

Module M2.4 Introducing scalars and vectors

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1 Opening items

1.1 Module introduction

Many of the quantities that physicists use are easy to specify; measurements of mass, length, time, area, volume and temperature can all be expressed as simple numbers together with appropriate units of measurement. Such quantities are known as *scalar quantities* or *scalars*. However, some quantities are more tricky to deal with. For instance, if you want to travel from one place to another you will not only want to know how far apart the two places are, you will also need to know the direction that leads from one to the other. The physical quantity that combines distance and direction is called *displacement* and is clearly more complicated than distance alone. Displacement is a simple example of a large class of physical quantities known collectively as *vector quantities* or *vectors*. A detailed understanding of vectors and how to use them is crucial to many parts of physics, particularly mechanics and electromagnetism.

This module provides an introduction to the mathematical treatment of vectors. Section 2 gives a definition of vector quantities that separates them from simpler scalar quantities — such as mass and distance — that are also defined. This section also describes the graphical representation of vectors and the notation used to distinguish vectors from scalars, both in print and in handwritten work. Section 3 introduces some of the basic operations of *vector algebra* such as *scaling* (multiplication by a scalar), *vector addition* and *vector subtraction*. It aims to make clear the meaning of vector equations such as $3\mathbf{a} - 2\mathbf{b} = \mathbf{0}$. The section ends with a discussion of the way in which a given vector may be split-up (*resolved*) into *component vectors* — a process that is of importance in many practical problems.

Throughout the module the emphasis is on basic ideas and geometric (graphical) methods. The algebraic methods that are more frequently used by those already familiar with vectors are mentioned briefly in a short conclusion (Section 4) but their full development is left to other modules. Since it is those methods that are best suited to tackling three-dimensional problems, most of the questions in this module are restricted to two-dimensional situations on idealized planes or flat surfaces. However, even though the discussion is mainly restricted to two dimensions, the significance of vectors is amply demonstrated as is the importance of always using vector notation clearly and correctly.

Study comment Having read the introduction you may feel that you are already familiar with the material covered by this module and that you do not need to study it. If so, try the [Fast track questions](#) given in Subsection 1.2. If not, proceed directly to [Ready to study?](#) in Subsection 1.3.

1.2 Fast track questions

Study comment Can you answer the following *Fast track questions*? If you answer the questions successfully you need only glance through the module before looking at the *Module summary* (Subsection 5.1) and the *Achievements* listed in Subsection 5.2. If you are sure that you can meet each of these achievements, try the *Exit test* in Subsection 5.3. If you have difficulty with only one or two of the questions you should follow the guidance given in the answers and read the relevant parts of the module. However, *if you have difficulty with more than two of the Exit questions you are strongly advised to study the whole module.*

Question F1

In terms of a strict interpretation of vector and scalar quantities, what is wrong with each of the following statements?

- (a) The velocity of light travelling through a vacuum (usually denoted by c) is approximately $3 \times 10^8 \text{ ms}^{-1}$.
- (b) The acceleration due to gravity (usually denoted by g) near the surface of the Earth is approximately 10 ms^{-2} .
- (c) The mass of your head pushes down on your neck.



Question F2

a is a displacement of 4.32 km due north, and ***b*** is a displacement of 2.40 km due east.

- (a) What is the magnitude and the direction of the displacement $3\mathbf{a} - 5\mathbf{b}$?
- (b) Given a specified line, any vector may be expressed as the sum of two (orthogonal) *component vectors*, one parallel to the given line and the other at right angles to that line. What is the magnitude of the (orthogonal) component vector of $3\mathbf{a} - 5\mathbf{b}$ that points in the direction 20° west of north?



Study comment Having seen the *Fast track questions* you may feel that it would be wiser to follow the normal route through the module and to proceed directly to [*Ready to study?*](#) in Subsection 1.3.

Alternatively, you may still be sufficiently comfortable with the material covered by the module to proceed directly to the [*Closing items*](#).

1.3 Ready to study?

Study comment Relatively little background knowledge is required to study this module. Terms such as mass, electric charge and temperature are used and it is assumed that you know something about measuring such quantities using SI units, but you do not need to know precise definitions or exact meanings. It is also assumed that you have met Newton's laws of motion before, but again you do not need to know those laws in detail in order to study the module. (Indeed, this module would be good preparation for a thorough study of Newton's laws.) There are some mathematical topics, however, with which you will need to be familiar before starting to study; they are the subject of the following questions. Consult the Glossary for references to any terms that are unfamiliar.

