## Module M1.4 Solving equations

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## 1 Opening items

### 1.1 Module introduction

When we do anything in physics, there comes a point when we have to produce numbers; either as the result of an experimental measurement, or as the prediction of a theory, or both. Almost always this means having to do some calculations; not just adding numbers, or multiplying them, but manipulating them in some way so as to produce an answer; often this involves solving equations. In this module we will look at how to solve the simplest (and commonest) equations, and outline some of the more generally applicable ways in which we can solve more difficult problems.

Initially, we will look at solving linear equations (in Section 2), i.e. equations in which the independent variable appears only as the first power. In Section 3 the discussion is extended to the solution of quadratic equations by various methods, e.g. factorizing, completing the square and using the quadratic equation formula. Finally, in Section 4 graphical and numerical procedures for solving polynomial equations are described.

The first thing you must do before starting this module is to PUT YOUR CALCULATOR AWAY. You will not need it for most of this module, and you will be told when you do need it. Calculators are very useful for doing calculations to great accuracy, but they can stop you thinking about the important features of calculations; they are often a good means of making mistakes quickly!

Study comment Having read the introduction you may feel that you are already familiar with the material covered by this module and that you do not need to study it. If so, try the Fast track questions given in Subsection 1.2. If not, proceed directly to Ready to study? in Subsection 1.3.

### 1.2 Fast track questions

Study comment Can you answer the following Fast track questions? The answers are given in Section 6. If you answer the questions successfully you need only glance through the module before looking at the Module summary (Subsection 5.1) and the Achievements listed in Subsection 5.2. If you are sure that you can meet each of these achievements, try the Exit test in Subsection 5.3. If you have difficulty with only one or two of the questions you should follow the guidance given in the answers and read the relevant parts of the module. However, if you have difficulty with more than two of the Exit questions you are strongly advised to study the whole module.

## Question F1

Find the value of $x$ that satisfies the following equation:

$$
\frac{x-1}{x+1}=2
$$

2

## Question F2

Solve the following equations:

$$
\begin{aligned}
& x+2 y=4 \\
& 2 x-y=3
\end{aligned}
$$

## Question F3

Find the roots of the following equation

$$
x^{2}+x-1=0
$$

leaving the square roots in the answer. How can you verify that your answers are right?

Study comment Having seen the Fast track questions you may feel that it would be wiser to follow the normal route through the module and to proceed directly to Ready to study? in Subsection 1.3.

Alternatively, you may still be sufficiently comfortable with the material covered by the module to proceed directly to the Closing items.

### 1.3 Ready to study?

Study comment In order to study this module you will need to understand the following terms: constant, equation, function, graph, powers, square root and variable. If you are uncertain about any of these terms you should review them now by referring to the Glossary, which will indicate where in FLAP they are developed. In addition, you will need to be able to expand, simplify and evaluate simple algebraic expressions that involve brackets. You will also need to be able to sketch and plot the graphs of simple functions. If you are uncertain about your ability to perform these operations, you should again refer to the Glossary for further information. The following Ready to study questions will help you to check that you have the required skills and knowledge before embarking on this module.

## Question R1

Expand the following expressions:
(a) $(a-2 b)(a+2 b)$
(b) $(x+3)(x-4)$
(c) $(p-3)^{2}$
$\triangle$

## Question R2

Simplify the following expression:

$$
2\left(x^{2}+3 x-2\right)-\left(2 x^{2}-6 x-4\right)
$$

## Question R3

If $f(x)=3 x+2$, write down expressions for $f(2 y)$ and $f(y-1)$.

## Question R4

What are the square roots of $x^{4}$ ?


## Question R5

Sketch $\underline{\underline{40} \text { gro }}$ graphs of the following functions from $x=-1$ to $x=3$.

$$
f(x)=2 x-1 \quad \text { and } \quad g(x)=x^{2}-3 x+2
$$

## Question R6

Rewrite, with a single denominator, $\xrightarrow{18 y}$ the expression

$$
\frac{1}{1-x}-\frac{1}{1+x}
$$

## 2 Solving linear equations

When we attempt to solve an equation for an unknown variable, $x$ say, we aim to find an expression which looks something like this

$$
\begin{equation*}
x=? ? ? \tag{1}
\end{equation*}
$$

where the right-hand side does not involve $x$ and is something we can calculate immediately. When we have reached this point the mathematics is done and arithmetic takes over.

In this section we shall look at linear equations, which only involve the first power of the unknown variable, $x$, rather than $x^{2}$ or any higher power. In other words we will look at equations such as

$$
3 x+2=5 x-2
$$

which are said to be in linear form. We shall attempt to find the value (or values) of the unknown variable for which the equation is true. In the case of the above equation it is easy to see that the value $x=2$ makes the equation true since for this value the left-hand side becomes $3 \times 2+2=8$, while the right-hand side is $5 \times 2-2=8$. We call this process solving the equation, and we speak of finding the solution, or finding the root of the equation. Later we will extend these notions to more than one equation. Equations that can be solved are sometimes said to be soluble, while those that can't are insoluble.

### 2.1 Simple linear equations; solving by rearrangement

Suppose you set your stop-watch to zero as you start cycling along a road at 10 mph . After you have travelled 15 miles you stop for a rest. How long has this taken you?

The distance you have travelled after a period of time $t$ is

$$
d=\mathrm{V} t \quad \underline{\boxed{12} 8}
$$

where V is your speed. In this example we know what $d$ and V are, and we have to find $t$. Dividing both sides of the above equation by V we see that

$$
t=\frac{d}{\mathrm{~V}}=\left(\frac{15}{10}\right) \text { hours }
$$

Thus we have reached the point at which arithmetic takes over and the final answer is 1.5 hours. This simple problem was solved by rearranging the equation.
Now for a slightly more complicated problem. Forget for the moment that $d=15$ miles and $\mathrm{V}=10 \mathrm{mph}$, and think about the following problem in purely algebraic terms. Suppose that having cycled a distance $d$ at speed V you then rested for a time $t_{\mathrm{R}}$, and that following your rest you set out again on a second ride at speed V . If you kept cycling until your total journey time (including the rest) was $T$, what distance would you have covered in that time?

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The time spent on the second ride is found by subtracting the rest time and the time of the first ride from the total journey time, so it is ( $T-d / \mathrm{v}-t_{\mathrm{R}}$ ); and the total distance $D$ covered in both rides is just the sum of the distances covered in each ride, so

$$
\begin{equation*}
D=d+\mathrm{V}\left(T-d / \mathrm{v}-t_{\mathrm{R}}\right) \tag{2}
\end{equation*}
$$

It is easy to rearrange this equation to find an expression for $T$ in terms of the total distance travelled, the speed, the distance $d$ and the rest time. The form of this expression may already be obvious to you, but being systematic and spelling everything out in detail we can start by trying to isolate the bracket that contains $T$. To do this note that from Equation 2

$$
D-d=\mathrm{V}\left(T-d / \mathrm{v}-t_{\mathrm{R}}\right)
$$

so that $\left(T-d / v-t_{\mathrm{R}}\right)=(D-d) / \mathrm{v}$
Thus, $\quad T=d / \mathrm{v}+t_{\mathrm{R}}+(D-d) / \mathrm{v}=d / \mathrm{v}+t_{\mathrm{R}}+D / \mathrm{v}-d / \mathrm{v}$
You can see that $d / \mathrm{v}$ cancels out on the right-hand side of the last equation, so

$$
T=t_{\mathrm{R}}+D / \mathrm{v}
$$

(3)

Now we can substitute in some numbers; suppose the total distance travelled is 35 miles and $\mathrm{V}=15 \mathrm{mph}$; if the rest time was half an hour then $t_{\mathrm{R}}=0.5$ hours and

$$
T=\left(\frac{1}{2}+\frac{35}{10}\right) \text { hours }=(0.5+3.5) \text { hours }=4 \text { hours }
$$

You may be wondering why we didn't put these numbers in at the beginning. There are three reasons for this; first, if you look again at Equation 3 you will see that the variable $d$ has disappeared,

$$
T=t_{\mathrm{R}}+D / v
$$

so less calculation is needed than if we had used numbers all the way through. Second, having worked out the general equation, we can change the distances, or the speed, and still find the time taken without having to go through the whole calculation again. Finally, it is often easier to check a calculation that involves algebraic variables rather than numerical values.

