Module M2.2 Introducing co-ordinate geometry

1 Opening items

1.1 Module introduction

1.2 Fast track questions

1.3 Ready to study?

2 Straight lines

2.1 Cartesian coordinates

2.2 Equation of a straight line — gradient and *y*-intercept

2.3 Straight line with a given gradient through a given point

2.4 Other forms of the equation for a straight line

2.5 Intersection of two lines

2.6 Distance between two points

3 Circles

3.1 Equation of a circle, given its centre and radius

3.2 Tangents to a circle

3.3 Polar coordinates

4 Three-dimensional geometry

4.1 Points in three dimensions

4.2 Planes in three dimensions

4.3 Lines in three dimensions

5 Closing items

5.1 Module summary

5.2 Achievements

5.3 Exit test

Exit module

1 Opening items

1.1 Module introduction

You probably think of geometry as a subject that involves pictures of points, lines, curves, surfaces and so on; and indeed the origins of the topic are intimately related to the figures that one might draw on a piece of paper. However, if you stick to this classical approach to the subject, you very soon find that the pictures become exceedingly complicated and difficult to visualize; and particularly so in three dimensions. A break-through came in the 17th century when René Descartes (1596–1650) showed that algebraic methods could be applied to geometry, and formulated the subject that we know today as coordinate (or Cartesian) geometry. Of course we continue to use diagrams, but we are no longer bound by them, and most of our arguments are entirely algebraic.

You have almost certainly met the idea of a *Cartesian coordinate system* in school (although you may not have called it that), and so our first task in this module will be to review some of the fundamental ideas. In Sections 2 and 3 we are concerned solely with geometry in two dimensions, but in Section 4 we discuss briefly how some of the ideas can be extended to three dimensions.

The essence of the Cartesian approach to geometry is to specify the position of a point by distances, in mutually perpendicular directions from a given fixed point. In a Cartesian coordinate system two distances are needed to specify the location of a point on a flat surface (known as a *plane*) and these distances are called *coordinates*. The given fixed point is called the *origin* (of coordinates) and the two directions are called *axes*: one of these is known as the x-axis and the other as the y-axis and the coordinates are written as an *ordered pair* of numbers. Thus the point represented by the pair (2, 3) has x-coordinate equal to 2 and y-coordinate equal to 3.

Having specified the points in a plane by a Cartesian coordinate system, it is natural to ask how we can determine the distance between two points in terms of their coordinates. We answer this question in Subsection 2.6.

Apart from a point, a straight line is the simplest geometric entity, and we show, in Subsection 2.2 that it can be represented by a simple equation relating values of y for points on the line to corresponding values of x. There is just one equation for a given line, but in Subsections 2.3 and 2.4 we will see that this equation can be manipulated into various equivalent forms in order to provide the information that we require.

In Section 3 we take an introductory look at the equation of a circle. In Subsection 3.1 we first derive the equation for a circle with its centre at the origin; then we generalize this equation to cover the cases where the centre is at any point in the plane. In Subsection 3.2 we derive equations for the tangents to a circle, in cases where the tangent passes through a given point outside the circle and in cases where the point of contact on the circle is specified. Subsection 3.3 deals briefly with an important alternative coordinate system — the *polar coordinate* system.

Section 4 extends some of the earlier results to three dimensions. Subsection 4.1 deals with points in three dimensions and the calculation of the distance between any two of them. Subsection 4.2 presents the equation of a plane surface which has similarities with the general form of the equation of a line in two dimensions. Finally, Subsection 4.3 considers the specification of a line in three dimensions.

Study comment Having read the introduction you may feel that you are already familiar with the material covered by this module and that you do not need to study it. If so, try the *Fast track questions* given in Subsection 1.2. If not, proceed directly to *Ready to study?* in Subsection 1.3.

1.2 Fast track questions

Study comment Can you answer the following *Fast track questions*? If you answer the questions successfully you need only glance through the module before looking at the *Module summary* (Subsection 5.1) and the *Achievements* listed in Subsection 5.2. If you are sure that you can meet each of these achievements, try the *Exit test* in Subsection 5.3. If you have difficulty with only one or two of the questions you should follow the guidance given in the answers and read the relevant parts of the module. However, *if you have difficulty with more than two of the Exit questions you are strongly advised to study the whole module*.

Question F1

Find the equations of the following straight lines:

(a) the line with gradient -2 and y-intercept 3,

(b) the line with gradient 3 passing through the point $(-1, 5)$,

(c) the line through the points $(-2, -3)$ and $(-3, 1)$.

(d) the line with intercepts -1 and 2 on the x- and y-axes, respectively.

Question F2

Find the equations of the circles:

(a) centred at the origin and of radius 5,

(b) centred at (–1, –1) and of radius 2.

(c) Find the equations of the tangents to the first circle which pass through the point (4, 4).

(d) Find the equation of the tangent to the first circle at the point (–3, 4) on the circle.

Question F3

(a) What is the distance between the points (1, 2, 3) and (–1, 3, –2)?

(b) Write down the equations which determine the line joining these points.

(c) Does this line lie in the plane $x - 3y - z = -8$?

Study comment Having seen the *Fast track questions* you may feel that it would be wiser to follow the normal route through the module and to proceed directly to *Ready to study?* in Subsection 1.3.

Alternatively, you may still be sufficiently comfortable with the material covered by the module to proceed directly to the *Closing items*.

1.3 Ready to study?

Study comment

In order to study this module you will need to be familiar with the following terms: *circle*, *dimensions*, *distance*, *equation*, *function*, *intersection*, *inverse trigonometric functions* (*arcsin* (x), *arccos* (x) and *arctan* (x)), *line*, *modulus* $|x|$, *plane*, *point*, *simultaneous equations*, *tangent line*, *trigonometric ratios* and *variable.* You will need to be proficient in manipulating algebra and be able to apply *Pythagoras's theorem* (and its converse); and you will also need to be able to find the area of a triangle, and to solve *quadratic equations* and state the condition that ensures that such equations have two real roots. In addition, you will need to be familiar with *SI units*. One question requires the use of certain *trigonometric identities*, but these are provided in the text. If you are uncertain of any of these terms you can review them now by referring to the *Glossary* which will indicate where in *FLAP* they are developed. The following *Ready to study questions* will allow you to establish whether you need to review some of the topics before embarking on this module.

Note that in this module $\sqrt{x}$ means the *positive* square root of the non-negative number x.

Question R1

Which of the following are correct statements?

(a) $(x - 1)(y - 2) - 3(x - 5)(y - 4) - 12 = -2(x - 7)(y - 5)$ for all values of x and y.

(b) $\dfrac{3a + b}{a} = 3 + b$ for all values of a and b.

(c) The quadratic equation $x^2 + 2x + 1 = 5$ has just one real root.

(d) $(m^2 + 3m + 2)^2 = m^4 + 9m^2 + 4$ for all values of m.

Question R2

If two sides of a right-angle triangle are 5 m and 12 m, does it follow that the third side is 13 m?

Question R3

What is the definition of a circle ?

Question R4

What does it mean to say that ‘a line is a tangent to a circle’?

2 Straight lines

2.1 Cartesian coordinates

The framework that we use to draw pictures of curves is provided by a **Cartesian coordinate system**, consisting of a pair of straight lines at right angles, called **coordinate axes**, as shown in Figure 1. The two lines are infinitely long, but we draw only a portion of each near the point where they cross, which is known as the **origin**. The horizontal line is called the x-axis and the vertical line the y-axis, and both lines are scaled to indicate distance from the origin along the line. Distances to the right of the origin along the x-axis are positive while those to the left are negative; along the y-axis, distances above the origin are positive while those below are negative. It is usual to place arrowheads on the axes to show the direction of increasing x and y (since occasionally we may wish to change the orientation of the axes).

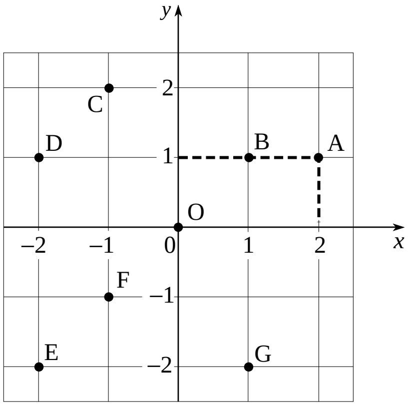

Figure 1 A Cartesian coordinate system and some points.

Any pair of values for x and y can be used to determine a point in the coordinate system. Thus, for example, the **ordered pair** (2, 1) is shown as the point A in Figure 1; a vertical line from A to the x-axis meets it at $x = 2$, while a horizontal line from A to the y-axis meets it at $y = 1$. Similarly, the points B and C in Figure 1 represent the ordered pairs of numbers (1, 1) and (−1, 2), respectively.

❖ What can you say about the coordinates of a point, (a) on the x-axis, (b) on the y-axis?

Question T1

Which values of (x, y) correspond to the points D, E, F, G, O in Figure 1? ❏

The points where a given line crosses the x- and y-axes are known as the x- and y-**intercepts**, respectively (although some authors refer to the y-intercept as *the* intercept).

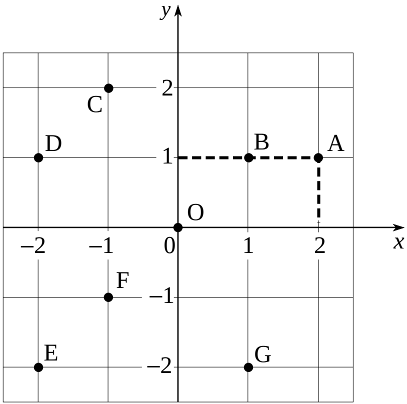

Figure 1 A Cartesian coordinate system and some points.