Question R1

Draw a set of *two-dimensional Cartesian axes*. Label the horizontal axis x and the vertical axis y . Using the axes you have drawn, plot the three points A, B and C the *coordinates* of which are given in Table 1. Draw the *right-angled triangle* that has the points A, B and C at its corners. Write down the lengths of sides AB and BC of that triangle, and use *Pythagoras's theorem* to work out the length of the side CA.

Table 1 See Question R1.

point	x -coordinate	y -coordinate
A	1	1
B	-3	1
C	-3	4



Question R2

Using the triangle given in the answer to Question R1 (Figure 15), and indicating the length of side CA by c and the angle \hat{CAB} by θ , write down expressions for the lengths of sides AB and BC in terms of the length c , the angle θ , and the basic [*trigonometric ratios*](#). What is the value (in degrees) of the angle θ ?



Question R3

The [*modulus*](#) of a quantity x is written $|x|$ and represents the [*absolute value*](#) of x . Evaluate the following expressions and complete the [*equations*](#).

(a) $|3| =$ (b) $|-3| =$ (c) $|(-2)(3.1)| =$ (d) $|(-2.4)/(2)| =$



2 Scalars and vectors

2.1 Scalars and scalar quantities

Many important physical quantities, such as length, mass and temperature, can be completely specified by a single number together with an appropriate unit of measurement. For instance, it makes perfectly good sense to say that the length of an object is 1.42 m or that the mass of an object is 12.2 kg. Quantities that can be specified in this simple and straightforward way are called *scalar quantities*. Thus:

Scalar quantities are physical quantities that can be completely specified by a single number together with an appropriate unit of measurement.

Different values of a given scalar quantity may be easily added, subtracted, multiplied or divided. In fact, all the elementary operations of arithmetic apply to values of a scalar quantity just as they do to ordinary numbers. So, if a , b and c are three values of a given scalar quantity (three masses, say) then we know that

$$a + b = b + a \quad (1)$$

$$ab = ba \quad (2)$$

$$a + 0 = a \quad (3)$$

$$a \times 1 = a \quad (4)$$

$$(a + b) + c = a + (b + c) \quad (5)$$

$$a(bc) = (ab)c \quad (6)$$

$$a(b + c) = ab + ac \quad (7)$$



Scalar quantities are often referred to simply as [scalars](#). There is nothing very surprising about the behaviour of scalars; their properties seem obvious and, indeed, they are obvious. What is much more interesting is that there are many physical quantities that are *not* scalars.

2.2 Vectors and vector quantities

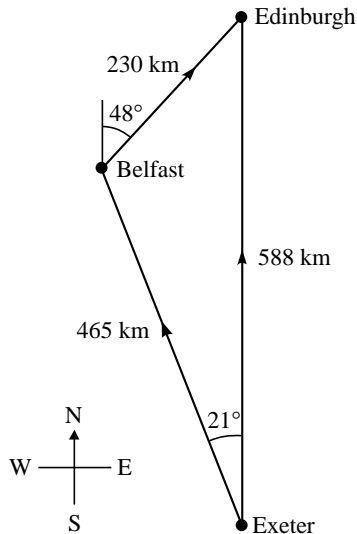
Figure 1 shows the relative locations of Belfast, Edinburgh and Exeter, three of the major cities in the UK. If you were asked to describe the location of Edinburgh relative to Exeter what would you say? You might start by stating that Edinburgh is 588 km from Exeter, but that would not answer the question. Although you would have defined the *distance* from Exeter to Edinburgh you would still not have fully defined the relative *position*. To provide a full description of the location you have to give the distance *and* the direction.

The physical quantity that describes the location of one point (Edinburgh, say) with respect to another point (Exeter, say) is called **displacement**. The displacement from Exeter to Edinburgh is 588 km in the direction due north.

◆ What is the displacement from Edinburgh to Exeter?



Figure 1 An idealized map showing the relative locations of Belfast, Edinburgh and Exeter.



Displacement, unlike distance, cannot be specified by a number together with an appropriate unit of measurement; an additional ingredient is required — a direction. Of course, the direction can be specified in a number of different ways, but its presence is unavoidable.

