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Optical pulses with spatial dispersion – solitons & relativity

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The slowly varying envelope approximation (SVEA) and the ensuing Galilean boost to a local time frame are near-universal features of conventional scalar pulse models. They have enjoyed unbridled longevity in the literature over the past four decades for two principal reasons. Firstly, they often provide an adequate description of the phenomena under consideration; secondly, a large body of knowledge now exists on how to solve the resultant parabolic governing equations. Here, we consider the implications of relaxing both the SVEA and the Galilean boost. The result is a generalized Helmholtz-type equation that, somewhat surprisingly, can be analyzed and solved exactly for bright and dark solitons.

1. Introduction

It can be safely said that optical soliton pulses are one of the most thoroughly investigated and well-understood phenomena in nonlinear photonics. Since the seminal works of Hasegawa and Tappert [1], and later the experiments of Mollenauer *et al.* [2], the cornerstone of many investigations has been the slowly-varying envelope approximation (SVEA). The SVEA, in combination with a subsequent Galilean boost to a local time frame, tends to reduce the complexity of the longitudinal (spatial) part of the wave operator, with temporal effects left unchanged [3]. While this approach has advantages, there are some physical effects that fall outside its remit. One such effect is spatial dispersion [4], recently introduced in the context of some semiconductor waveguides.

2. Helmholtz Pulse Model

We begin by considering a scalar electric field $E(t,z) = A(t,z)\exp[i(k_0 z - \omega_0 t)]$ + c.c. that is travelling down the longitudinal axis z of a waveguide, and where t denotes time in the laboratory. Here, $A(t,z)$ is the envelope modulating a carrier wave with optical frequency ω_0 and propagation constant $k_0 = n_0\omega_0/c$, where n_0 is the linear refractive index of the core medium at ω_0 and c is the vacuum speed of light. By substituting $E(t,z)$ into the corresponding Maxwell equations and Fourier transforming to the temporal frequency domain, it can be shown that [3]

$$\left(\frac{\partial^2}{\partial z^2} + i2k_0\frac{\partial}{\partial z}\right)a(\omega,z) + \left(k^2 - k_0^2\right)a(\omega,z) = 0 \qquad \text{...(1)}$$

where $a(\omega,z)$ denotes the Fourier transform of the pulse envelope. At this juncture, one should recognize that **the double-z operator $\partial^2/\partial z^2$ appears naturally in the governing equation**; it is this term that is routinely neglected in analyses of pulse propagation phenomena [1,2,5] by assuming that $|\partial^2 a/\partial z^2| << |2k_0\partial a/\partial z|$. The parameter k^2 is the mode eigenvalue (obtained by solving Maxwell equations for the transverse part of the confined field). The factor $(k^2 - k_0^2)$ is often approximated by $2k_0(k - k_0)$, and the remaining linear term $k \equiv k(\omega)$ is Taylor-expanded around ω_0 according to $k(\omega) \approx k_1(\omega - \omega_0) + (k_2/2)(\omega - \omega_0)^2 + \Delta k_{NL}$, where $k_0 \equiv k(\omega_0)$, and $k_j \equiv (\partial^j k/\partial\omega^j)_{\omega 0}$ for $j = 1, 2$ [these two expansion coefficients parameterize the group velocity and group-velocity dispersion, respectively]. The nonlinear correction is $\Delta k_{NL} = n_2 I\omega_0/c$, where n_2 is the Kerr coefficient and I is the light intensity. After inverse-Fourier transforming, one finds

$$\frac{1}{2k_0}\frac{\partial^2 A}{\partial z^2} + i\left(\frac{\partial A}{\partial z} + k_1\frac{\partial A}{\partial t}\right) - \frac{k_2}{2}\frac{\partial^2 A}{\partial t^2} + \gamma|A|^2 A = 0,$$

where the coefficient of the nonlinear term is $\gamma = n_2/2n_0$.