- Suppose you cycled non-stop for 36 miles at 12 mph . How long would this have taken?

This example provides us with a rule that applies throughout mathematics and physics. To avoid unnecessary effort and to reduce the chance of mistakes:

Don't introduce numbers until it is essential; leave all the arithmetic until the end.

Let us now return to the business of solving equations. The following method applies to any equation in which the unknown variable appears to the first power only, and only in the numerator $\stackrel{[188}{ }$. The procedure in general is much like the one described above; for brevity let us suppose the unknown variable is $x$.

To solve a linear equation, e.g. $3(x-1)=2 x+5(2-x)$
(a) if $x$ appears in more than one bracket, expand the brackets;

$$
3 x-3=2 x+10-5 x
$$

(b) collect together all the terms involving $x$ on one side of the equation and the remaining terms on the other;

$$
3 x+5 x-2 x=10+3
$$

(c) combine the coefficients of $x$ as far as possible (i.e. simplify all the terms multiplying $x$ ), and then simplify all the remaining terms;

$$
6 x=13
$$

(d) you should now have an equation that can be written in the form of Equation 1 with no more than one more step.

$$
x=\frac{13}{6}
$$

With practice it is possible to condense this process by combining the steps, but be warned, doing too many steps in your head is likely to lead to mistakes $\underline{\text { 뗭 }}$. Try the next four questions, all of which can be solved by the above method.

## Question T1

Solve the equation $2(x+1)=x-5$.

## Question T2

Solve the equation

$$
(z-2)+3(2 z+1)-(5 z-7)=0
$$

## Question T3

If $a=1$ and $b=4$, find the value of $p$ that satisfies

$$
p-a=3 p+2 b
$$

## Question T4

If $a=4$ and $b=-2$, find the value of $p$ that satisfies

$$
p-a=3 p+2 b
$$

It is sometimes possible to solve what appear to be more complicated problems in the way described above. The general principle is to reduce more complicated equations to a form that we know how to solve. For example,

$$
\begin{equation*}
\frac{x-1}{x+2}=3 \tag{4}
\end{equation*}
$$

can be rewritten, if both sides are multiplied by $(x+2)$, as

$$
x-1=3(x+2)
$$

and then treated as before. However, it is not always possible to reduce such an equation to linear form.
$\rightarrow$ Try to simplify the following:

$$
\frac{x-1}{x+2}=\frac{3}{x}
$$

There are many equations that appear at first sight to be beyond the realm of linear equations, but which can be simplified to an unexpected degree. Look, for example, at the following

$$
\frac{x-1}{x+2}=\frac{x+3}{x-1}
$$

If we multiply both sides of the equation by $(x+2)(x-1)$ we have

$$
(x-1)^{2}=(x+3)(x+2)
$$

which implies

$$
x^{2}-2 x+1=x^{2}+5 x+6
$$

i.e. $7 x=-5$
so $\quad x=-5 / 7$
However, events like this should be regarded as just a lucky accident, and are not to be relied on - though it is legitimate to hope!

## Question 15

Which of the following equations can be reduced to a linear form? Solve those which can.
(a) $\frac{1}{x-1}=6$
(b) $x^{2}-1=\frac{2}{x+1}$
(c) $x^{2}+2=x(x-3)$
(d) $\frac{(x+2)(x-2)}{x^{2}-4}=x$

In real life it is not always obvious (as we can see from what we have just discussed) which equations can be reduced to the form of Equation 1. There is no general method which will allow us to take an arbitrary equation and say 'yes, this can be solved like this', and then write down the answer. Some equations can be solved easily, some with more difficulty, and some are impossible to solve. One of the advantages that comes from practice is the ability to recognize the category to which a particular equation belongs, so that you don't waste time trying to solve something insoluble, and you can go straight to the right method for something soluble.

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### 2.2 Equations linear in a function of $\boldsymbol{x}$

There is a slight and fairly obvious generalization of the linear equation that arises if the variable is, say $x^{2}$, rather than $x$. For example,

$$
x^{2}+4=2 x^{2}-12
$$

can be solved for the unknown variable $x^{2}$ to give $x^{2}=16$, i.e. $x= \pm 4$. A similar method can be used in the following example.
$\leftrightarrow$ Solve the following equation $1 /(x-1)+2=2 /(x-1)-4$.

We could have reduced the chance of a mistake in copying, by substituting $y=1 /(x-1)$ at the start of this question. This trick is well worth remembering - try it on the next exercise.

## Question T6

Solve the following equation:

$$
2\left(\frac{x^{2}+1}{x^{2}-1}+1\right)=\frac{x^{2}+1}{x^{2}-1}+5 \quad\left(\text { hint }: \text { put } y=\frac{x^{2}+1}{x^{2}-1}\right)
$$

### 2.3 Simultaneous linear equations

Let us reconsider the cycling problem discussed in Subsection 2.1. Again suppose that you cycled along a road at a steady 10 mph . One hour after you left, a friend set off in a car at a steady 30 mph to catch you up. Where and when will you meet?

Let us take the distance to the meeting-point to be $d_{\mathrm{m}}$, and the time on your stop-watch to be $t_{\mathrm{m}}$ when you meet. To keep things as general as possible, take the time at which the car starts to be $T_{0}$, and the speeds of the bicycle and the car to be V and $V$, respectively. At the instant you meet you will each have travelled the same distance, and so we have two equations analogous to Equation 2:
distance travelled by you $=$ your speed $\times$ time spent cycling
i.e. $d_{\mathrm{m}}=\mathrm{V} t_{\mathrm{m}}$
and
distance travelled by your friend $=$ your friend's speed $\times$ time spent driving
i.e. $d_{\mathrm{m}}=V\left(t_{\mathrm{m}}-T_{0}\right)$
so $\quad d_{\mathrm{m}}=V t_{\mathrm{m}}-V T_{0}$
$\square 5$

We know the values of $v$ and $V$ in these equations, and we can calculate $V T_{0}$; but we do not know the distance $d_{\mathrm{m}}$ or the time $t_{\mathrm{m}}$, and we cannot work them out from either Equation $5\left(d_{\mathrm{m}}=\mathrm{V} t_{\mathrm{m}}\right)$ or Equation $6\left(d_{\mathrm{m}}=V t_{\mathrm{m}}-V T_{0}\right)$ alone $\underline{[198 \text {. }}$. Equations 5 and 6 are called simultaneous linear equations since they must be solved together if they are to be solved at all.