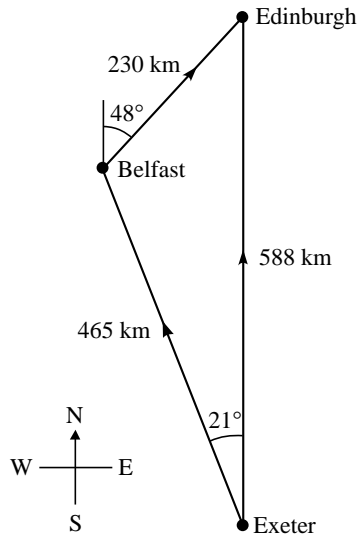
Question T1

What is the displacement:

- (a) from Exeter to Belfast?
- (b) from Belfast to Edinburgh? ☐



Figure 1 An idealized map showing the relative locations of Belfast, Edinburgh and Exeter.

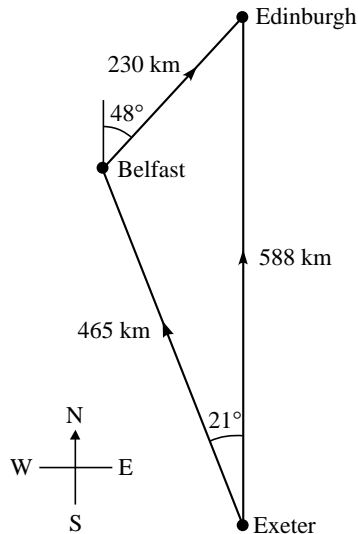


It is important to note that displacements are *entirely* specified by *two* items of information:

- A distance (this is called the **magnitude** of the displacement and must be a non-negative scalar quantity).
- A direction (in a two-dimensional plane this is often specified in terms of compass bearings, but other methods can be used; in three dimensions other methods *must* be used).

The two end-points of the displacement do not enter into this specification and are immaterial to the displacement itself. Thus, a displacement of 588 km due north starting from Exeter is the same displacement as one of 588 km due north from Belfast, even though one ends in Edinburgh and the other in the sea. In fact, any two displacements are equal if they have the same magnitude and the same direction.

Figure 1 An idealized map showing the relative locations of Belfast, Edinburgh and Exeter.



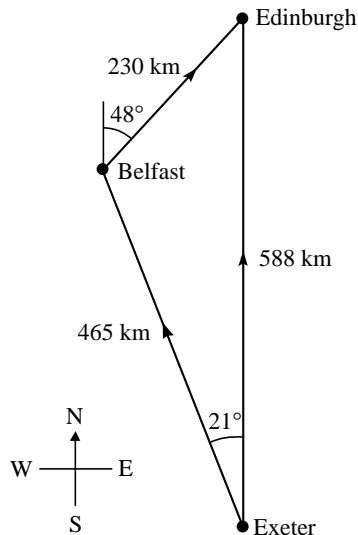
Question T2

Which of the following displacements are equal to the displacement from Edinburgh to Belfast?

- (a) A displacement of magnitude 230 km, 48° east of north.
- (b) A displacement of magnitude -230 km, 48° west of south.
- (c) A displacement of magnitude 230 km, 42° south of west. ☐



Figure 1 An idealized map showing the relative locations of Belfast, Edinburgh and Exeter.



Displacements are just one example of a large class of physical quantities known as *vector quantities*



The common characteristics of all vector quantities can be summarized as follows:


Vector quantities are physical quantities that can be completely specified by a magnitude and a direction.

Note that the general ‘magnitude’ referred to here is always a non-negative scalar quantity, just as distance (the magnitude of a displacement) must always be a non-negative scalar quantity. The magnitude of any vector quantity can be thought of as the ‘size’ or ‘length’ of that vector.

Apart from displacement, other important vector quantities include *velocity*, *acceleration*, and *force*. You may not be familiar with all of these quantities, so a few words about each should help to define it and to underline its vector nature.

The velocity of an object is a vector quantity that describes how fast the object is moving *and* the direction in which it is travelling. The *magnitude* of the velocity is called the speed and is a non-negative scalar quantity. The *direction* of the velocity is the direction of motion.