2.2 Equation of a straight line — gradient and *y*-intercept

A *linear function* is a *function* of the form $f(x) = mx + c$ where m and c are constants. For example, $f(x) = 2x + 1$ when $m = 2$ and $c = 1$, and, if we let y denote the values of $f(x)$, we may plot the graph of $y = 2x + 1$ as in Figure 2. It should come as no surprise that the graph is a straight line; this is after all why such functions are called *linear*.

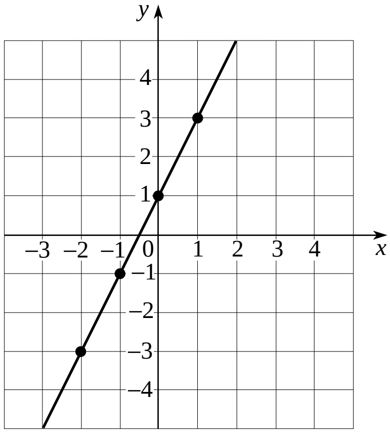

Figure 2 The graph of $y = 2x + 1$.

You may have used a *table of values* (as in Table 1) to construct such a graph.

In fact, once you know that the graph must be a straight line, it is quite unnecessary to tabulate all these values. You just need to know two well spaced points, such as (–3, –5) and (3, 7), then join them with a straight line (although it is wise to calculate a third point and check that it lies on the line that you have drawn).

It is essential to realize that the equation $y = 2x + 1$ is a true statement if, and only if, we choose a point (x, y) that lies on the line. For example, the values $x = 2.5$ and $y = 6$ correspond to the point (2.5, 6) which lies on the line because $6 = 2 \times 2.5 + 1$; on the other hand, the point (3.5, 7) does *not* lie on the line because $7 \neq 2 \times 3.5 + 1$. (Notice that we do not need to draw the graph in order to reach these conclusions.)

x	y
–3	–5
–2	–3
–1	–1
0	1
1	3
2	5
3	7

Table 1 A table of values for $f(x) = 2x + 1$.

✦

Do the following points lie on the line $y = 2x + 1$:

(a) (0.5, 2), (b) (989.13, 1979.26), (c) (0.0013, 1.0025)?

The two constants, m and c, which characterize a linear function are called the **gradient** (or sometimes the **slope**) and the ***y*-intercept**, respectively.

Look again at Figure 2. Note that the straight line $y = 2x + 1$ crosses the y-axis at the point where $y = 1$, which is the value of c (i.e. the y-intercept). Also, from Table 1, we see that a unit increase in the value of x gives rise to an increase of two units in the value of y. This factor of 2 corresponds to the value of m, the gradient of the linear function, and, in graphical terms, it is this value that determines how steep the line will be. In the case of $m = 2$, one step to the right implies two steps up. ☞

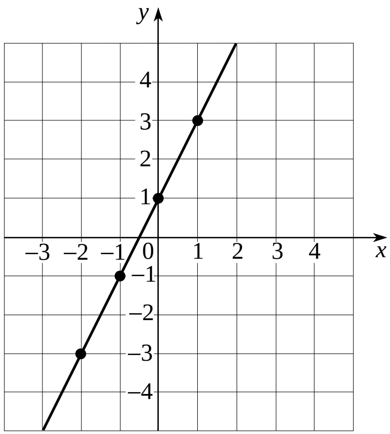

Figure 2 The graph of $y = 2x + 1$.

The **gradient–intercept form** of the equation of a straight line is

$$y = \underbrace{m}_{\text{gradient}}\, x + \underbrace{c}_{y\text{-intercept}} \qquad (1)$$

where m and c are constants.

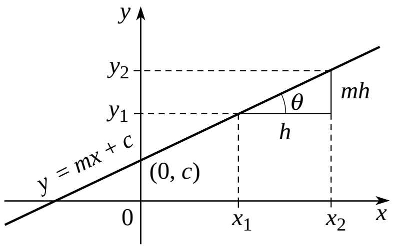

Figure 3 The gradient–intercept form of the straight line $y = mx + c$.

✦ Sketch the straight lines corresponding to the following choices:
(a) $m = 1$ and $c = 0$, (b) $m = 0$ and $c = 1$.

Figure 3 represents an arbitrary line with equation $y = mx + c$. The constant c is known as the y-intercept because the point $(0, c)$ on the y-axis lies on this line.

Suppose now that (x_1, y_1) and (x_2, y_2) are any two points lying on the line, then

$$y_1 = mx_1 + c$$

and $$y_2 = mx_2 + c$$

Subtracting these equations we have

$$y_2 - y_1 = mx_2 - mx_1$$

and it follows that $$m = \frac{y_2 - y_1}{x_2 - x_1} \qquad (2)$$

If we let $x_2 - x_1 = h$, as in Figure 3, then $y_2 - y_1 = mh$ and $\tan\theta = m$.

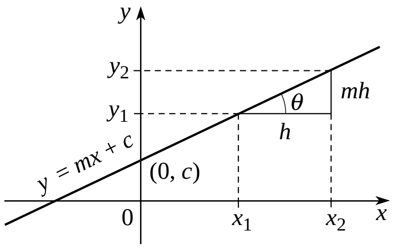

Figure 3 The gradient–intercept form of the straight line $y = mx + c$.

Figures 4a and 4b show, respectively, the effects on the line of different values of m and of c.

Varying the value of m, and keeping c fixed, will rotate the line about a fixed point (the intercept) on the y-axis. If m is positive then the graph slopes upwards from left to right, whereas if m is negative then the graph slopes downwards from left to right. Note that the graph is horizontal when $m = 0$.

Varying the value of c, and keeping m fixed, produces lines with the same gradient (they are parallel to one-another). If $c = 0$ the line passes through the origin; increasing c raises the line, while decreasing c lowers it.

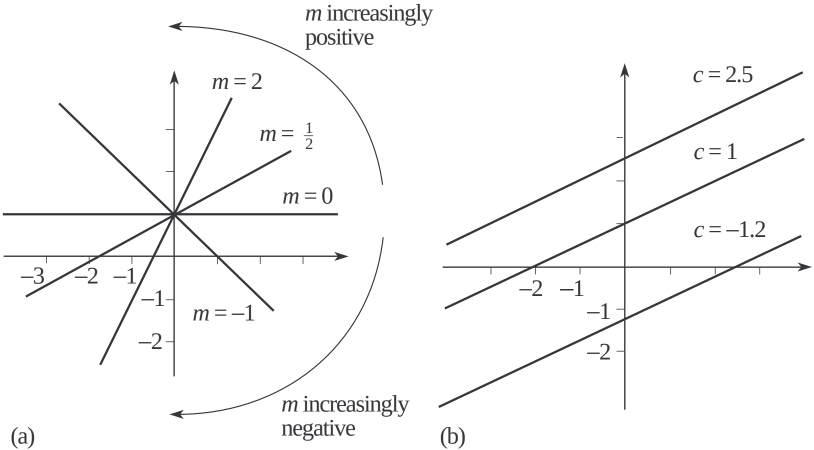

Figure 4 (a) Varying the gradient m, (b) varying the intercept c.

❖ What is the gradient of the line that passes through the points (–1.3, 2.8) and (–3.4, 1.2)?

❖ What is the gradient–intercept equation of the line that passes through the points (–1.3, 2.8) and (–3.4, 1.2)?

Equations of lines need not be in gradient–intercept form, for example the equation $x = 5y + 3$ represents a line, but its gradient and y-intercept are not immediately obvious. However, if we rearrange this equation into the form $y = \dfrac{1}{5}x - \dfrac{3}{5}$ it becomes clear that the gradient is 1/5 while the y-intercept is –3/5.

Question T2

What are the gradients and y-intercepts of the following lines?

(a) $y = 2 - x/3$ (b) $2y = 4x - 5$ (c) $2x + 3y = 5$

(d) $y = 3(x - 1)$ (e) $\dfrac{y-3}{x+2} = 1$ (f) $\dfrac{2}{x-1} = \dfrac{1}{y+3}$ ❏

2.3 Straight line with a given gradient through a given point

Suppose that we wish to find the equation of the line through the point (1, –2) with gradient 0.5. We know the gradient, so the equation must be of the form

$$y = 0.5x + c$$

and it remains to find the value of c. Using the fact that the point (1, –2) lies on the line we have $-2 = 0.5 + c$

from which it follows that $c = -2.5$

and so the required equation is $y = 0.5x - 2.5$.

Alternatively, we could use Equation 2, letting the known point (1, –2) be the first point on the line, and an arbitrary point (x, y) on the line be the second, then

$$0.5 = \frac{y - (-2)}{x - 1}$$

which can be rearranged to give $y = 0.5x - 2.5$ as before.

To obtain a general formula we can let the given point have coordinates (x_1, y_1) and let the gradient of the line be m, then for an arbitrary point (x, y) on the line, using Equation 2, we have $m = \dfrac{y - y_1}{x - x_1}$ which can be rearranged into

The **point–gradient form** of the equation of a straight line

$$y - y_1 = m(x - x_1) \tag{3}$$

i.e. $y = mx + y_1 - mx_1$

from which it is easy to see that the y-intercept is $y_1 - mx_1$.

❖ Given that a line has gradient −5 and passes through the point (1, 3) use the following methods to find its equation:

(a) by finding the value of c in the equation $y = mx + c$,

(b) and then by using Equation 2.