The basic method of solving a pair of equations of this kind involves substitution. This means that we find an expression for one of the variables using one equation, and we then substitute that expression for the selected variable in the other equation. For instance, Equation 5 tells us that we can replace $d_{\mathrm{m}}$ by $\mathrm{V} t_{\mathrm{m}}$; making this substitution in Equation 6 gives

$$
\mathrm{V} t_{\mathrm{m}}=V t_{\mathrm{m}}-V T_{0}
$$

i.e. $\mathrm{v} t_{\mathrm{m}}-V t_{\mathrm{m}}=-V T_{0}$
i.e. $(\mathrm{V}-V) t_{\mathrm{m}}=-V T_{0}$
so $\quad t_{\mathrm{m}}=\frac{-V T_{0}}{\mathrm{v}-V}=\frac{V T_{0}}{V-\mathrm{v}}$

This is part of the solution we seek since it expresses one of the unknown quantities $\left(t_{\mathrm{m}}\right)$ in terms of known quantities. (It also makes good sense physically. Notice that $t_{\mathrm{m}}$ increases as $T_{0}$ increases, which we should expect, because the later your friend leaves, the longer it will be before you meet. Furthermore, if $V=\mathrm{V}$ the bicycle and the car are travelling at the same speed, so the denominator becomes zero, $t_{\mathrm{m}}$ becomes infinite, and you never meet - which is again what we expect.) All that remains now is to find $d_{\mathrm{m}}$, which we can do directly by substituting for $t_{\mathrm{m}}$ in Equation 5:

$$
d_{\mathrm{m}}=\mathrm{v} V T_{0} /(V-\mathrm{v})
$$

Now we can put in the numerical values $\mathrm{V}=10 \mathrm{mph}, V=30 \mathrm{mph}, T_{0}=1 \mathrm{~h}$, and find

$$
t_{\mathrm{m}}=\left(\frac{30 \times 1}{30-10}\right) \text { hours }=1.5 \text { hours } \quad \text { and } \quad d_{\mathrm{m}}=(10 \times 1.5) \mathrm{miles}=15 \mathrm{miles}
$$

Finally, we should check to see that these values do actually satisfy the original equations. We have used Equation 5 directly to find $d_{\mathrm{m}}$, but Equation 6 correctly gives

15 miles $=30 \mathrm{mph} \times(1.5-1) \mathrm{h}=15$ miles
You will have noticed that we have put great emphasis on checking the answers; many wrong answers are due to bad arithmetic and algebraic slips. Always check that your answers satisfy the original equations and that they make sense. For example, an answer $t_{\mathrm{m}}=-3$ hours would not be sensible.

Notice that the problem we have just solved had two equations and two unknown variables $t_{\mathrm{m}}$ and $d_{\mathrm{m}}$. In general, a pair of simultaneous linear equations may look more like this:

$$
\begin{align*}
& x+2 y=8  \tag{7}\\
& 2 x-3 y=-5 \tag{8}
\end{align*}
$$

and we may be required to solve these equations for the two variables $x$ and $y$.
We can express $x$ in terms of $y$ (using Equation 7)

$$
\begin{equation*}
x=-2 y+8 \tag{9}
\end{equation*}
$$

and then substitute for $x$ in Equation 8 to obtain

$$
2(-2 y+8)-3 y=-5
$$

so that $-4 y+16-3 y=-5$
i.e. $\quad 7 y=21$
giving $\quad y=3$

If we substitute this value for $y$ in Equation 9

$$
\begin{equation*}
x=-2 y+8 \tag{Eqn9}
\end{equation*}
$$

we obtain

$$
x=(-2 \times 3)+8=2
$$

Now we should check to see that these answers satisfy the original equations. We have just used Equation 7 (in the form of Equation 9) to find $x$, so we check by substituting both values into the other equation, Equation 8,

$$
\begin{equation*}
2 x-3 y=-5 \tag{Eqn8}
\end{equation*}
$$

and we find

$$
(2 \times 2)-(3 \times 3)=4-9=-5
$$

which is correct.
Notice that for checking we always use the original equation (before we have had a chance to make any mistake), and we don't use the equation that has just been used to find one of the variables.

You can apply this same procedure to any pair of linear equations. However, if the coefficients are not the simple integers we have here, the algebra may soon start to look complicated, and it is useful to have some sort of drill for the process of solution. Look again at the equations we have just solved.

$$
\begin{align*}
& x+2 y=8 \\
& 2 x-3 y=-5 \tag{Eqn8}
\end{align*}
$$

(Eqn 7)

To get an equation involving just $y$, multiply the equations by suitable factors so that both have the same coefficient of $x$ - in this case we multiply both sides of Equation 7 by 2 and leave Equation 8 unchanged:

$$
\begin{align*}
& 2 x+4 y=16 \\
& 2 x-3 y=-5 \tag{Eqn8}
\end{align*}
$$

Now subtract the second equation from the first, to get

$$
(2 x+4 y)-(2 x-3 y)=16-(-5)
$$

i.e. $0 x+7 y=21$
so, as before

$$
y=3
$$

This process is called elimination, we speak of eliminating $x$ from the equations, though it's really just a systematic way of implementing substitution.
We now proceed as before and use Equation 7 to give $x=2$ again. After this we should use Equation 8 to carry out the same check.

Alternatively, we could have eliminated $y$ from Equations 7 and 8; and the simplest way to do that is to multiply Equation 7 by 3, and Equation 8 by 2 to obtain

$$
\begin{aligned}
& 3 x+6 y=24 \\
& 4 x-6 y=-10
\end{aligned}
$$

Now we may $a d d$ the above equations to obtain

$$
7 x+0 y=14
$$

which gives $x=2$
We can then use either Equation 7 or Equation 8 to find $y$, and the other to check the answer.

So we have seen three alternative methods of solving Equations 7 and 8:

- substituting for $x$ (found from one equation) in terms of $y$ (in the other)
- eliminating $x$
- eliminating $y$.

뇽 In general we try to choose the method that is most convenient.

## Question 17

Solve the following pairs of equations:
(a) $5 x+2 y=0$
(b) $4 x+y=10$
$2 x+5 y=21$
$3 x-2 y=13$

We can also work out the general solution to a pair of simultaneous equations using symbols rather than numbers. For example, for the pair of equations nas

$$
\begin{align*}
& a x+b y=p  \tag{10}\\
& c x+d y=q \tag{11}
\end{align*}
$$

the solution (provided $a d-b c \neq 0$ ) is

$$
\begin{align*}
& x=(d p-b q) / D  \tag{12}\\
& y=(a q-c p) / D \tag{13}
\end{align*}
$$

where $\quad D=a d-b c$
This general solution is not particularly helpful, and certainly not worth remembering. However, you may meet it again when you study the theory of sets of three or more simultaneous equations.

## Question T8

Show that Equations 12-14 provide the correct solution to Equations 7 and 8.

Before leaving the subject of simultaneous linear equations it's worth noting that such equations are not always soluble. Here are two situations where Equations 10 and 11 do not have a solution.

## Case 1

This is the case in which the equations are not independent - i.e. one equation is a multiple of the other, so the second equation provides no new information; an example would be the pair of equations

$$
\begin{array}{ll} 
& x+2 y=8 \\
\text { and } & 2 x+4 y=16
\end{array}
$$

If we try to eliminate $x$ from these equations we shall just end up with $0=0$, a statement which is true, but not much use.

## Case 2

This is the case in which the equations are inconsistent - they cannot both be true at the same time. For example,

$$
x+2 y=8
$$

and $\quad 2 x+4 y=15$
If we try to solve this pair of equations we shall find $1=0$, which is neither useful nor true.

The first case occurs when we don't have enough information to solve a problem, the second when some of the information is wrong or inconsistent. Note that in either case the general solution of Equations 12-14 does not apply since $D=0$.

In this section we have been concerned with two linear equations, and two unknowns; however, there are many occasions when one has to deal with three or more unknowns. In general:

If there are $n$ unknowns, then $n$ consistent independent linear equations are needed to solve the equations for the unknowns.

If we are given more than $n$ consistent linear equations, they cannot all be independent, so we are free to select $n$ of them that are independent and use those to find the $n$ unknowns. Finally, it should be noted that when seeking a set of three or more independent linear equations it is not enough to require that no equation should be a multiple of any other; rather we have to demand that none of the equations can be expressed as a sum of multiples of the others. It is actually fairly easy to determine whether or not a given set of equations satisfies this requirement, but it requires techniques beyond the scope of this module.