Recently, it has been shown for the first time that spatial dispersion (by way of field-exciton coupling) in some semiconductor materials (such as ZnCdSe/ZnSe superlattices) can provide a second contribution to the coefficient of $\partial^2 A/\partial z^2$ [4]. Here, we take the two contributions to be additive, arriving at a new lumped coefficient:

$$\frac{1}{2k_0} + \frac{n_0\Gamma\Delta\varpi_0}{2\delta\omega^2 c},$$

Propagation contribution (inherent to any electromagnetic mode) ← $\frac{1}{2k_0}$; $\frac{n_0\Gamma\Delta\varpi_0}{2\delta\omega^2 c}$ → Material contribution (spatial dispersion from field-exciton coupling)

where $\Gamma \equiv \hbar/2M_x^*$, M_x^* is the effective exciton mass, ϖ_0 is a resonant frequency, Δ is a dimensionless parameter related to the oscillator strength for the coherent exciton-photon interaction, and $\delta\omega$ is a frequency detuning. A salient point is that **the coefficient of $\partial^2 A/\partial z^2$ can, in principle, become negative when $M_x^* < 0$** [4]. After rescaling, the following governing equation for the dimensionless envelope u may be derived:

$$\kappa\frac{\partial^2 u}{\partial\zeta^2} + i\left(\frac{\partial u}{\partial\zeta} + \alpha\frac{\partial u}{\partial\tau}\right) + \frac{s}{2}\frac{\partial^2 u}{\partial\tau^2} + |u|^2 u = 0 \qquad \text{...(2)}$$

The normalized space and time coordinates are $\zeta = z/L$ and $\tau = t/t_p$, respectively, where t_p is the duration of a reference pulse and $L = t_p^2/|k_2|$. The sign of the group velocity dispersion is flagged by $s = \pm 1 = -\text{sgn}(k_2)$ (+1 for anomalous; –1 for normal), $\alpha \equiv k_1 t_p/|k_2|$ is a material parameter, and $\kappa = \kappa_0 + D$, where $\kappa_0 \equiv 1/2k_0 L = c|k_2|/2n_0\omega_0 t_p^2$ and $D \equiv n_0\Gamma\Delta\omega_0/2\delta\omega^2 cL = |k_2|n_0\Gamma\Delta\omega_0/2\delta\omega^2 c t_p^2$. Finally, $u = A/A_0$, where $A_0 = (\gamma L)^{-1/2} = (2n_0|k_2|/n_2 t_p^2)^{1/2}$.

3. Local-Time vs. Laboratory Frames

When considering problems involving pulse propagation, one typically follows a prescribed route to get from the more general nonlinear-Helmholtz governing equation to the more straightforward nonlinear-Schrödinger model. Firstly, one typically invokes the SVEA by arguing that the term in $\kappa\partial^2 u/\partial\zeta^2$ is small. A Galilean boost to a frame moving at the group speed $1/\alpha$ along the $+z$ axis is then implemented by defining coordinates $(\tau_{loc}, \zeta_{loc}) = (\tau - \alpha\zeta, \zeta)$ so that in this local frame, u satisfies the familiar canonical equation $[i\partial/\partial_{loc} + (s/2)\partial^2/\partial\tau_{loc}^2 + |u|^2]u = 0$ [5].

The natural question to ask is, **"what happens if one keeps the $\kappa\partial^2 u/\partial\zeta^2$ term when implementing the Galilean boost?"** In that case, the governing equation takes on a cross-derivative operator which can hinder a straightforward physical interpretation:

$$\kappa\frac{\partial^2 u}{\partial\zeta_{loc}^2} + i\frac{\partial u}{\partial\zeta_{loc}} + \frac{1}{2}\left(s + 2\kappa\alpha^2\right)\frac{\partial^2 u}{\partial\tau_{loc}^2} - 2\kappa\alpha\frac{\partial^2 u}{\partial\zeta_{loc}\partial\tau_{loc}} + |u|^2 u = 0.$$

To proceed, one might, for instance, consider only those families of solutions where $\kappa\alpha^2 << O(1)$, which enables the coefficient of the temporal dispersion term to be simply $s/2$. One could also present a case, based on order-of-magnitude considerations, for omitting the cross-derivative term. In so doing, one ends up with the approximate model of Biancalana and Creatore [4], which is a temporal analogue of the spatial Helmholtz equation [6].

The nub of the problem is that if one wishes to keep the $\kappa\partial^2 u/\partial\zeta^2$ term, then the Galilean boost results in a local governing equation that is actually *more* complicated than the original equation! So at the outset, the conventional coordinate transformation serves no useful purpose. This leaves one with a fairly stark choice. One could either use approximate models, *or* **one can abandon the near-universal Galilean transformation and remain in the laboratory frame**. In choosing the latter option – which distinguishes our work from other analyses – we have been able to make very encouraging progress with the theory of **Helmholtz soliton pulses**.