While it is quite acceptable to find the equation of the line by calculating the value of c, it is perhaps slightly easier to use Equation 2.

$$m = \frac{y_2 - y_1}{x_2 - x_1} \qquad \text{(Eqn 2)}$$

✧ What is the equation of the straight line through the point (–1, –2) with gradient 3?

Question T3

The equation $5 = \dfrac{y+2}{x-1}$ is the equation of a line.

(a) What is the gradient of the line?

(b) Does the line pass through the point (–1, 2)?

(c) What is the equation of this line in gradient–intercept form?

(d) What is the y-intercept of this line? ❏

2.4 Other forms of the equation for a straight line

Given two points on the line

Suppose that we are given two points, say (2, –5) and (7, 3), and that we are asked to construct the equation of the line joining them. A straightforward method would be to first use Equation 2 to find the gradient, and then to find c, the y-intercept. From Equation 2 we have

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3 - (-5)}{7 - 2} = \frac{8}{5}$$

If the equation of the line is $y = mx + c$, and using either one of the points on the line, (7, 3) say, we have $3 = \frac{8}{5} \times 7 + c$ so that $c = -41/5$.

Finally we can write the equation of the line is $y = \frac{8}{5}x - \frac{41}{5}$.

The above method is quite acceptable, but it is slightly easier to use Equation 2 twice.

$$m = \frac{y_2 - y_1}{x_2 - x_1} \qquad \text{(Eqn 2)}$$

First we find the gradient of the line to be 8/5 as before, but then we choose an arbitrary point (x, y) on the line, and use Equation 2 again with say (7, 3) as the first point and (x, y) as the second, which gives

$$m = \frac{y - 3}{x - 7}$$

Since we know the value of m to be 8/5, this gives

$$\frac{8}{5} = \frac{y - 3}{x - 7}$$

which can be rearranged to give $y = \dfrac{8}{5}x - \dfrac{41}{5}$.

We will now generalize this result, using the second of these methods.

Let the coordinates of two given points A and B be (x_1, y_1) and (x_2, y_2) respectively, ☞ and let P(x, y) be a general point on the line, as in Figure 5.

Using Equation 2 and the coordinates of points A and B we have

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

but using Equation 2 with the coordinates of points A and P gives us

$$m = \frac{y - y_1}{x - x_1}$$

Therefore equating these two expressions for m gives us

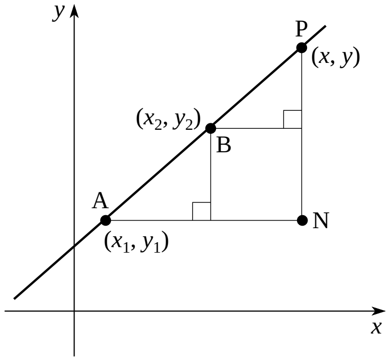

Figure 5 Graph of a straight line through two given points.

The **two-point form** of the equation of a straight line

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1} \tag{4}$$ ☞

✥ Find the equation of the straight line passing through the points A(2, 5), and B(−1, 3).

In physics we are often confronted with relationships which are certainly not linear, for example the periodic time T for small vibrations of a simple pendulum is given by the formula $T = 2\pi\sqrt{l/g}$ where l is the length of the pendulum and g is the magnitude of the acceleration due to gravity.

Such equations can sometimes be simplified by a change of variable, and in this case if we let $y = T$ and $x = \sqrt{l}$ then the equation becomes $y = (2\pi/\sqrt{g})x$ which is the equation of a line through the origin with gradient $2\pi/\sqrt{g}$.

Such an expression could form the basis for an experiment to determine the value of g. We could perhaps vary the length l of the pendulum and measure the corresponding times T, then plot y against x and measure the gradient. Once we have estimated this value it would be then a simple matter to estimate the value of g, as the following exercise shows.

✥ In the experiment described above only two measurements were made (allowing 10 cycles of the pendulum in each case) giving the following values.

$T = 1.4\,\text{s}$ when $l = 0.5\,\text{m}$, and $T = 1.7\,\text{s}$ when $l = 0.75\,\text{m}$.

Fit a straight line to the data in the manner described above, and hence estimate the value of g.

Given the intercepts on the axes

❖ Where does the line $\dfrac{x}{5}+\dfrac{y}{3}=1$ meet the axes?

The form of the above equation makes it particularly easy to find the two intercepts; but the reverse is also true, if the intercepts are given it is straightforward to write down the equation of the line.

❖ Which of the following equations is the equation of the line with intercepts (0, 2) and (7,0)?

(a) $\dfrac{x}{2}+\dfrac{y}{7}=1$ or (b) $\dfrac{x}{7}+\dfrac{y}{2}=1$

In Figure 6 the intercepts of the line on the *x*-and *y*-axes are A(a, 0) and B(0, b), respectively, and the equation of the line may be written using

The **intercept form** of the equation of a straight line

$$\frac{x}{a} + \frac{y}{b} = 1 \tag{5}$$

❖ What is the gradient of the line $\dfrac{x}{a} + \dfrac{y}{b} = 1$?

❖ Find the equation of the line with intercepts A(1/2, 0) and B(0, −6). What is its gradient?

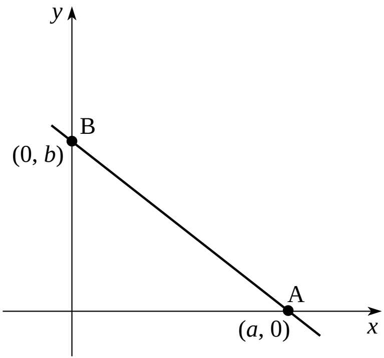

Figure 6 Graph of the straight line with given intercepts on the axes.

We can illustrate the use of Equation 5 with the following example.

$$\frac{x}{a} + \frac{y}{b} = 1 \qquad \text{(Eqn 5)}$$

At time t, the velocity v_x of an object moving in a straight line with constant acceleration a_x is given by $v_x = u_0 + a_x t$, where u_0 is its initial velocity in the x-direction.

✣ A particle is moving in a straight line with constant acceleration, and its velocity is found to be $v_{x1} = 10\,\mathrm{m\,s^{-1}}$ at time $t_1 = 3\,\mathrm{s}$, and $v_{x2} = 2.5\,\mathrm{m\,s^{-1}}$ at time $t_2 = 5\,\mathrm{s}$. Find the acceleration of the particle, estimate the time when its velocity will be zero, and also the initial velocity of the particle.

General form of the equation of a straight line

The previous forms of the equation of a straight line have a deficiency which is not immediately apparent, as the following example is intended to show.

Example 1 Find the equation of the straight line passing through the points (1, 2) and (1, 4). Draw the line.

Solution If we attempt to use Equation 4

$$\frac{y-y_1}{x-x_1}=\frac{y_2-y_1}{x_2-x_1} \qquad \text{(Eqn 4)}$$

we obtain $\dfrac{y-2}{x-1}=\dfrac{4-2}{1-1}=\dfrac{2}{0}$

but 2 divided by zero is undefined!

On the other hand, we might try to find the constants m and c in the general form $y = mx + c$, in which case the point (1, 2) on the line gives

$$2 = m + c$$

while the point (1, 4) on the line gives the equation

$$4 = m + c$$

Clearly it is impossible for $m + c$ to be equal to both 2 and 4, and we conclude that the line cannot be written in the form $y = mx + c$.

The form of the equation of the line becomes clear if we list some points that lie on it. Each of the points (1, 0), (1, –2), (1, 2), (1, 2.3), (1, 99) lies on the line, and for each of them the x value is 1; in fact this is the property which characterizes such points and the equation of the line is $x = 1$, and the line is parallel to the y-axis. ❏

Any line parallel to the y-axis has an equation $x = b$ for some constant b.

It is often useful to have a general form for the equation of a line that includes *all* possible lines. Thus

The general equation of a line in two dimensions is

$$ax + by + c = 0 \tag{6}$$

Note that by putting $b = 0$ we obtain the equation $ax + c = 0$ from which we find that $x = -\dfrac{c}{a}$ which is a line parallel to the y-axis.

✦ If $b \neq 0$ find the gradient and intercept of the line $ax + by + c = 0$.

Question T4

Write the equation $y = -3x - 2$ in each of the three forms described in this subsection, corresponding to Equations 4, 5 and 6.

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1} \qquad \text{(Eqn 4)}$$

$$\frac{x}{a} + \frac{y}{b} = 1 \qquad \text{(Eqn 5)}$$

$$ax + by + c = 0 \qquad \text{(Eqn 6)}$$

(For Equation 4 you will need to identify two points on the line; choose these to be the points with x-coordinates 1 and –1, respectively.) ❏

2.5 Intersection of two lines

Parallel lines and perpendicular lines

If two lines have the *same gradient* then they are *parallel*. For example, the lines with equations $y = 2x + 3$ and $y = 2x - 1$ are parallel.

✦ Are the lines $y = x + 1$, $y = -x + 1$ parallel?

Example 2 What is the relationship between the lines OA and OC in Figure 7?

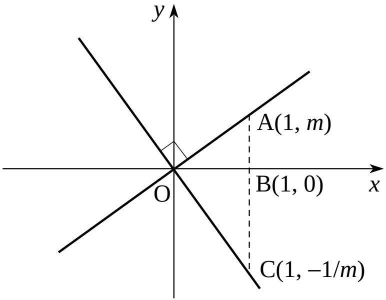

Figure 7 Two perpendicular lines through the origin.