The **acceleration** of an object is a vector quantity that describes the rate of change of the object's velocity. Acceleration is a vector quantity because it involves a *change* in a vector quantity, velocity, and such a change requires both a magnitude and a direction for its complete specification. For example, a car travelling in a fixed direction at ever increasing speed is accelerating because the *magnitude* of its velocity is changing, whereas a car travelling in a circle at a fixed speed is accelerating because the *direction* of its velocity is changing. Any change in the magnitude or the direction of the velocity, or any combination of the two, constitutes an acceleration. Similarly, the rate of change of any other vector quantity must be a vector quantity.

According to [*Newton's laws of motion*](#), when an object accelerates it does so because an unbalanced () **force** acts upon it. Roughly speaking, force is the 'push' or 'pull' that causes (or tends to cause) acceleration. [*Newton's second law*](#) asserts that the acceleration of a body of fixed mass is proportional to the force that acts upon it. We know that the 'effect' of a force depends on its direction as well as its strength, so force is a vector. The 'effect' of a force, according to Newton, is an acceleration, which is also a vector. More specifically, the magnitude of the acceleration is proportional to the magnitude of the force and the direction of the acceleration is the same as the direction of the force.

Question T3

Which of the following are vector quantities; energy, distance, time, electric charge, rate of change of acceleration, altitude above sea level? ☐



Just as scalar quantities are often referred to as scalars, so vector quantities are often called **vectors**. It is possible to draw a distinction between vectors and vector quantities on the one hand and scalars and scalar quantities on the other. Vector and scalar quantities are *physical* quantities while vectors and scalars are the mathematical entities that are used to represent those physical quantities. Although it is important to distinguish between real things and their mathematical representations we shall not pursue that distinction here. In what follows the terms vector and vector quantity will be used interchangeably, as will scalar and scalar quantity.

Question T4

How would you counter the following assertions (both of which are wrong)?

- (a) The velocity of a particular object (such as my car) cannot be a vector since it is not completely specified by a magnitude and a direction — its complete specification also requires the identification of the moving object (my car).
- (b) Displacement cannot be a vector since its complete specification requires the definition of the various compass bearings (or something similar) and not merely the statement of a magnitude and a direction. ☐



2.3 Representing scalars and vectors

Everybody knows how to represent a scalar quantity such as mass or length when writing equations or drawing diagrams; just choose an appropriate letter or symbol to represent the quantity concerned (m or l say) and write it down wherever the scalar quantity is to be represented. It is taken for granted that the chosen symbol indicates the appropriate units of measurement as well as the number of those units, so it makes good sense to write statements such as:

If length $l = 6 \text{ m}$, then $l^2 = 36 \text{ m}^2$

Unfortunately, dealing with vectors is a good deal more difficult since directions are involved as well as magnitudes. When drawing diagrams the problem is not too bad since, as in Figure 1, a vector can be represented by a line labelled with the appropriate magnitude and pointed in the appropriate direction. The direction of the line is usually indicated by an arrowhead at the end of the line or by a pointer somewhere along the line. For this reason, such lines are usually referred to as *arrows* or [directed line segments](#).

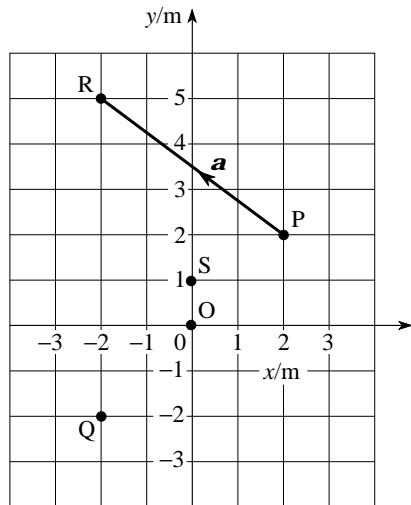


Figure 2 A representation of the locations of the five points O, P, Q, R and S in a two-dimensional plane.

Figure 2 shows five points O, P, Q, R and S, located in a two-dimensional plane. The positions of the five points can be specified by their *Cartesian coordinates* (x, y) with respect to the axes x and y shown in the figure. (Note that all measurements are expressed in metres.) Also included in the figure is a directed line segment from P to R. This represents the displacement from P to R.

One obvious way of representing the displacement from P to R symbolically is to write it as \overrightarrow{PR} . This is a clear if somewhat cumbersome notation that may easily be applied to other displacements.