4. Velocity Combination Rule

When investigating the properties of Eq. (2) and its solutions under transformations in space-time, it is convenient to adopt the notation routinely deployed for spatial solitons [6]. Under the coordinate change $\tau = (\tau' - V\zeta')/(1 + 2s\kappa V^2)^{1/2}$ and $\zeta = (2s\kappa V\tau' + V\zeta')/(1 + 2s\kappa V^2)^{1/2}$, the covariance of Eq. (2) is guaranteed so long as u transforms as

$$u(\tau,\zeta) = \exp\left[-i\frac{sV\tau'}{\sqrt{1+2s\kappa V^2}} + \frac{i}{2\kappa}\left(1 - \frac{1}{\sqrt{1+2s\kappa V^2}}\right)\zeta'\right]\exp\left[-is\alpha\frac{\tau' - V\zeta'}{\sqrt{1+2s\kappa V^2}} + is\alpha\tau'\right]u'(\tau',\zeta').$$

Here, V plays a role analogous to the transverse velocity of optical beams (see Fig. 1). The velocity combination rule for two velocities V and V_0 is then

$$W = \frac{V_0 + V}{1 - 2s\kappa VV_0}.$$

→ strongly reminiscent of relativistic kinematics! (though W, V and V_0 are, strictly, *inverse* velocities)

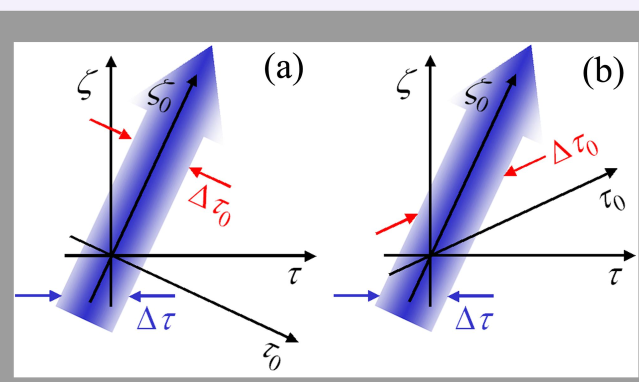


FIG. 1. The space-time geometry of Helmholtz soliton pulses depends upon $\text{sgn}(s\kappa)$. The coordinates (τ_0,ζ_0) denote the rest frame of the pulse. One can identify relativistic-type effects, but where ζ (the evolution variable) plays the role of "time", while τ is equivalent to "space." When $\text{sgn}(s\kappa) = +1$ [(a)], transformations in the (τ,ζ) plane correspond to *rotations* [6]. This leads to a "time dilation" effect (which can be interpreted as "length dilation", an effect not found in relativity theory). However, when $\text{sgn}(s\kappa) = -1$ [(b)], transformations correspond to *skews* (c.f. Lorentz transformation). This leads to a "time contraction" effect (which can be regarded as analogous to "length contraction" in relativity theory).

5. Exact Bright & Dark Solitons

Exact analytical bright solitons of Eq. (2) exist when $s = +1$; they can be written as

$$u(\tau,\zeta) = \eta\,\text{sech}\left(\eta\frac{\tau \mp W\zeta}{\sqrt{1+2\kappa W^2}}\right)\exp\left[i\Omega\tau \pm i\sqrt{1+4\kappa\beta - 4\kappa\Omega\left(\alpha + \tfrac{1}{2}\Omega\right)}\frac{\zeta}{2\kappa}\right]\exp\left(-i\frac{\zeta}{2\kappa}\right),$$

where η is the peak amplitude, $\beta \equiv \eta^2/2$ characterizes the nonlinear phase shift, and Ω represents a (normalized) measure of the frequency deviation of the pulse envelope from the carrier frequency ω_0. The net velocity W takes on a compact, solution-specific form $W = (\alpha + \Omega)/[1 + 4\kappa\beta - 4\kappa\Omega(\alpha + \Omega/2)]^{1/2}$. Note that bright solitons have an intrinsic velocity $V_0 = W(\Omega = 0) = \alpha/(1 + 4\kappa\beta)^{1/2}$ because they are moving with respect to an observer in the laboratory frame. Dark solitons exist when $s = -1$. They comprise a "dip" on top of a (modulationally stable) continuous wave solution with (real) amplitude u_0,