Solution The line OA in Figure 7 joins the points (0, 0) and (1, m), and its equation is

$$y = mx$$

The line OC joins the points (0, 0) and (1, $-1/m$) and its equation is

$$y = -x/m$$

Notice that applying Pythagoras's theorem to triangle OBA gives us

$$\mathrm{OA}^2 = \mathrm{OB}^2 + \mathrm{BA}^2 = 1 + m^2$$

and applying the same theorem to triangle OBC gives us

$$\mathrm{OC}^2 = \mathrm{OB}^2 + \mathrm{BC}^2 = 1 + \frac{1}{m^2}$$

It follows that

$$\mathrm{AC}^2 = (\mathrm{AB} + \mathrm{BC})^2 = \left(m + \frac{1}{m}\right)^2 = (1 + m^2) + \left(1 + \frac{1}{m^2}\right) = \mathrm{OA}^2 + \mathrm{OC}^2$$

and therefore (from the converse of Pythagoras's theorem) the angle AOC is a right angle. ❏

The essential point to notice from Example 2 is that we have two lines through the origin, $y = mx$ and $y = -x/m$, which must be perpendicular to each other for any value of m. This means, for example, that we can see instantly that the two lines $y = 2x$ and $y = -0.5x$ are perpendicular because the product of their gradients is -1.

We quote without proof the general result that if the equations of two lines are

$y = m_1x + c_1$ and $y = m_2x + c_2$ then:

> Two lines of gradient m_1 and m_2 are *perpendicular* if
>
> $$m_2m_1 = -1 \qquad (7)$$

❖ Are the lines $y = 2x + 3$ and $x + 2y = 4$ perpendicular?

Note that the lines $x = a$, $y = b$, where a and b are constants, are perpendicular. We cannot use Equation 7 in this case (but we do not need to).

Intersection of two lines

Two lines that are not parallel intersect at a point. In order to find the coordinates of the point of intersection we could certainly draw the graphs of the lines on the same set of axes, and find where they cross. The lines $y = x - 1$ and $x + y = 3$, which appear to intersect at (2, 1), are shown in Figure 8.

A graphical approach is sometimes useful, but it can be laborious and rather inaccurate, so we would prefer an algebraic method.

The equation $y = x - 1$ is a true statement only for points (x, y) lying on the first line, while the equation $x + y = 3$ is a true statement only for points (x, y) lying on the second line. If we are told that *both* equations are true, then we must be referring to the particular point (x, y) that lies on both lines.

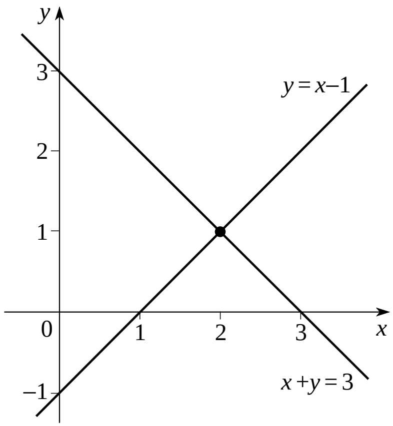

Figure 8 Intersection of two lines.

A pair of equations such as

$$\left.\begin{array}{rr} & y = x - 1 \\ \text{and} & x + y = 3 \end{array}\right\}$$

which are true for the same values of x and y are known as *simultaneous equations*, and they can be solved to find these values of x and y. In this instance we already suspect (from the graph) that the values $x = 2$ and $y = 1$ are the solution, and we have merely to verify that this is so by substituting these values into the equations. We do indeed find that both equations are satisfied, because $1 = 2 - 1$ (the first equation) and $2 + 1 = 3$ (the second equation).

Algebraic solution

There are several methods of solving a pair of simultaneous equations, but they all reduce to essentially the same idea — we use one of the equations to eliminate one of the variables, then solve the resulting equation in the remaining variable.

We will use the equations

$$y = x + 1 \qquad \text{(i)}$$

$$2x + y = 3 \qquad \text{(ii)}$$

to illustrate the general methods. Notice that we have numbered the equations in order to clarify the subsequent algebraic steps, and you should do likewise in your own work.

Method 1 — solving by direct substitution

We substitute for y using Equation (i) into Equation (ii) to obtain

$$2x + (x + 1) = 3$$

so that $\quad 3x + 1 = 3$

i.e. $\quad 3x = 2$

and therefore $\quad x = \dfrac{2}{3}$

Substituting this value for x into Equation (i) we find that $y = \dfrac{5}{3}$.

We then check that these values also satisfy Equation (ii),

and we find $\quad 2x + y = 2 \times \dfrac{2}{3} + \dfrac{5}{3} = \dfrac{4}{3} + \dfrac{5}{3} = \dfrac{9}{3} = 3$.

Method 2 — solving by indirect substitution

We re-write the two equations as

$$-x + y = 1 \qquad \text{(i)}$$

$$2x + y = 3 \qquad \text{(ii)}$$

We eliminate y by subtracting Equation (i) from Equation (ii) to obtain

$$2x - (-x) + y - y = 3 - 1$$

i.e. $3x = 2$ as before.

Then we can use either Equation (i) or Equation (ii) to calculate y and use the other equation as a check.

Alternatively, we could begin by eliminating x, and multiply Equation (i) by 2 and add the result to Equation (ii).

$$-x + y = 1 \qquad \text{(i)}$$

$$2x + y = 3 \qquad \text{(ii)}$$

$$-2x + 2x + 2y + y = 2 \times 1 + 3$$

i.e. $$3y = 2 + 3 = 5$$

so that $$y = \frac{5}{3}$$

Either Equation (i) or (ii) yields $x = 2/3$ and the other equation can be used as a check.

✥ Use the value $y = 5/3$ in Equation (ii) to obtain the value $x = 2/3$ and check the result using Equation (i).

The choice of method is often a matter of taste, although sometimes the numbers involved ensure that one method is quicker than another. It can be helpful to sketch the graphs of the two lines in order to get a rough idea of where they intersect, and perhaps to give an extra check on the results.

Strange cases

There are some particular cases that may lead you to some strange results. Suppose we were asked to find the point of intersection of the lines

$$x - 2y = 3 \quad \text{and} \quad 2x - 4y = 6$$

and let us suppose that inspiration strikes and you guess the solution $x = 5$ and $y = 1$. You try these values in the equations, and you find that they work; so everything is fine and you have found the solution. Right?

However, the values $x = -1$, $y = -2$ also satisfy both equations.

How can this be? How can a pair of lines intersect at more than one point?

In fact, the two equations both represent the *same* straight line. (You can obtain the second equation by multiplying the first equation by 2.) *Any* point on the common line has values of x and y which satisfy both equations.

Question T5

Find where the following pairs of lines intersect:

(a) $y = 3x - 1$, $y = -2x + 2$,

(b) $x + 2y = -1$, $y - 3x = 4$,

(c) $2x + 5y = 10$, $-x + 4y = 2$,

(d) $2x + 4y = 8$, $y = 2 - (1/2)x$. ❏

Ill-conditioned equations

Example 3 Solve the (rather unpleasant) pair of simultaneous equations

$$x - \pi y = 1 \qquad \text{(i)}$$

and $$2000x - 6283y = 1995 \qquad \text{(ii)}$$

Solution As a first step simplify Equation (i) a little by replacing π by 3.14 say, so that Equation (i) becomes

$$x - 3.14y = 1 \qquad \text{(iii)}$$

now solving Equations (ii) and (iii) (and we won't bore you with the details) you would find a solution $x = \dfrac{187}{30} \approx 6.23$ and $y = \dfrac{5}{3} \approx 1.67$, and everything appears to be fine. However, what if you were to choose a slightly better approximation for π, will this make much difference to the final result?

Suppose that you replace π by 3.141 59, then Equation (i)

$$x - \pi y = 1 \qquad \text{(i)}$$

becomes

$$x - 3.14159y = 1 \qquad \text{(iv)}$$

and solving Equations (ii) and (iv)

$$2000x - 6283y = 1995 \qquad \text{(ii)}$$

(and again we won't bother with the details) you would find

$$x = -\frac{310\,559}{3600} \approx -86.27 \text{ and } y = -\frac{250}{9} \approx -27.78$$

This appears to be a disaster! A minor difference in the approximation for π gives startlingly different results. Why has it happened? ❏

Perhaps another example will make things clearer.

Example 4 Consider the equations

$$x + 2y = 4 \qquad \text{(i)}$$

$$20x + 40.1y = 80.1 \qquad \text{(ii)}$$

Solution If we subtract 20 times Equation (i) from Equation (ii) we obtain

$$0.1y = 0.1$$

so that $y = 1$ and from Equation (i) we obtain $x = 2$. This is the (exact) unique solution. The lines intersect at the point (2, 1). However, if you look at Figure 9 you will see that the two lines are very close together and it is hard to detect precisely where the lines meet.

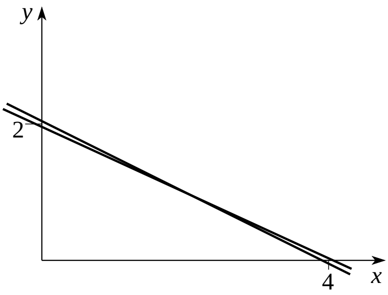

Figure 9 Graphs corresponding to ill-conditioned equations.

