to the knowledge of soliton dynamics in 2nd-order non-integrable models*. Algebraic solitons in such systems have also been identified.
- On the experimental side, the gNNLS equation pertains directly to known material media, such some semiconductor-doped glasses [2] and non-linear polymers [3,4].
- Numerical perturbative techniques have been used to study the dynamical properties of the gNNLS solitons, and a new stability criterion proposed.
- The novel gNNLS solitons presented here have been to shown to behave predominantly as **LIMIT CYCLE ATTRACTORS** of the system, with self-persistent oscillations occurring in the long-term dynamics. In some cases (e.g. Type-I competing non-linearity far from threshold) they can act as **STABLE FIXED POINTS**, but this behaviour is highly sensitive to both material parameters and the inverse beam width μ.

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