### 2.4 Graphical solutions

There is one other way of solving a pair of simultaneous equations that is worth mentioning - a graphical method. Let us consider Equations 7 and 8 yet again. If we write these in the equivalent form

$$
\begin{aligned}
& y=\frac{8-x}{2} \\
& y=\frac{2 x+5}{3}
\end{aligned}
$$

we can plot each of them as a straight-line graph, as in Figure1. On the first line, $y=(8-x) / 2$, we have all the points $(x, y)$ which make this equation a true statement. Similarly, on the second line, $y=(2 x+5) / 3$, we have all the points $(x, y)$ which make this equation a true statement. If both statements are true for a particular point $(x, y)$ then this point must be where the two lines intersect, for it is only there that both equations are true simultaneously.


Figure 1 Graphical solution of Equations 7 and 8.

This method can be used to find an approximate solution to any pair of equations (linear or not) and it is often a good starting point, particularly if you have access to a graph-drawing computer or calculator.

## Question 19

Use a graphical method to find approximate solutions of the simultaneous equations

$$
x+2 y=k \quad \text { and } \quad 2 x-3 y=-5
$$

for $k=2,4,6$ and 8 . (You can make use of Figure 1.)

## 3 Solving quadratic equations

Up to now we have dealt with linear equations and their roots, that is, equations in which the independent variable appears only as the first power. Let us now advance to quadratic equations, which involve the square of the independent variable. The general form of a quadratic equation is

$$
\begin{equation*}
a x^{2}+b x+c=0 \tag{15}
\end{equation*}
$$

where $a, b$ and $c$ are constants. There is a simple general method of solving such equations which will be explained shortly, but it is instructive to examine the problem of solving quadratic equations from a number of other viewpoints, since it sheds a good deal of light on the general problem of solving equations.

### 3.1 The solutions as zeros of a quadratic function

A quadratic function is any function $f(x)$ that may be written in the form

$$
\begin{equation*}
f(x)=a x^{2}+b x+c \tag{16}
\end{equation*}
$$

where $a, b$ and $c$ are constants. Thus, when any given quadratic equation is written in the style of Equation 15, its left-hand side will define a corresponding quadratic function $f(x)$. Moreover, it follows from Equations 15 and 16 that $f(x)=0$ for each value of $x$ that satisfies the given quadratic equation.

For this reason the roots (or solutions) of a quadratic equation are also referred to as the zeros of the corresponding quadratic function.

If we set $y=f(x)$ and plot the graph of a quadratic function the resulting curve is called a parabola. Six different parabolas (corresponding to different values for $a$, $b$ and $c$ ) are shown in Figure 2. If $a>0$ the parabolas are as shown on the left of the figure, while the case $a<0$ corresponds to those on the right.

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As you can see in Figure 2, the graph of any given quadratic function may meet the $x$-axis once or twice or not at all - it all depends on the values of $a, b$ and $c$. However, if the graph does meet the $x$-axis it does so when $y=f(x)=0$. It follows that each value of $x$ at which the graph meets the $x$-axis is a zero of the quadratic function and therefore a solution (root) of the corresponding quadratic equation. The possibilities illustrated by the three parts of Figure 2 are as follows:
(a) there are two unequal roots, and the parabola crosses the $x$-axis in two distinct points;
(b) there is one root, at the point where the parabola just touches the $x$-axis;
(c) there are no roots, because the parabola never reaches the $x$-axis.

Case (b), where there is just one root, is really a special limiting case of (a).

(a)

(c)

Figure 2 Graphs of six different quadratic functions.

If we move either of the parabolas in Figure 2a away from the $x$-axis its points of intersection with the $x$-axis (and hence the roots of the corresponding quadratic equation) will get closer and closer together, so that when the parabola just touches the $x$-axis, the two will have merged. For this reason we usually say that the root in (b) is a repeated root or that the equation has coincident roots.
These possibilities are the central feature of the process of solving quadratic equations, as we shall see.


Figure 2 Graphs of six different quadratic functions.

### 3.2 Sums and products of roots: factorizing the quadratic

Throughout this section our aim is to solve the quadratic equation $a x^{2}+b x+c=0$, by finding values for its roots, $\alpha$ and $\beta$ say. But suppose for the moment that we know $\alpha$ and $\beta$ and we ask instead 'what is the quadratic equation that has $\alpha$ and $\beta$ as its roots?'

An answer to this question is given by the equation

$$
\begin{equation*}
(x-\alpha)(x-\beta)=0 \tag{17}
\end{equation*}
$$

It is easy to see that if we put either $x=\alpha$ or $x=\beta$ in this equation then the left-hand side is zero (if $\alpha$ and $\beta$ are equal, then we have a repeated root). Moreover, expanding the left-hand side of Equation 17 gives the quadratic equation

$$
\begin{equation*}
x^{2}-(\alpha+\beta) x+\alpha \beta=0 \tag{18}
\end{equation*}
$$

In other words, the quadratic equation $x^{2}-(\alpha+\beta) x+\alpha \beta=0$ has $\alpha$ and $\beta$ as its roots.
Now, compare Equation 18 with the general quadratic equation we are trying to solve:

$$
\begin{equation*}
a x^{2}+b x+c=0 \tag{Eqn15}
\end{equation*}
$$

We can make the general equation look more like Equation 18 by dividing both sides by $a$, which gives

$$
\begin{align*}
& x^{2}+(b / a) x+(c / a)=0  \tag{19}\\
& x^{2}-(\alpha+\beta) x+\alpha \beta=0 \tag{Eqn18}
\end{align*}
$$

It then becomes clear that Equations 18 and 19 are the same, provided that

$$
\begin{equation*}
\alpha+\beta=-b / a \quad \text { and } \quad \alpha \beta=c / a \tag{20}
\end{equation*}
$$

Thus, we can make the following connections between the coefficients of a quadratic and its roots.
Given a general quadratic equation $a x^{2}+b x+c=0$ with roots $\alpha$ and $\beta$

- the sum of its roots is $\alpha+\beta=-b / a$
- and the product of its roots is $\alpha \beta=c / a$

Unfortunately it is not possible to use these equations directly to solve the equation; if you try to do so you will just end up with the original equation! However, they may help us to factorize the original equation, that is to reexpress it in the form of Equation 17 and hence identify its solutions. This is often the quickest method of solution, especially if we know, or suspect, that the roots are integers (i.e. whole numbers).

- Solve the equation $x^{2}-5 x+6=0$.

In the above solution we have used the process known as factorization to find the roots of the equation. We can check that our answer is correct by expanding the brackets to give

$$
(x-2)(x-3)=x^{2}-2 x-3 x+6=x^{2}-5 x+6
$$

If we had to solve the almost identical equation

$$
x^{2}-5 x-6=0
$$

we would have $\alpha+\beta=5$ and $\alpha \beta=-6$ and the roots would still involve 2 and 3 , or 1 and 6 , but this time the two numbers in each pair would have opposite signs, since the product $\alpha \beta$ is negative. We must choose the pair +6 and -1 . We can check by expanding as before, and this time:

$$
(x-6)(x+1)=x^{2}-6 x+x-6=x^{2}-5 x-6
$$

## Question T10

Solve the following quadratic equations by factorization:
(a) $x^{2}-3 x+2=0$
(b) $x^{2}+x-2=0$
(c) $x^{2}+7 x+12=0$
(d) $x^{2}-x-12=0$
(e) $x^{2}-2 x-15=0$
(f) $x^{2}-4 x-12=0$
$\square$

The above equations all had the coefficient of $x^{2}$ equal to 1 . It is a little more complicated if $a$ is not 1 ; consider, for example, the expression

$$
2 x^{2}+5 x+3
$$

If this can be factorized, then it must be possible to express it as

$$
(2 x+?)(x+?)
$$

Trial and error shows us that the factorization is

$$
(2 x+3)(x+1)
$$

Thus, to solve the equation $2 x^{2}+5 x+3=0$ we rewrite it as $(2 x+3)(x+1)=0$ and then it is clear that the roots are $-3 / 2$ and -1 . $\square$
It is important to check your answer by substituting the values you have obtained back into the original equation to verify that they are indeed roots of the equation.