If the coefficient 40.1 is altered to 40, which is a change of about 1 in 400, the second equation becomes $20x + 40y = 80.1$, yet when we multiply the first equation by 20 we obtain $20x + 40y = 80$, which is a direct contradiction — therefore the equations have no solution (the lines are parallel). If, in addition, the value 80.1 is altered to 80 then the second equation becomes $20x + 40y = 80$ which is exactly 20 times the first equation; the two lines coincide. ❏

In such cases, where the lines are almost parallel, very slight changes in the given values of the constants can lead to massive changes in the calculated values of the solution. Such systems of equations are said to be **ill-conditioned**.

Question T6

Solve the following pairs of equations:

(a) $2x + y = 3$ and $x + 0.501y = 6$

(b) $2x + y = 3$ and $x + 0.499y = 6$ ❏

2.6 Distance between two points

We have already seen in Example 2 how Pythagoras's theorem can be used to find distances, but here we discuss the distance between two arbitrary points P and Q, as shown in Figure 10. The point R has the same y-coordinate as that of P and the same x-coordinate as that of Q.

The distance PR is the difference in the x-coordinates of P and R, i.e. $(x_2 - x_1)$. The distance RQ is the difference in the y-coordinates of R and Q, i.e. $(y_2 - y_1)$. Applying Pythagoras's theorem to the triangle PQR gives the equation $(\mathrm{PQ})^2 = (\mathrm{PR})^2 + (\mathrm{RQ})^2$, i.e. $(\mathrm{PQ})^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$.

Hence the distance we require is given by

$$\mathrm{PQ} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \qquad (8)$$

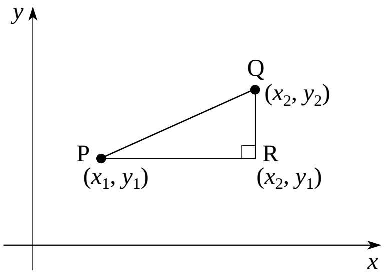

Figure 10 Distance between two points.

❖ Find the distance between the points P(2, –1) and Q(3, 4).

Question T7

Which two of the points (2, 3), (–1, 2), (3, –1) and (–2, –2) are (a) closest to each other (b) furthest apart ? ❏

3 Circles

3.1 Equation of a circle, given its centre and radius

Our objective in this section is to determine the general equation of a circle, but first we consider a simple case, when the centre of the circle is at the origin. The point P lies on the circumference of the circle as shown in Figure 11a. Since OP is the radius of the circle its length will be the same whatever the choice of $P(x, y)$ on the circumference, i.e. $OP = R$ where R is the radius of the circle. But, using Equation 8, $(OP)^2 = (x - 0)^2 + (y - 0)^2 = x^2 + y^2$ and therefore the equation of the circle is

$$x^2 + y^2 = R^2 \tag{9}$$

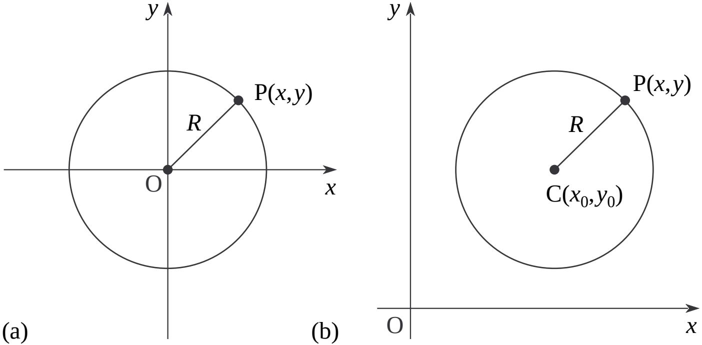

Figure 11 Circles.

❖ What is the radius of the circle $x^2 + y^2 = 4$?

In the general case where the centre of the circle, C, is the point (x_0, y_0) in Figure 11b, it is the distance CP which is constant and equal to R. Therefore, for an arbitrary point $P(x, y)$ on the circle, $(CP)^2 = R^2$ and so:

> The standard **equation of a circle** with centre (x_0, y_0) and radius R is
>
> $$(x - x_0)^2 + (y - y_0)^2 = R^2 \qquad (10)$$

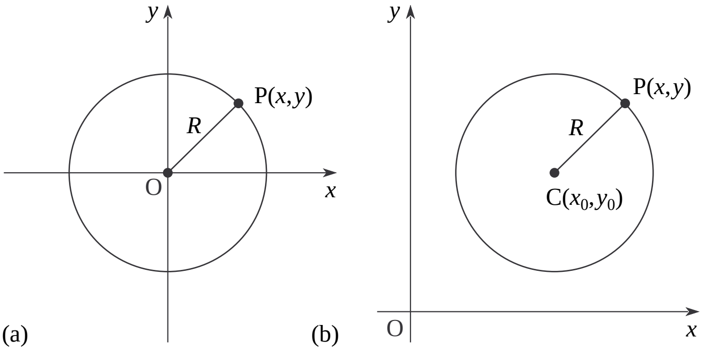

Figure 11 Circles.

❖ Write down the equations of the following circles:

(a) with centre (1, 2) and radius 5; (b) with centre (–3, –2) and radius 3.

✦ Where does the line $y = x + 1$ meet the circle $(x - 1)^2 + (y - 2)^2 = 4$?

Suppose that we wish to find the points of intersection of the circles

$$x^2 + y^2 = 1 \qquad \text{(i)}$$

and $$(x - 1)^2 + (y + 1)^2 = 4 \qquad \text{(ii)}$$

As a first step we may expand Equation (ii) to give

$$x^2 - 2x + y^2 + 2y = 2 \qquad \text{(iii)}$$

We can now subtract (i) from (iii) to give

$$-2x + 2y = 1 \qquad \text{(iv)}$$

which is the equation of a line. *These equations are all true provided that we are referring to the points* (x, y) *where the circles meet*. This means that the points where the circles meet must also lie on the line (iv). Instead of completing the calculation by finding these points we now invite you to carry out some similar calculations for yourself.

Question T8

Where do the circles $(x - 1)^2 + (y - 2)^2 = 25$ and $(x + 3)^2 + (y + 2)^2 = 9$ meet? ❑

3.2 Tangents to a circle

In this subsection we are mainly concerned with circles whose centres are at the origin.

We consider the intersections of a straight line QP with such a circle. As depicted in Figure 12, the line could

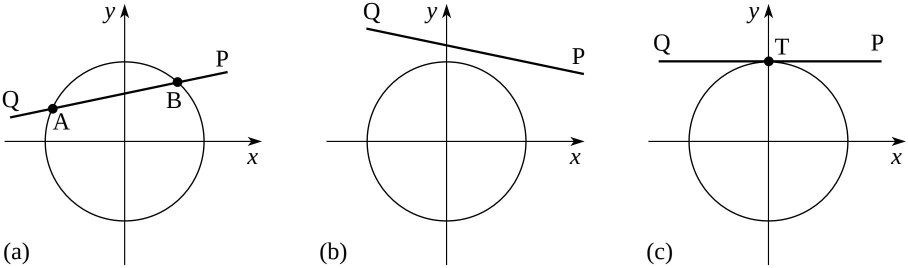

Figure 12 The intersection of a line and a circle.

- cut the circle at two points A and B as in diagram (a),
- not intersect the circle as in diagram (b),
- or touch the circle at one point *T* as in diagram (c).

In this last case the line is a **tangent** to the circle.

Let the equation of the straight line be

$$y = mx + c \qquad \text{(i)}$$

and the equation of the circle be

$$x^2 + y^2 = R^2 \qquad \text{(ii)}$$

To find where the line meets the circle we solve Equations (i) and (ii) simultaneously.

Substituting the expression for y from Equation (i) into (ii)

$$x^2 + (mx + c)^2 = R^2$$

which can be rearranged to give a quadratic equation in x:

$$(1 + m^2)x^2 + 2mcx + (c^2 - R^2) = 0 \qquad \text{(iii)}$$

Now, we know that the general quadratic equation

$$\alpha x^2 + \beta x + \gamma = 0$$

- has two real roots if $\beta^2 > 4\alpha\gamma$
- no real roots if $\beta < 4\alpha\gamma$
- one repeated real root if $\beta^2 = 4\alpha\gamma$

So, for the particular case of Equation (iii)

$$(1 + m^2)x^2 + 2mcx + (c^2 - R^2) = 0 \qquad \text{(iii)}$$

there will be one root and therefore one point of contact if $(2cm)^2 = 4(1 + m^2)(c^2 - R^2)$

i.e. $$4c^2m^2 = 4c^2 + 4m^2c^2 - 4R^2(1 + m^2)$$

i.e. $$c^2 = R^2(1 + m^2) \tag{11}$$

which can be rearranged to give

$$m^2 = \frac{c^2 - R^2}{R^2} \tag{12}$$

From Equation 11

$$c^2 = R^2(1 + m^2) \qquad \text{(Eqn 11)}$$

we see that for any given value of m there are two values of c; $c = R\sqrt{1+m^2}$ and $c = -R - \sqrt{1+m^2}$. These values of c together with the given value of m determine, via Equation (i), the two straight lines of gradient m that are tangential to a circle of radius R centred on the origin.

From Equation 12

$$m^2 = \frac{c^2 - R^2}{R^2} \qquad \text{(Eqn 12)}$$

we see that there are two values of m for any value of c provided that $c < -R$ or $c > R$. These values, $m = \sqrt{c^2 - R^2}\big/R$ and $m = -\sqrt{c^2 - R^2}\big/R$, together with the given value of c, determine, via Equation (i), the two straight lines passing through $y = (0, c)$ that are tangential to a circle of radius R centred on the origin. (Note that the condition $c < -R$ or $c > R$ ensures that the point $(0, c)$ is outside the circle and thus guarantees that there *are* two tangents passing through that point.)