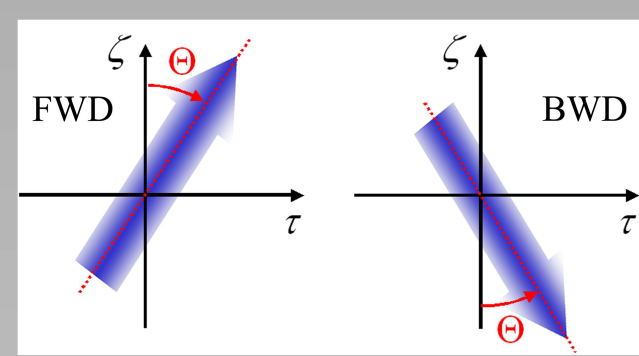
$$u(\tau,\zeta) = u_0\left[A\tanh\left(u_0 A\frac{\tau \mp W\zeta}{\sqrt{1-2\kappa W^2}}\right) + iF\right]\exp\left[-i\Omega\tau \pm i\sqrt{1+4\kappa u_0^2 + 4\kappa\Omega\left(\alpha + \tfrac{1}{2}\Omega\right)}\frac{\zeta}{2\kappa}\right]\exp\left(-i\frac{\zeta}{2\kappa}\right),$$

where F is the traditional greyness parameter, and $A^2 + F^2 = 1$. The intrinsic velocity V_0 of the grey dip, and the traditional transverse velocity $V(\Omega)$, are given by

$$V_0 = \frac{u_0 F\sqrt{1 + 2\kappa u_0^2\left(2 + F^2\right) - 2\kappa\alpha^2} + \alpha\sqrt{1 + 4\kappa u_0^2}}{1 + 2\kappa u_0^2\left(2 + F^2\right)} \quad \text{and} \quad V = \frac{(\Omega + \alpha)\sqrt{1 + 4\kappa u_0^2 + 4\kappa\Omega\left(\alpha + \tfrac{1}{2}\Omega\right)} - \alpha\sqrt{1 + 4\kappa u_0^2}}{1 + 4\kappa u_0^2 + 2\kappa(\Omega + \alpha)^2}.$$

The upper (lower) signs correspond to pulses that are travelling in the forward (backward) longitudinal direction (see Fig. 2). Physically meaningful solutions must have $W > 0$, thus ensuring that the pulse is always moving **forwards in time** (irrespective of its evolution in z).

FIG. 2. Schematic diagram illustrating forward (FWD) and backward (BWD) Helmholtz pulses in the laboratory frame. For physically meaningful solutions, the FWD (BWD) soliton [upper (lower) signs in the solutions] must always span the 1st and 3rd (2nd and 4th) quadrants of the (τ, ζ) plane. This condition, which is captured by $W > 0$, ensures that the centre of the pulse is always moving forwards in time. Straight (dotted) lines show trajectories $\tau \mp W\zeta = 0$. Physically meaningful solutions require $-\alpha < \Omega < \Omega_{max}$. However, the temporal dispersion description [i.e., $k^2 - k_0^2 \approx 2k_0(k - k_0)$] can become invalid for sufficiently large frequency deviations (for instance, where $\Omega \rightarrow \Omega_{max}$).



In a simultaneous multiple limit, the predictions of conventional pulse theory must be recovered from the full Helmholtz model. All contributions from the $\kappa\partial_{\zeta\zeta}$ operator in Eq. (2) can be neglected when $\kappa \rightarrow 0$, $\kappa\eta^2 \rightarrow 0$, $\kappa u_0^2 \rightarrow 0$, $\kappa\Omega \rightarrow 0$ and $\kappa W^2 \rightarrow 0$. When applied to the forward bright pulse, this simultaneous four-fold limit leads to the approximate solution

$$u(\tau,\zeta) \simeq \eta\,\text{sech}\left\{\eta\left[(\tau - \alpha\zeta) - \Omega\zeta\right]\right\}\exp\left[i\Omega(\tau - \alpha\zeta) + i\left(\beta - \frac{\Omega^2}{2}\right)\zeta\right],$$

which corresponds to an exact soliton of the conventional model [i.e., Eq. (2) with $\kappa\partial^2 u/\partial\zeta^2$ is neglected]. Similarly, when applied to the forward dark pulse, one recovers

$$u(\tau,\zeta) = u_0\left\{A\tanh\left[\Theta(\tau,\zeta)\right] + iF\right\}\exp\left[-i\Omega(\tau - \alpha\zeta) + i\left(u_0^2 + \frac{\Omega^2}{2}\right)\zeta\right],$$

where $\Theta(\tau,\zeta) = u_0 A[(\tau - \alpha\zeta) - (\Omega + u_0 F)\zeta]$. Of note is the much simpler dependences of the velocities, V_0 and V, on the frequency shift Ω, namely $V_0 \approx u_0 F + \alpha$ and $V \approx \Omega$. In the local frame $(\tau_{loc}, \zeta_{loc}) = (\tau - \alpha\zeta, \zeta)$, these solutions become the classic solutions of conventional pulse theory. When the same multiple limit is applied to the backward Helmholtz solitons, a rapid phase term of the form $\exp[-i2(\zeta/2\kappa)]$ survives. This compounds the fact that the conventional (parabolic) model has no counterpart to the Helmholtz backward pulses.