## Question T11

Solve the following equations by factorization:
(a) $2 x^{2}+9 x-5=0$
(b) $3 x^{2}-11 x-4=0$
(c) $3 x^{2}-x-2=0$
(d) $4 x^{2}-3 x-1=0$
(e) $4 x^{2}-4 x+1=0$
(f) $6 x^{2}+5 x-6=0$

With practice this method can become very quick, but unfortunately it does not always work. The quadratic equation

$$
x^{2}-3 x+1=0
$$

cannot be factorized in the same way, because the roots are not integers. We must therefore find a more generally applicable method even if it is somewhat slower.

### 3.3 The quadratic equation formula: completing the square

The quadratic equation

$$
x^{2}-7=0
$$

is easily solved, for we may simply add 7 to both sides of the equation and then take square roots of both sides to obtain

$$
x^{2}=7, \text { so that } x= \pm \sqrt{7}
$$

The quadratic equation

$$
(x+3)^{2}-5=0
$$

is hardly more difficult, since we can add 5 to both sides, then take the square root of both sides to get $x+3= \pm \sqrt{5}$ so that $x=-3 \pm \sqrt{5}$.

If we could reduce the general quadratic equation to something like this form we should have another method of solution that might work more generally. Introducing such a method is the main purpose of this subsection, but first we must examine how one might write any quadratic equation in such a form. Another example may make the problem clear.

To solve the equation

$$
3 x^{2}-10 x+1=0
$$

we first extract the factor 3 to give

$$
\begin{equation*}
3\left(x^{2}-\frac{10}{3} x+\frac{1}{3}\right)=0 \tag{21}
\end{equation*}
$$

and we thus ensure that the coefficient of $x^{2}$ inside the brackets is 1 . Next we arrange for the term $x^{2}-\frac{10}{3} x$ to arise from a term of the form $(x+k)^{2}$ for some value $k$. (Such a term is called a perfect square.)
Since $(x+k)^{2}=x^{2}+2 k x+k^{2}$ we will be forced to choose $k$ so that $2 k=-10 / 3$ in this case, in other words $k=-5 / 3$, which gives $k^{2}=25 / 9$.

$$
\begin{equation*}
3\left(x^{2}-\frac{10}{3} x+\frac{1}{3}\right)=0 \tag{Eqn21}
\end{equation*}
$$

Now we add and subtract $25 / 9$ to the term in the brackets in Equation 21 to get

$$
\begin{equation*}
3\left(x^{2}-\frac{10}{3} x+\frac{25}{9}-\frac{25}{9}+\frac{1}{3}\right)=0 \tag{22}
\end{equation*}
$$

The advantage of this step is that the first three terms in the bracket can be written as a perfect square, so Equation 22 becomes

$$
3\left[\left(x-\frac{5}{3}\right)^{2}-\frac{25}{9}+\frac{1}{3}\right]=0
$$

and this can be simplified to

$$
3\left[\left(x-\frac{5}{3}\right)^{2}-\frac{22}{9}\right]=0
$$

so that $\left(x-\frac{5}{3}\right)^{2}=\frac{22}{9} \quad$ and therefore $\quad x=\frac{5}{3} \pm \frac{\sqrt{22}}{3}$
The above process is known as completing the square; try the same method in the following exercises.

## Question T12

Find the roots of the following equations by completing the square:
$\begin{array}{ll}\text { (a) } x^{2}-x-2=0 & \text { (b) } x^{2}-2 x-8=0\end{array}$
(c) $2 x^{2}+x-10=0$
(d) $3 x^{2}-18 x+24=0$

To see how this method can be applied in general, let us repeat the procedure for the general quadratic.

$$
\begin{aligned}
a x^{2}+b x+c & =a\left(x^{2}+\frac{b}{a} x+\frac{c}{a}\right) \\
& =a\left(x^{2}+\frac{b}{a} x+\frac{b^{2}}{4 a^{2}}-\frac{b^{2}}{4 a^{2}}+\frac{c}{a}\right) \\
& =a\left[\left(x+\frac{b}{2 a}\right)^{2}-\frac{b^{2}-4 a c}{4 a^{2}}\right]
\end{aligned}
$$

Now if we put this expression equal to zero, we can divide both sides by the multiplying factor $a$, and get

$$
\left(x+\frac{b}{2 a}\right)^{2}-\frac{b^{2}-4 a c}{4 a^{2}}=0 \quad \text { so that }\left(x+\frac{b}{2 a}\right)^{2}=\frac{b^{2}-4 a c}{4 a^{2}} \quad \text { and } \quad x+\frac{b}{2 a}= \pm \sqrt{\frac{b^{2}-4 a c}{4 a^{2}}}
$$

$$
\begin{equation*}
\text { thus } \quad x=\frac{-b \pm \sqrt{b^{2}-4 a c}}{2 a} \tag{23}
\end{equation*}
$$

$$
x=\frac{-b \pm \sqrt{b^{2}-4 a c}}{2 a} \quad(\text { Eqn 23) }
$$

This is a very important formula, which is used so often that you should commit it to memory.
(The author was taught the piece of doggerel in the margin; you may, or may not, find it helpful.)
In practice this formula provides the simplest and quickest way of solving most quadratic equations.

## Question T13

Use Equation 23 to solve the problems you met in Question T12; i.e find the roots of the following equations:
(a) $x^{2}-x-2=0$
(b) $x^{2}-2 x-8=0$
(c) $2 x^{2}+x-10=0$
(d) $3 x^{2}-18 x+24=0$


From square of $b$ Take $4 a c$, Square root extract, Then $b$ subtract. Proceed in this way, And divide by $2 a$.


## Question T14

$$
x=\frac{-b \pm \sqrt{b^{2}-4 a c}}{2 a}
$$

(Eqn 23)
Use Equation 23 to solve the following:
$\begin{array}{ll}\text { (a) } x^{2}-6 x+1=0 & \text { (b) } 3 x^{2}-9 x+2=0\end{array}$
Do not use your calculator, but leave the answer in the simplest form possible, with the square roots not evaluated. Check that the sum and product of the roots are correct.

Not only does Equation 23 simplify the process of solving quadratic equations, it also has a feature which enables us to say something about the general properties of the quadratic equation without actually having to solve it.
This feature is known as the discriminant, and is given by the expression inside the square root, $b^{2}-4 a c$.

The sign of the discriminant determines some useful properties of the roots of the equation:

- If $b^{2}-4 a c$ is positive, then we can find its square root, and the quadratic equation will have two different roots. This corresponds to the case (a) of Figure 2, where the parabola crosses the $x$-axis twice. (For reasons that will become clear in the next subsection, we also say that the equation has two distinct real roots.)
- If $b^{2}-4 a c$ is equal to zero, then the two roots of the equation will be repeated and the quadratic equation can be written as $a(x+k)^{2}=0$. This corresponds to the special case (b) of Figure 2 in which the parabola just touches the $x$-axis and we have just one (repeated) real root.
- If $b^{2}-4 a c$ is negative, then we cannot take the square root, and we say that there are no real roots to the equation. This case corresponds to case (c) of Figure 2, when the parabola does not intersect the $x$-axis at all.