✧ Given the circle $x^2 + y^2 = 1$ and the line $y = 2x + c$ find the values of c for which the line is a tangent to the circle and interpret your answer geometrically.

✦ Given the circle $x^2 + y^2 = 25$ and the point P(0,–10), find the equations of the lines through P that are tangents to the circle.

Equation of the tangent at a given point on a circle

We can obtain an interesting result if we consider an arbitrary point (x_1, y_1) and the points of intersection of the line

$$x_1x + y_1y = R^2 \qquad \text{(i)}$$

☞)

with the circle $\quad x^2 + y^2 = R^2 \qquad$ (ii)

The equation of the line (i) can be rewritten as $y = -\dfrac{x_1}{y_1}x + \dfrac{R^2}{y_1}$

(so that $m = -\dfrac{x_1}{y_1}$ and $c = \dfrac{R^2}{y_1}$) and this line will be a tangent to the circle if Equation 11 is satisfied,

i.e. if $$\left(\frac{R^2}{y_1}\right)^2 = R^2\left[1 + \left(-\frac{x_1}{y_1}\right)^2\right]$$

which reduces to $\quad x_1^2 + y_1^2 = R^2$

In other words:

the line $\quad x_1x + y_1y = R^2 \quad$ (13)

is a tangent to the circle $x^2 + y^2 = R^2$ provided that the point (x_1, y_1) lies on the circle.

✥ What is the equation of the tangent to the circle $x^2 + y^2 = 25$ at the point (4, 3) on the circle?

Question T9

(a) Show that the line $y = 33.8 - 2.4x$ is a tangent to $x^2 + y^2 = 169$.

(b) Where is the point of contact?

(c) Write down the equation of the parallel tangent and find its point of contact. ❑

So far we have only considered circles centred on the origin. However, by introducing a new system of coordinates, with axes X and Y and a different origin from that of the usual (x, y) system, we can investigate the equation of the tangent to a circle that is not centred on the origin. This is what we do in the next example.

Example 5 Given a circle of radius 5, centred on the point (–3, –4), what is the equation of a tangent to the circle through a given point on the circle.

Solution If we make a change of coordinates to $X = x - 3$ and $Y = y - 4$ then the origin in the (x, y) coordinate system becomes the point $(-3, -4)$ in the new (X, Y) coordinate system. If we rearrange the coordinate relations to give $x = X + 3$ and $y = Y + 4$, the circle $x^2 + y^2 = 25$, with radius 5 and centre at the origin in the (x, y) coordinate system, becomes the circle $(X + 3)^2 + (Y + 4)^2 = 25$ in the new coordinate system. In other words, its radius remains unchanged, but its centre is now at the point $(-3, -4)$ in the (X, Y) coordinate system.

We can similarly transform the tangent $xx_1 + yy_1 = 25$, where (x_1, y_1) is a point on the circle in the (x, y) coordinate system, to obtain

$$(X + 3)x_1 + (Y + 4)y_1 = 25$$

then if we put $x_1 = X_1 + 3$ and $y_1 = Y_1 + 4$ we obtain the equation of the tangent

$$(X + 3)(X_1 + 3) + (Y + 4)(Y_1 + 4) = 25$$

to the circle $(X + 3)^2 + (Y + 4)^2 = 25$, where (X_1, Y_1) is a point on the circle in the (X, Y) coordinate system (because $(X_1 + 3)^2 + (Y_1 + 4)^2 = x_1^2 + y_1^2 = 25$).

This method of obtaining results by making a simple transformation is often a very convenient means of obtaining useful results (see Question E5). ❏

Question T10

In this question we assume that the Earth is a perfect sphere. (You will need to use the trigonometric identities $\cos^2 A + \sin^2 A = 1$ and $\cos A \cos B + \sin A \sin B = \cos(A - B)$.)

(a) At time t, a point P has coordinates $x = a\cos(\omega t)$ and $y = a\sin(\omega t)$, where $a > 0$. Show that the point lies on a circle of radius a.

(b) Imagine now that you are looking down at the Earth from a point high above the North Pole, so that the Earth is rotating beneath you. You imagine the x- and y-axes to be two fixed directions (each perpendicular to the axis of the Earth). If P is a particular point on the Earth's equator (so that it rotates with the Earth), how long does it take the point P to move through a complete circle, and what is the value of its *angular speed* ω (in radians s^{-1})?

(c) A satellite S is placed in a circular orbit above the Earth's equator so that at time t its position is given by the coordinates

$$x = R\cos(\Omega t) \quad \text{and} \quad y = R\sin(\Omega t)$$

Take a to be the radius of the Earth. What condition involving a tangent to the Earth's equator must be satisfied whenever P is moving directly towards or away from the satellite? At what time t is this condition satisfed? ❏

3.3 Polar coordinates

Cartesian coordinates provide a valuable method of specifying the location of a point relative to a given origin, but they are not the only way in which such information can be specified. Other kinds of coordinate system exist and may be better suited to specific problems. A case in point is that of two-dimensional **polar coordinates**, which are particularly useful when distance from a fixed origin is of paramount importance.

The polar coordinate system is illustrated in Figure 14. The origin is represented by the point O and the point of interest by P. The line segment from O to P (sometimes called the **radius vector** of P) is of length r and is inclined at an angle θ, measured in the anticlockwise direction, from an arbitrarily chosen ray, called the **polar axis**, that emanates from O. The length r is called the **radial coordinate** and the θ the **polar angle** of P.

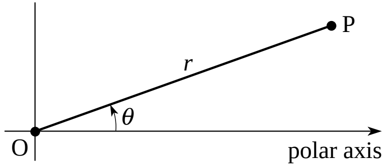

Figure l4 Polar coordinate system.

The *polar coordinates* r and θ of the point P and are usually combined so that P is represented by the ordered pair (r, θ).

Figure 15 shows four points P_1, P_2, P_3 and P_4 together with their polar coordinates.

Notice that an *anticlockwise* rotation is taken to be positive so that, for example, the point P_4 has polar coordinates (4, 315°). Since clockwise rotation is negative, and we could reach P_4 by a clockwise rotation of 45° from the polar axis, its polar coordinates could be written (4, −45°).

❖ What is an alternative polar representation of P_3?

The polar coordinates of a given point are not uniquely determined, because we can add or subtract any multiple 360° to the angle. We sometimes prefer polar coordinates to be completely specified, in which case the angle θ may be restricted to lie in the range $-180° < \theta \leq 180°$; (although there are occasions when the range $0 \leq \theta < 360°$ is more convenient).

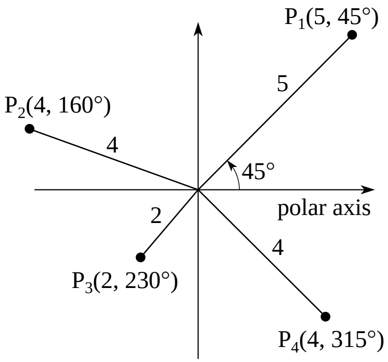

Figure 15 Some points and their polar coordinates.

Linking Cartesian and polar coordinates

If the origin of polar coordinates O coincides with the origin of Cartesian coordinates, and if the polar axis is chosen as the positive *x*-axis, we may link the Cartesian and polar coordinates as in Figure 16.

Since r is the hypotenuse of a right-angled triangle, it is clear that

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta \tag{14}$$

and these equations enable us to convert from polar to Cartesian coordinates.

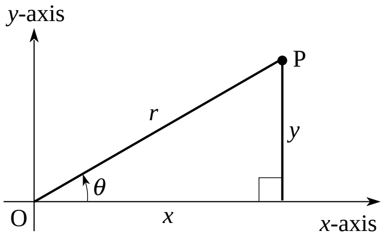

Figure 16 Polar and Cartesian coordinates.

✧

The polar coordinates of a point P are (5, 30°); what are its Cartesian coordinates?

It is also possible to deduce from Figure 16 that

$$r^2 = x^2 + y^2, \quad \sin\theta = y/r \quad \text{and} \quad \cos\theta = x/r \tag{15}$$

and these equations enable us to convert from Cartesian to polar coordinates. Since r is a positive quantity, its determination presents no difficulty; the same is not true for θ.

Example 6 Find the polar coordinates of the point P with Cartesian coordinates (3, 4).

Solution Since $r^2 = 3^2 + 4^2 = 25$, we have $r = 5$. It remains to find an angle θ such that $\sin\theta = 4/5 = 0.8$ and $\cos\theta = 3/5 = 0.6$.

If you use a calculator to find arcsin(0.8) and arccos(0.6) you will find that both give the same result $\theta \approx 53.13°$. Therefore the polar coordinates of P are approximately (5, 53.13°). ❏

Example 7 Find the polar coordinates of the point P with Cartesian coordinates (–3, 4).

Solution Since $r^2 = (-3)^2 + 4^2 = 25$, we have $r = 5$, as in Example 6. Also we have $\sin\theta = 4/5 = 0.8$ and $\cos\theta = 3/5 = -0.6$, and now when you use your calculator to find the angle you will find arcsin(0.8) ≈ 53.13° as before, but arccos(–0.6) ≈ 126.87°. This illustrates the limitations of your calculator, rather than a real problem in the mathematics.

The correct answer is $\theta \approx 126.87°$ as you can very easily verify. If you use your calculator to evaluate sin(126.87°) and cos(126.87°) you will soon see that sin(126.87°) ≈ 0.8 and cos(126.87°) ≈ –0.6.