6. Soliton Stability

The exact analytical solitons discussed above are stationary states of model (2). However, one must now establish how stable these new solutions are against perturbations to their local shape. To this end, we inject pulses $u(\tau,0) = \eta\,\text{sech}(\eta\tau)\exp(i\Omega\tau)$ and $u(\tau,0) = u_0[A\tanh(u_0 A\tau) + iF]\exp(-i\Omega\tau)$ into the waveguide. These initial data correspond to exact solutions of the conventional model. Simulations [7] show that the pulses self-reshape (and shed a small amount of radiation in the process), evolving toward stationary solutions as $\zeta \rightarrow \infty$. This indicates that Helmholtz soliton pulses can be interpreted as robust fixed-point attractors of the system (see Fig. 3). By choosing the appropriate frame of reference (e.g., the rest frame of the injected pulse), one can deploy inverse-scattering techniques [8] to predict the asymptotic soliton parameters.

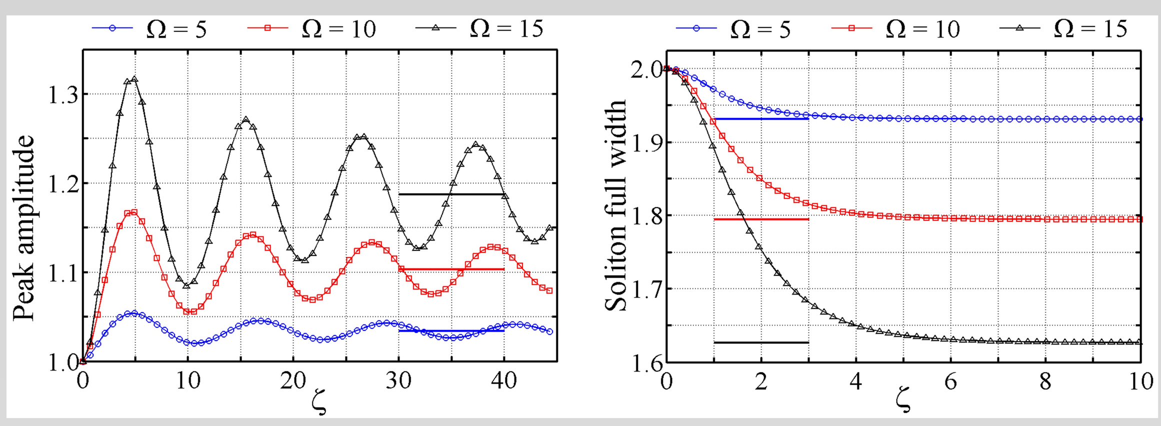


FIG. 3. Self-reshaping of perturbed Helmholtz solitons toward stationary states. *Left*: a bright pulse with $\eta = 1.0$ and $\kappa = -10^{-3}$. *Right*: a dark (black, $F = 0$) pulse with $u_0 = 1.0$ and $\kappa = +10^{-3}$ (grey pulses, where $|F| > 0$ exhibit qualitatively similar self-reshaping characteristics). The horizontal bars denote predictions of inverse-scattering theory [8] regarding the asymptotic soliton parameters.

7. Conclusions

We have taken the first steps toward understanding nonlinear optical pulses from a new perspective by studying their behaviour from the laboratory frame. In the course of our work, we have found that this frame is the natural frame from which to describe pulses. Furthermore, the internal inconsistencies introduced by the classic Galilean boost can be quite subtle. **We have discovered, what we believe to be, a compact and elegant framework for describing optical pulses**. The framework is exact [in the sense that no further approximation beyond Eq. (2) is required] and self-consistent. A clever choice of reference frame allows one to use inverse-scattering theory to predict the asymptotic parameters of perturbed solitons. We believe that the Helmholtz pulse modelling approach may also find application in other nonlinear pulse contexts, such as fluid dynamics and plasma physics.

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