(b)


Figure 2 Graphs of six different quadratic functions.

## Question T15

Use the discriminant to establish how many real roots the following equations have:
(a) $4 x^{2}-20 x+25=0$
(b) $x^{2}+x+6=0$
(c) $x^{2}+x-6=0$

Although we now seem to have a recipe that will solve all our problems with quadratic equations, and make the whole business purely mechanical, this is not quite true. It often happens that when we have to substitute numbers in formulae, things are not quite as straightforward as appears at first sight. For example, consider the equation:

$$
x^{2}-2 x+10^{-12}=0
$$

and notice that the constant term, $c$, is very small.
If the constant term $10^{-12}$ were not there, the equation would be $x^{2}-2 x=0$ which has roots at $x=2$ and $x=0$, so we would expect the given equation to have roots near these values.
Applying the formula (Equation 23) gives us the roots

$$
x=\left(2 \pm \sqrt{4-4 \times 10^{-12}}\right) / 2
$$

which can only be evaluated as 2 or 0 to the accuracy of most calculators.
The first root will certainly be as close to 2 as most people could wish, but to get the second root we have to take the difference of two numbers that are very close together, which means that we have lost a lot of precision. However, there is an alternative. If we calculate the larger root (say $\alpha$ ) using Equation 23, we find a value near to 2 as before. We can then use the fact that the product of the roots $(\alpha \beta)$ is equal to $c / a$, i.e. $10^{-12}$, to obtain the value $10^{-12} / 2=5 \times 10^{-13}$ for the smaller root (say $\beta$ ). $\qquad$

### 3.4 Roots for all equations: complex numbers

Let us look again at what happens when the discriminant $b^{2}-4 a c$ is negative. Consider the equation

$$
x^{2}-x+1=0
$$

for which $a=1, b=-1$ and $c=1$, so that $b^{2}-4 a c=-3$.
If we try to solve this equation using Equation 23

$$
x=\frac{-b \pm \sqrt{b^{2}-4 a c}}{2 a} \quad(\text { Eqn 23) }
$$

we get

$$
x=\frac{1}{2}(1 \pm \sqrt{-3})
$$

which does not seem to make sense, since a negative number like -3 has no square root.

Yet if we substitute one of these 'roots' into the left-hand side of the equation we find:

$$
\begin{aligned}
x^{2}-x+1 & =\left[\frac{1}{2}(1-\sqrt{-3})\right]^{2}-\frac{1}{2}(1-\sqrt{-3})+1 \\
& =\frac{1}{4}\left[1-2 \sqrt{-3}+(\sqrt{-3})^{2}\right]-\frac{1}{2}(1-\sqrt{-3})+1 \\
& =\frac{1}{4}(1-2 \sqrt{-3}-3)-\frac{1}{2}(1-\sqrt{-3})+1 \\
& =-\frac{1}{2}-\frac{1}{2} \sqrt{-3}-\frac{1}{2}(1-\sqrt{-3})+1=0
\end{aligned}
$$

So treating the quantity $\sqrt{-3}$ as an 'ordinary' number such that its square is -3 appears to give us a solution to the equation.

## Question T16

Show that

$$
x=\frac{1}{2}(1+\sqrt{-3})
$$

also satisfies the equation

$$
x^{2}-x+1=0
$$

in the same way.

Both 'roots' satisfy the equation, even if they do not mean anything! Since there seems to be no contradiction, this is an invitation to mathematicians to invent something new.

A quantity called $i$ is introduced, which is used in place of $\sqrt{-1}$; it has the property that $i \times i=-1$. $\underline{\text { nas }}$ This enables us to deal with the square root of any negative number; for example
$\sqrt{-3}=\sqrt{-1 \times 3}=\sqrt{-1} \times \sqrt{3}=i \sqrt{3}$. The number $i$ can be given a perfectly sound mathematical definition, but that is not our concern here, we merely wish to use it.

For example, we may write

$$
\begin{aligned}
& i^{4} \\
&=\left(i^{2}\right)^{2}=(-1)^{2}=1 \\
& \text { or } i^{3} \\
&=i \times i \times i=-1 \times i=-i \\
& \text { or } \frac{1}{i}
\end{aligned}=\frac{i}{i \times i}=\frac{i}{-1}=-i
$$

The quantity $i$ and all its multiples, such as $i \sqrt{3}$, are examples of imaginary numbers. These should be contrasted with 'ordinary numbers' such as 35 or -2.6 that do not involve $i$ and which are formally referred to as real numbers. By adding together real and imaginary numbers it is possible to create so-called complex numbers of the form

$$
a+b i
$$

where $a$ and $b$ are both real numbers. For example, $3+5 i$, would be a typical complex number.
The real number $a$ is called the real part, and the real number $b$ the imaginary part, of the complex number. The real part of $3+5 i$ is 3 while its imaginary part is 5 。 $\underline{\text { deg }}$. 'Real' and 'imaginary' are perhaps unfortunate names to have been chosen, since they suggest that $a$ is more important than $b$, whereas we must treat both on an equal footing. However, the names are now too well established to be changed easily.

## Question T17

Using complex numbers, find the solutions of the following equations:
$\begin{array}{ll}\text { (a) } x^{2}+25=0 & \text { (b) } x^{2}+x+6=0\end{array}$

Complex numbers play an important part in mathematics and physics, and are the subject of a whole block of FLAP modules.

We cannot go into all their applications here, but it is worth noting that if we allow roots to be complex, and if we count repeated roots twice, then every quadratic equation has two roots, the values of which are given by Equation 23.

$$
\begin{equation*}
x=\frac{-b \pm \sqrt{b^{2}-4 a c}}{2 a} \tag{Eqn23}
\end{equation*}
$$

## 4 Solving more complicated equations

The equations we have dealt with so far are a very small sample of the variety that you will meet. We mentioned simultaneous equations with more than two unknown quantities in Subsection 2.3, but you will also meet equations that involve higher powers of the unknown variable than the square, including cubic equations such as

$$
x^{3}+x^{2}+x+1=0
$$

and quartic equations such as

$$
x^{4}+x^{3}+x^{2}+x+1=0
$$

Such equations, together with the linear and quadratic equations we considered earlier, are special cases of a class of polynomial equations which have the general form

$$
a_{0}+a_{1} x+a_{2} x^{2}+\ldots+a_{n-1} x^{n-1}+a_{n} x^{n}=0
$$

where $n$ is an integer called the degree of the polynomial, and the $n+1$ constants $a_{0}, a_{1}, a_{2}, \ldots a_{n-1}$ and $a_{n}$ are the coefficients of the polynomial. The left-hand side of such an equation constitutes a polynomial expression and may be used to define a corresponding polynomial function.

There is a general result for roots of polynomial equations:
The fundamental theorem of algebra states that every polynomial equation of degree $n$ has precisely $n$ roots as long as complex roots and repeated roots are counted.

Cubic and quartic equations can be solved using formulae similar to that used for solving quadratic equations, but these formulae are so complicated that they are rarely used. However, there are no general formulae for the roots of equations of the 5th and higher degrees, and in such cases we must usually be satisfied with numerical approximations. Many of the numerical methods that are used for finding these approximations can also be applied to non-polynomial equations and we shall examine such methods shortly. First, however, let us look at methods that allow us to solve some polynomial equations.