So the polar coordinates are (5, 126.87°). ❏

You may have noticed that the two equations

$$\sin\theta = y/r \quad \text{and} \quad \cos\theta = x/r$$

can be combined to give $\tan\theta = y/x$, but specifying the tangent of an angle does not uniquely determine the angle, so this does not resolve the difficulty.

Example 8 Determine the polar coordinates of the points P_1 and P_2 with Cartesian coordinates (2, –2) and (–2, 2), respectively.

Solution In each case $r^2 = 2^2 + 2^2 = 8$ so that $r = 2\sqrt{2}$.

In each case $\tan\theta = -1$, and my calculator gives $\arctan(-1) = -45°$, but P_1 and P_2 are clearly different points. The answer to our problems is to construct a sketch and plot the points, see Figure 17.

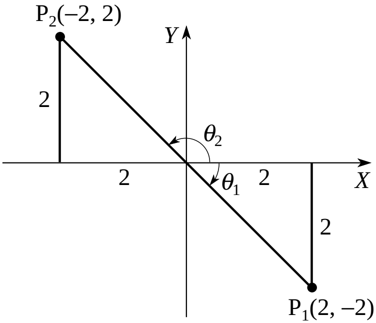

Figure 17 Points whose polar angles differ by 180°.

If we plot the point (2, –2) we can see at once that the angle θ_1 is 45°, and, since it is measured clockwise from OX the polar coordinates of P_1 are $(2\sqrt{2}, -45°)$, or equivalently $(2\sqrt{2}, 315°)$. The angle θ_2 is 180° – 45° = 135° and the polar coordinates of P_2 are $(2\sqrt{2}, 135°)$. ❏

The message is clear:

> When determining the polar angle θ it is wise to *sketch* a diagram to show the position of the point.

4 Three-dimensional geometry

4.1 Points in three dimensions

We need two pieces of information to determine the position of a point in a plane, but in three dimensions we need to provide three *independent* pieces of information.

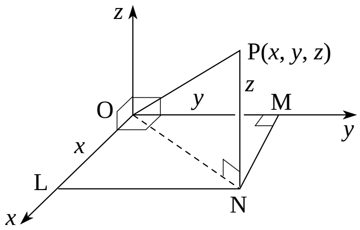

Figure 18 Three-dimensional Cartesian coordinate system.

Figure 18 shows a Cartesian coordinate system in three dimensions. The point N is 'vertically' below the point P so that the line NP is parallel to the z-axis.

A point O is chosen as the origin of coordinates and three mutually perpendicular rays x, y and z are drawn as shown. The position of P is specified by three coordinates: x, y and z as indicated; and we refer to the point as P(x, y, z). Having decided on the x- and y-axes, there are two possible directions, opposite to one another, in which the z-axis can be directed, so that it is normal (i.e. perpendicular) to both the x- and y-axes.

To remove any ambiguity about the choice of direction of the z-axis, a **right-handed Cartesian coordinate system** is almost invariably used. There are various ways of describing this system, but one simple method only is included here.

If the thumb and first two fingers of the right hand are arranged so that they are approximately mutually perpendicular as in Figure 19, then, if the first and second finger point along the x- and y-axes, respectively, the thumb points along the z-axis in a right-handed system.

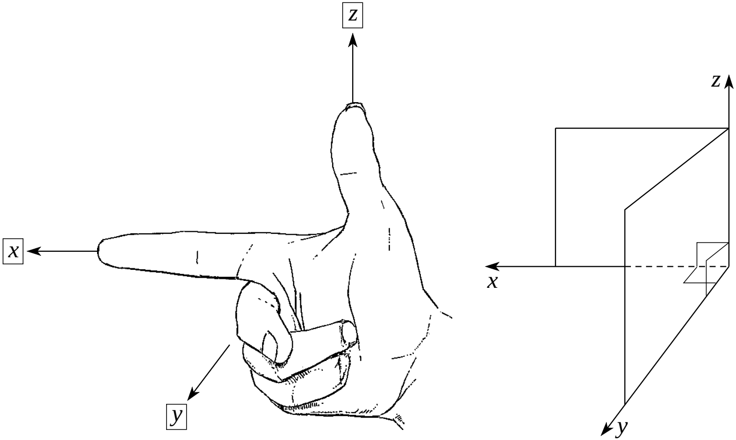

Figure 19 Picture of a hand showing the directions of the axes in a right-handed Cartesian coordinate system.

✧ Which of the following is a right-handed coordinate system:

(a) x points west, y points north and z points vertically upward,

(b) x is vertically downward, y points south and z points west?

Distance between two points

The distance OP in Figure 18 can be found by applying Pythagoras's theorem twice. In the horizontal plane which contains the x- and y-axes,

$$(\mathrm{ON})^2 = (\mathrm{OL})^2 + (\mathrm{LN})^2$$

In the vertical plane which contains the lines ON, NP and OP,

$$(\mathrm{OP})^2 = (\mathrm{ON})^2 + (\mathrm{NP})^2 = (\mathrm{OL})^2 + (\mathrm{LN})^2 + (\mathrm{NP})^2$$

Hence $\quad (\mathrm{OP})^2 = x^2 + y^2 + z^2$

Now we consider the distance between two general points.

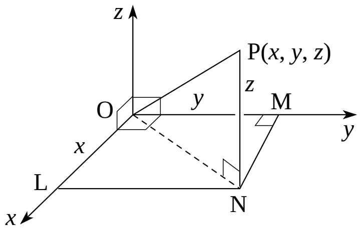

Figure 18 Three-dimensional Cartesian coordinate system.

Let the two points be $\mathrm{P}_1(x_1, y_1, z_1)$ and $\mathrm{P}_2(x_2, y_2, z_2)$. We can 'move' the point P_1 to the origin by subtracting x_1 from its first coordinate, y_1 from its second coordinate and z_1 from this third coordinate (this has the same effect as moving the origin to the point P_1). The same transformation will move P_2 to a point whose coordinates are $(x_2 - x_1, y_2 - y_1, z_2 - z_1)$. If we call this point P we can apply the result just obtained to determine the distance between P_1 and P_2. Thus the distance d between the points (x_1, y_1, z_1) and (x_2, y_2, z_2) is given by

$$\boxed{d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 \qquad (16)}$$

✦ What is the distance between the points (1, 2, –3) and (–2, 0, 4)?

It is worth noting that the formula for the distance between two points in two dimensions (Equation 8)

$$\mathrm{PQ} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \qquad \text{(Eqn 8)}$$

is the special case of the formula for three dimensions

$$d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 \qquad \text{(Eqn 16)}$$

with z_1 and z_2 both put equal to zero.

Question T11

Which of the following points are (a) closest together, (b) furthest apart (1, 2, 3), (–1, –2, –1), (2, 2, –2), (3, 0, 1)? ❑

4.2 Planes in three dimensions

A plane is a flat surface which has the property that a straight line joining any two points in the plane lies entirely in the plane itself.

The general **equation of a plane** in three dimensions is often written as

$$ax + by + cz = d \tag{17}$$

where a, b, c and d are constants. Note that if we choose $a = 1$ with $b = c = d = 0$ the equation becomes $x = 0$ which is the (y, z) plane.

✦ What values of a, b, c, d correspond to (a) the (x, y) plane, (b) the (z, x) plane?

Other planes can be described by choosing suitable combinations of the constants a, b, c and d. The plane $z = 2$ is parallel to the plane $z = 0$ (the (x, y) plane) and cuts the z-axis at $z = 2$, i.e. the point $(0, 0, 2)$. The equation $z = 2$ corresponds to Equation 17 with $a = 0$, $b = 0$, $c = 1$ and $d = 2$.

✥ The three points (1, 3, 5), (1, –2, 7), (1, –1, 0) lie in a plane.

What is the equation of the plane?

Note that if $d = 0$ in Equation 17

$$ax + by + cz = d \qquad \text{(Eqn 17)}$$

then the plane will pass through the origin, since $x = y = z = 0$ will satisfy the equation of the plane.

Intersection of two planes

If two planes are not parallel then they meet in a line. Imagine two consecutive pages of a book. Hold the pages taut and you see that they meet in the line of the binding connecting them. For example the (x, y) plane meets the (y, z) plane on the y-axis, this means that the y-axis corresponds to the points where both x and z are zero; in other words, we can see instantly that a point such as (0, 5, 0) must lie on the y-axis.

✥ Which axis corresponds to the intersection of the two planes $x = 0$ and $y = 0$?

❖ Where does the plane $x + y + 2z = 6$ meet the x-, y- and z-axes?

If we keep the values of a, b and c fixed in Equation 17,

$$ax + by + cz = d \qquad \text{(Eqn 17)}$$

but double the value of d, we obtain a plane that is parallel to the original plane (but twice as far from the origin). Compare the results of the following exercise with those of the previous exercise.

❖ Where does the plane $x + y + 2z = 12$ meet the x-, y- and z-axes?

Question T12

Find the equation of a plane that is parallel to the plane $3x - 4y + z = 2$ and passes through the point (1, 2, 3). ☞ ❏

4.3 Lines in three dimensions

In three dimensions, straight lines are often specified by the intersection of two planes. Of course we try to choose the planes so that the specification is as neat as possible; for example, we could specify a line as the intersection of the planes

$$3x - 2y = 5$$

and $$5y - 3z = -11$$

but this specification is not as neat as it might be.