25

### 4.1 Solving polynomial equations

Sometimes, if we are fortunate, we are able to guess one, or more, solutions of a cubic, or higher order, equation and this makes the task of finding the remaining roots considerably easier. For example, if we substitute $x=1$ into the left-hand side of the following equation:

$$
\begin{equation*}
x^{3}-6 x^{2}+11 x-6=0 \tag{24}
\end{equation*}
$$

we see immediately that this value of $x$ makes the equation true. It follows that $(x-1)$ is a factor of the left-hand side of Equation 24 so that

$$
x^{3}-6 x^{2}+11 x-6=(x-1) \times(\text { a quadratic expression in } x)
$$

i.e. $\left(\frac{x^{3}-6 x+11 x-6}{x-1}\right)=$ a quadratic expression in $x$

We can find this quadratic expression by algebraic division as follows:

$$
\begin{aligned}
& \frac{x^{2}-5 x+6}{x - 1 \longdiv { x ^ { 3 } - 6 x ^ { 2 } + 1 1 x - 6 }} \\
& \frac{x^{3}-x^{2}}{-5 x^{2}+11 x} \\
& \frac{-5 x^{2}+5 x}{+6 x-6} \\
& +6 x-6
\end{aligned}
$$

so that $x^{3}-6 x^{2}+11 x-6=(x-1)\left(x^{2}-5 x+6\right)$.
The quadratic expression on the right is then easily factorized to give

$$
x^{3}-6 x^{2}+11 x-6=(x-1)(x-2)(x-3)
$$

and now we can see that the roots of Equation 24 are 1, 2 and 3.

This method is quite restricted in its applications because it relies on our ability to guess a root, but it can still be very useful.
neq

## Question $T 18$

Find the roots of $x^{3}+x^{2}-4 x-4=0$. (Hint: try some integer values of $x$.) $\square$

| FLAP M1.4 | Solving equations |  | S570 V1.1 |
| :--- | :--- | :--- | :--- |

### 4.2 Graphical solutions

In Subsection 2.4 we saw how we could use graphs to solve a pair of simultaneous equations, and in Section 3 we saw how a graph of a quadratic function could give us clues about the roots of the corresponding quadratic equation. Similar methods can be applied to a wide variety of equations. Consider, for example, the function

$$
\begin{equation*}
f(x)=x^{3}-4 x+1 \tag{25}
\end{equation*}
$$

the graph of which is shown in Figure 3.
The curve crosses the $x$-axis three times, so the equation $f(x)=0$ has three roots, which are, respectively, near -2 , between 0 and 1 , and near +2 . A larger scale graph would tell us that the values are nearer $-2.1,0.25$ and +1.9 . If this degree of precision is enough, then nothing further need be done. If greater accuracy is needed, then one has the choice of drawing a yet more accurate graph (or perhaps three separate graphs, one for the neighbourhood of each root) or of using another method.


Figure 3 The graph of Equation 25.

Since more function values must be calculated to plot the new graphs, it is probably better to go straight to one of the iterative methods discussed in the next subsection.
Any pair of simultaneous equations can be solved approximately by graphical methods. For example, the equations

$$
\begin{array}{ll} 
& y=x^{3}-4 x+1 \\
\text { and } & 4 y=3-2 x-x^{2} \tag{27}
\end{array}
$$

can be solved by plotting the graphs of $y=x^{3}-4 x+1$ and $y=\left(3-2 x-x^{2}\right) / 4$. The points of intersection correspond to the solutions of this pair of equations. Figure 4 shows that there are three possible solutions, corresponding to the three intersections, at approximately

$$
\begin{array}{ll}
x=-2, & y=0.7 \\
\text { and } \quad & x=0, \\
x=1.7, & y=-0.7
\end{array}
$$

which can be abbreviated to

$$
(-2,0.7),(0,0.7),(1.7,-0.8) \quad \text {, 運等 }
$$



Figure 4 The graphs of Equations 26 and 27.

### 4.3 Numerical procedures

Numerical procedures are methods of finding approximate solutions to a particular equation, in which all the constants are known numbers.

## The bisection method

The simplest procedure, called the bisection method, is not much more than trial and error. If we want to solve the equation $f(x)=0$, and we have some idea of the solution, perhaps from a graph, then we evaluate $f(x)$ on either side of our initial estimate so as to obtain a range of $x$ values over which the sign of $f(x)$ changes. As an illustration, consider the equation

$$
x^{3}-4 x+1=0
$$

corresponding to the function $f(x)=x^{3}-4 x+1$.
From Figure 3 we see that there is a root between $x=1$ and $x=2$.


Figure 3 The graph of Equation 25.

This is confirmed by noticing that

$$
f(1)=-2 \quad \text { and } \quad f(2)=1
$$

so that there is a sign change on the interval from $x=1$ to $x=2$.
At the mid-point, $x=1.5$, we have $f(1.5)=-1.625$ so that the sign change (and therefore the root) must lie between $x=1.5$ and $x=2$.

Similarly, we can show that the root lies between $x=1.75$ and $x=2$; and we can continue in this fashion, bisecting the interval and evaluating the function at the mid-point, as far as we please and so obtain an approximation to the root which is as accurate as we need.
$\square \leq$

## Regula falsi

The method of bisection is very reliable and simple to implement as a computer program. It is true that it is comparatively slow, but with the speed of modern computers this hardly matters for a single calculation. However, speed may become a critical factor if you need to perform a large number of evaluations of roots. A simple method of speeding up the method considerably is the variation known as regula falsi. Let us look in detail at a graph of the neighbourhood of the solution, in Figure 5.
If $x_{1}$ and $x_{2}$ are initial estimates for the root, and if $y_{1}=f\left(x_{1}\right)$ and $y_{2}=f\left(x_{2}\right)$ have opposite signs, then a straight line between the two points will intersect the $x$-axis at a


Figure 5 Regula falsi. point which is closer to the root. The equation of the straight line joining the two points $\left(x_{1}, y_{1}\right)$ and $\left(x_{2}, y_{2}\right)$ is

$$
\frac{y-y_{1}}{x-x_{1}}=\frac{y-y_{2}}{x-x_{2}}
$$

(28) 녕ㅇㅇ

If we put $y=0$ in this equation, then the corresponding value of $x$ will determine the point at which the line crosses the horizontal axis and will therefore give an improved estimate of the root. Thus,

$$
\frac{-y_{1}}{x-x_{1}}=\frac{-y_{2}}{x-x_{2}}
$$

which can be rearranged to give

$$
x=\frac{x_{2} y_{1}-x_{1} y_{2}}{y_{1}-y_{2}}
$$

This improved estimate can be used as the starting point for an even better estimate.
Finding a root by this method takes fewer steps, although more work is required in each step. In practice it can be a fine judgement as to which method is more efficient in a particular case.

## Question T19

The function defined in Equation 25 has a zero near $x=2$.

$$
\begin{equation*}
f(x)=x^{3}-4 x+1 \tag{Eqn25}
\end{equation*}
$$

Find this zero to an accuracy of five decimal places (i.e. so that the equation is satisfied to five decimal places) using one of the two numerical methods above. You may use your calculator.


## Iterative methods

An alternative class of methods that are often useful are the iterative methods; these are based on the idea that if one has an approximate solution to an equation, then it can be used to get a better value. You might consider either of the previous methods to be in this class, but we usually mean methods in which the next approximation is obtained by a special formula. Look again at the equation $x^{3}-4 x+1=0$, which we know, from Figure 3 , has a solution near $x=2$. We can rewrite the equation as

$$
\begin{equation*}
x=\frac{2 x^{3}-1}{3 x^{2}-4} \tag{29}
\end{equation*}
$$

and we can use this to provide an iteration formula that relates an approximate root $x_{n}$ to a more accurate approximate root $x_{n+1}$

$$
\begin{equation*}
x_{n+1}=\frac{2 x_{n}^{3}-1}{3 x_{n}^{2}-4} \tag{30}
\end{equation*}
$$

So, if we set $n=1$ in Equation 30 and take as our crude first approximation $x_{1}=2$, we can use Equation 30 to find a better approximation $x_{2}=15 / 8=1.875$. Setting $n=2$ in Equation 30 we can now repeat this process (starting with $x_{2}=1.875$ ) to find an even better approximation $x_{3}$.

Continuing in this way we can obtain a succession of approximations $x_{2}, x_{3}, x_{4}, x_{5}$ and so on, with each step, or iteration, being an improvement on the last.

$$
\begin{aligned}
& x_{2}=1.875 \\
& x_{3}=1.86097852 \\
& x_{4}=1.86080588 \\
& x_{5}=1.86080585 \\
& x_{6}=1.86080585
\end{aligned}
$$

$$
\underline{10 \% 8}
$$

We stop when two successive iterations are the same to the required number of decimal places, since we can expect no further improvement.

## Question T20

The equation $x^{3}+2 x-1=0$, which has a root near $x=0.45$, can be rearranged to give

$$
x=\frac{1-x^{3}}{2}
$$

Using the first approximation $x_{1}=0.45000$ and the iteration formula

$$
x_{n+1}=\frac{1-x_{n}^{3}}{2}
$$

find this root to five decimal place accuracy. (You may use your calculator.)

Unfortunately, this simple method must be used with great care, and careful choice of the iteration formula is often needed to produce an iteration that will result in a meaningful answer. However, if the method does work, then it has the great advantage that it is almost proof against a mistake, since any reasonable approximation will be improved; the worst that can happen is that the process is slowed down somewhat.
More advanced methods, beyond the scope of this module, enable one to produce very rapid convergence to the root.

## Question $\mathbf{T 2 1}$

The following iterative formula was used in the early days of computers to find the reciprocal of a given number a

$$
x_{n+1}=x_{n}\left(2-a x_{n}\right)
$$

Use this iteration to find $1 / 0.8$ (i.e. let $a=0.8$ in the above formula), starting with $x_{1}=1$.
(You may use your calculator.)

## 5 Closing items

### 5.1 Module summary

1 Given a function $f(x)$, a solution or root of the equation $f(x)=0$ is a value of $x$ that makes the equation true. Such a value is also a zero of the function and is represented graphically by any value of $x$ at which the graph of $f(x)$ meets the $x$-axis.
2 A linear equation in a single variable $x$ has the general form $a x+b=0$ and may be solved by rearrangement to give $x=-b / a$.
3 A pair of simultaneous linear equations may be solved by substitution; the method of elimination provides a standard routine for carrying out the substitution.
4 A set of simultaneous linear equations involving $n$ unknowns may be solved, provided that it contains at least $n$ independent equations and provided that none of the equations are inconsistent.
5 Simultaneous linear equations may also be solved graphically, by drawing appropriate straight lines and determining their point of intersection.
6 A quadratic equation $a x^{2}+b x+c=0$ has roots $x=\frac{-b \pm \sqrt{b^{2}-4 a c}}{2 a}$, as may be demonstrated by the process of completing the square.

7 If the discriminant, $b^{2}-4 a c$, of a quadratic equation is positive there are two real roots, if it is zero there is a repeated root, and if the discriminant is negative there are no real roots. In the latter case it is possible to find two complex roots of the form $a+b i$ where $i=\sqrt{-1}$.
8 If the roots of $a x^{2}+b x+c=0$ are $\alpha$ and $\beta$ then $\alpha+\beta=-b / a$ and $\alpha \beta=c / a$. Such a quadratic may be factorized and written in the form $(x-\alpha)(x-\beta)=0$.
9 According to the fundamental theorem of algebra, a polynomial equation of degree $n$ has $n$ roots, provided we count complex roots and repeated roots.
10 Equations in general can be tackled by graphical, numerical and iterative techniques, but each case must be treated on its merits.
11 When solving equations that involve physical quantities, don't introduce numbers until it is essential; leave all the arithmetic until the end.
12 Always check that your answers satisfy the original equations and that they make sense.

### 5.2 Achievements

After completing this module, you should be able to:
A1 Define the terms that are emboldened and flagged in the margins of the module.
A2 Solve simple linear equations by rearrangement.
A3 Recognize an equation that is linear in a function of the unknown, and rearrange it for solution.
A4 Solve linear simultaneous equations by substitution, provided they are consistent and independent.
A5 Use graphical methods to solve simultaneous linear equations.
A6 Recall the general form of a quadratic equation and sketch the graph of any given quadratic function.
A7 Relate the coefficients of a given quadratic equation to the sum and product of the roots of that equation.
A8 Factorize a simple quadratic function, and so solve the corresponding quadratic equation.
A9 Explain the method of completing the square, and use it to solve a quadratic equation.
A10 Write down the general formula for the roots of a quadratic equation, and use it to solve any particular equation.

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| :--- | :---: | :---: | :---: |
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A11 Recall the definition of the discriminant, and explain how it determines the nature of the solutions of a quadratic equation.
A12 Recall the nature and significance of complex numbers, and use them in the solution of quadratic equations.
A13 Solve simple polynomial equations by factorization.
A14 Describe the use of graphical methods in the solution of general equations, and use them to locate the roots of various equations.
A15 Use the methods of bisection and regula falsi in the solution of simple equations.
A16 Use iterative methods in the solution of equations given the iteration formula.

Study comment You may now wish to take the Exit test for this module which tests these Achievements. If you prefer to study the module further before taking this test then return to the Module contents to review some of the topics.
$\square>$

### 5.3 Exit test

Study comment Having completed this module, you should be able to answer the following questions, each of which tests one or more of the Achievements.

## Question E1

(A2) Find $x$ if $2(x-1)+3(x-6)=0$.

## Question E2

(A3) Find $x$ if

$$
\frac{1}{x-2}+2\left(\frac{1}{x-2}-6\right)=0
$$

## Question E3

(A4) Solve the following simultaneous equations:

$$
2 x+y=-4 \quad 3 x+5 y=1
$$

## Question E4

(A4 and A5) Solve the following simultaneous equations first by drawing suitable graphs, then algebraically

$$
x+y=k \quad 2 x-y=1
$$

for $k=2,5$ and 8 .

## Question E5

(A6 and A11) Sketch graphs to show quadratic functions with (a) two distinct zeros, (b) two coincident zeros, and (c) no real zeros.

## Question E6

(A7 and A8) Solve the following equations by factorizing the left-hand sides:
(a) $x^{2}+5 x-14=0$
(b) $2 x^{2}-21 x+27=0$

Check your answers by showing that the sum and product of the roots are correct in each case.

## Question E7

(A9 and/or A10) Solve the following equations:
(a) $x^{2}-2 x-1=0$
(b) $3 x^{2}+x-2=0$

## Question E8

(A11) State the conditions for the equation (in the variable $x$ )

$$
x^{2}-2 r x+s=0
$$

to have (a) two real roots, and (b) repeated roots.

## Question E9

(A10, A11 and A12) Write down the roots of the equation in Question E8 above if neither of the conditions quoted there is fulfilled (so that the equation has complex roots).

## Question E10

(A13) Guess one of the roots, and then solve the following equation by factorization:

$$
x^{3}-4 x^{2}+x+6=0
$$

## Question E11

(A14) Sketch the graph of the function

$$
g(x)=3 x^{3}-11 x^{2}+5 x+8
$$

and give approximate values for its zeros. (You may use your calculator.)

## Question E12

(A15) Use a numerical method (and the results of Question E11 above) to find the largest root of the equation $3 x^{3}-11 x^{2}+5 x+8=0$ to two decimal places. (You may use your calculator.)

Study comment This is the final Exit test question. When you have completed the Exit test go back to Subsection 1.2 and try the Fast track questions if you have not already done so.

If you have completed both the Fast track questions and the Exit test, then you have finished the module and may leave it here.