The first of these equations can be rewritten in the form $\dfrac{x-1}{2} = \dfrac{y+1}{3}$ while the second can be rewritten in the form $\dfrac{y+1}{3} = \dfrac{z-2}{5}$, so that the two equations can be combined into the particularly neat form

$$\frac{x-1}{2} = \frac{y+1}{3} = \frac{z-2}{5} \qquad (18)$$

☞

In this form we can see instantly that the point (1, –1, 2) must lie on the line. The values 2, 3 and 5 are also significant, since they can be used to determine the direction of the line (but it would take us beyond the scope of this module to explain how this is done).

$$\frac{x-1}{2} = \frac{y+1}{3} = \frac{z-2}{5} \qquad \text{(Eqn 18)}$$

If we put each of the fractions in Equations 18 equal to s, then they can be written in the equivalent form

$$\left.\begin{aligned} x &= 1 + 2s \\ y &= -1 + 3s \\ z &= 2 + 5s \end{aligned}\right\} \qquad (19)$$

The variable s is then a parameter that moves the point (x, y, z) up and down the line as we change its value.

It is often this form of the equations of a line which is most useful, as the following exercise illustrates.

✦ Where does the line $\dfrac{x-1}{2} = \dfrac{y+1}{3} = \dfrac{z-2}{5}$ meet the plane $x + 2y + 3z = 1$?

In three dimensions only lines that lie in a plane will meet in a point. Indeed, in three dimensions, it is more likely that a pair of lines chosen at random will not meet, in which case they are said to be **skew** lines.

In general, the **equations of a line in three dimensions**

$$\frac{x-a}{l} = \frac{y-b}{m} = \frac{z-c}{n} \tag{20}$$

specify a line through the point (a, b, c) and the constants l, m, n are known as the **direction ratios** (or **direction cosines**) of the line.

✥ Find the equations of a line through the points (1, 2, 3) and (6, 5, 4).

Question T13

Given the equations of a line in the form

$x = 1 + 2s$, $y = -2 + s$, $z = 2 - 3s$

(where s can take any value), show that the point P(1, –2, 2) lies on the line. Find the points on the line that are at a distance $\sqrt{14}$ from P. Make s the subject of each equation ☞ and hence find the direction ratios for the line. ❏

5 Closing items

5.1 Module summary

1 In two dimensions the *Cartesian system* of coordinates consists of two perpendicular lines, called axes, which meet at the *origin* of coordinates; one of these lines, usually drawn horizontally, is designated the x-axis, the other, which is usually drawn vertically, is the y-axis. The x-coordinate of a point is the perpendicular distance of that point from the y-axis, with a negative sign attached if the point lies to the left of the y-axis. The y-coordinate of the point is its perpendicular distance from the x-axis, with a negative sign attached if the point lies below the x-axis.

2 The equation of a straight line can appear in a variety of forms; the one chosen depends on the information available about the line and the information likely to be required about the line.

3 If the line has a known gradient m and a known *intercept* (on the y-axis) c then the equation relating the x- and y-coordinates of a point on it is

$$y = mx + c \qquad \text{(Eqn 1)}$$

4 If the line has a known gradient m and passes through the point (x_1, y_1) then the equation of the line is

$$y - y_1 = m(x - x_1) \qquad \text{(Eqn 3)}$$

5 If the line meets the x-axis at $x = a$ and the y-axis at $y = b$ then the equation of the line is

$$\frac{x}{a} + \frac{y}{b} = 1 \qquad \text{(Eqn 5)}$$

6 The so-called general form of the equation of a line is

$$ax + by + c = 0 \qquad \text{(Eqn 6)}$$

7 If two lines are parallel then their gradients are equal, and if two lines are perpendicular then the product of their gradients is -1.

8 If two lines intersect then the coordinates of the point of intersection can be found by solving simultaneously the equations of the lines.

9 The distance between the points $P(x_1, y_1)$ and $Q(x_2, y_2)$ is given by

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \qquad \text{(Eqn 8)}$$

10 The equation of a circle centred at the origin and of radius R is

$$x^2 + y^2 = R^2 \qquad \text{(Eqn 9)}$$

11 The equation of a circle centred at the point (x_0, y_0) and of radius R is

$$(x - x_0)^2 + (y - y_0)^2 = R^2 \qquad \text{(Eqn 10)}$$

12 The equation of the tangent to the circle $x^2 + y^2 = R^2$ at a point (x_1, y_1) on the circle is given by

$$x_1x + y_1y = R^2 \qquad \text{(Eqn 13)}$$

13 In the *polar coordinate system* a point is specified by its distance r from the origin and the *polar angle* θ which it makes with a given line through the origin. The link between the two systems is provided by the pair of equations

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta \qquad \text{(Eqn 14)}$$

and $$r^2 = x^2 + y^2, \quad \sin\theta = y/r\,, \quad \cos\theta = x/r \qquad \text{(Eqn 15)}$$

where we must take care to identify the correct value of θ.

(This can often be done by plotting the point on a diagram, and then using the fact that $\tan\theta = y/x$.)

14 The distance d between the points (x_1, y_1, z_1) and (x_2, y_2, z_2) in three dimensions is given by

$$d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 \qquad \text{(Eqn 16)}$$

15 The general equation of a plane in three dimensions is

$$ax + by + cz = d \qquad \text{(Eqn 17)}$$

If two planes intersect, they do so in a line.

16 The general equations of a line in three dimensions are

$$\frac{x - a}{l} = \frac{y - b}{m} = \frac{z - c}{n} \qquad \text{(Eqn 20)}$$

where (a, b, c) is a point on the line and l, m, n are constants, known as *direction ratios* (or *direction cosines*) of the line.

5.2 Achievements

Having complete this module, you should be able to:

A1 Define the terms that are emboldened and flagged in the margins of the module.

A2 Set up a Cartesian coordinate system and plot points in it.

A3 Recognize the different forms of equation of a straight line and produce the appropriate equation from the information given about the line, then derive the required information from it.

A4 Convert one form of the equation of a line into another.

A5 Write down the equation of a circle, given its centre and radius, and deduce the centre and radius of a circle from its equation.

A6 Produce the equation of tangents to a circle, both from a given point outside the circle and at a given point of contact on the circle.

A7 Set up the polar coordinate system, plot points in it and convert from Cartesian coordinates to polar coordinates and vice versa.

A8 Calculate the distance between two points in three dimensions.

A9 Recognize the equation of a plane, and use it to indentify points that lie in the plane.

A10 Recognize the equations of a line in three dimensions and use them to identify points that lie on the line and the point at which a line intersects a given plane.

Study comment You may now wish to take the *Exit test* for this module which tests these Achievements. If you prefer to study the module further before taking this test then return to the *Module contents* to review some of the topics.

5.3 Exit test

Study comment Having completed this module, you should be able to answer the following questions, each of which tests one or more of the *Achievements*.

Question E1

(*A2*) Plot the following points on a coordinate system: A(–1, 4), B(3, –1), C(–4, –1), D(0, 2), E(0, –3).

Question E2

(*A3*) Find the equations of the following lines:

(a) with gradient –5 and *y*-intercept –2;

(b) with gradient –2, passing through the point (–1, 4);

(c) passing through the points (2, – 3) and (–1, –1);

(d) with intercepts at –2 on both axes.

Question E3

(*A4*) Rewrite each of the equations in Question E2 in the general form $ax + by = c$.

Question E4

(*A3*) Find the gradient and y-intercept for the line of Question E2(c) and the gradient of the line in Question E2(d).

Question E5

($A3$) A line $ax + by - \mathrm{c} = 0$, with $a \neq 0$ and $b \neq 0$ meets the x- and y-axes at the points A and B, respectively.

(a) Find the area of the triangle AOB (where O is the origin).

(b) Find the length AB.

(c) Find the shortest length h from the line to the origin O.

(*Hint*: The shortest line will be perpendicular to AB, and what is the area of the triangle AOB when AB is the base?)

(d) Find the coordinates of the point N where the line $y = 2$ meets the line $x = 5$.

(e) If we make a change to the coordinate system by putting

$$X = x - 5 \quad \text{and} \quad Y = y - 2$$

What are the new coordinates of the point N (in the XY system)?

(f) We can find the equation of the line AB in the new coordinate system if we put

$$x = X + 5 \quad \text{and} \quad y = Y + 2$$

into the equation $ax + by - \mathrm{c} = 0$. What does this equation become?

(g) What is the shortest distance from the point (5, 2) to the line $ax + by - \mathrm{c} = 0$?

Question E6

(*A3* and *A4*) Which of the lines $2x + 3y = 2$ or $2x - 3y = 2$ is perpendicular to the line $6x + 4y = 5$? Find the equation of the line perpendicular to the line $6x + 4y = 5$ which passes through the point (4, 4).

Question E7

(*A5*) Write down the equations of the circles of radius 3 which have centres at (1, –2) and (2, 1), respectively, and find where they meet.

Question E8

(*A6*) Find the point of intersection of the tangents to the circle $x^2 + y^2 = 4$ at the points $(1, \sqrt{3})$ and (0, –2).

Question E9

(*A7* and *A8*) Show that the points (1, 2, 1), (2, −3, −2) and (−1, 0, −5) lie on the plane $2x + y - z = 3$. Which two are the closest together?

Where does the line $\dfrac{x-1}{3} = \dfrac{y-1}{5} = \dfrac{z}{2}$ meet the plane $2x + y - z = 3$?

Study comment This is the final *Exit test* question. When you have completed the *Exit test* go back to Subsection 1.2 and try the *Fast track questions* if you have not already done so.

If you have completed **both** the *Fast track questions* and the *Exit test*, then you have finished the module and may leave it here.