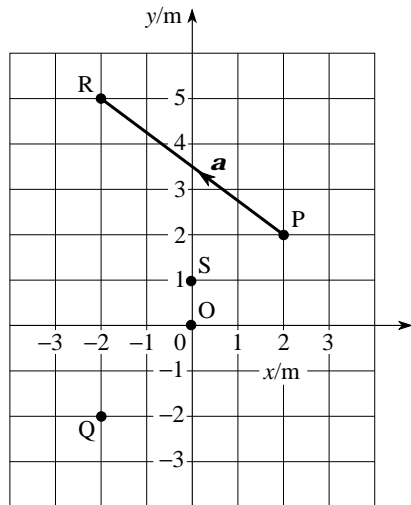


Figure 2 A representation of the locations of the five points O, P, Q, R and S in a two-dimensional plane.

Question T5

Using Figure 2:

- (a) Find the magnitude of \overrightarrow{PR} .
- (b) Describe the direction of \overrightarrow{PR} .
- (c) Find the magnitude of \overrightarrow{OQ} . \square

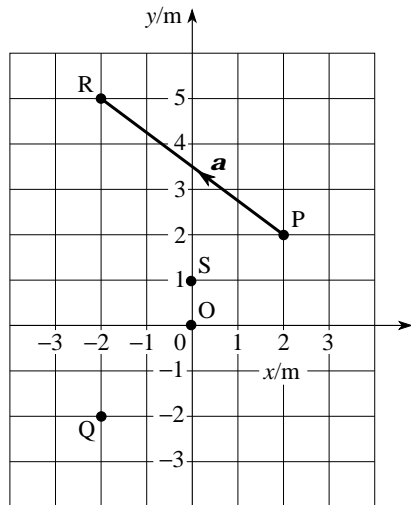


Figure 2 A representation of the locations of the five points O, P, Q, R and S in a two-dimensional plane.

Another way of representing vectors in printed material is to symbolize them by bold-face type, thus the bold ***a*** in Figure 2 is yet another way of representing the displacement \overrightarrow{PR} . Indeed, it makes sense to write

$$\mathbf{a} = \overrightarrow{PR}$$

This new symbol (the bold ***a***) does not have any obvious link with the points P and R, which may be regarded as an advantage or a disadvantage. However, one very definite bonus is that emboldening is applicable to all kinds of vector quantities (velocities, accelerations, forces, etc.) and not just displacements. The bold-typeface convention is the most widely used method of representing vectors and will be used throughout *FLAP*.

Distinguishing vectors by bold-face type is fine in print, but it presents a problem for those using pens or pencils. How can *you* show that ***a*** is a vector when *you* write it down? Fortunately there is an easy answer to this problem. When authors are preparing material to be printed they show that an item should be set in bold-face type by putting a wavy underline beneath it. Thus, what is written as \underline{a} will be printed as ***a***. You are encouraged to adopt this convention for distinguishing vectors so that you will never make the mistake of writing down a nonsensical equation such as

$$\underline{a} = 5 \text{ m}$$

❖ Why is this equation nonsensical?



Question T6

In Figure 2 the vectors ***b***, ***c***, ***d*** and ***e*** are defined as follows:

$$\mathbf{b} = \overrightarrow{PS}, \quad \mathbf{c} = \overrightarrow{OS}, \quad \mathbf{d} = \overrightarrow{QO}, \quad \mathbf{e} = \overrightarrow{QR}$$

- (a) Using a pencil, add directed line segments representing each of these vectors to Figure 2.
- (b) Label each of the directed line segments with the appropriate vector symbol. ☐

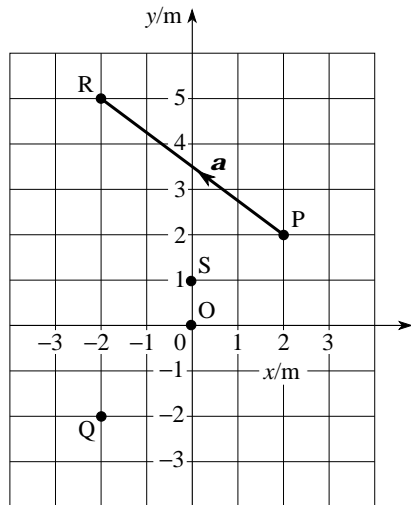


Figure 2 A representation of the locations of the five points O, P, Q, R and S in a two-dimensional plane.

Another advantage of the bold-typeface convention for vectors is that it provides a natural way of representing the magnitude of a vector. Given a vector ***a***, its magnitude is conventionally denoted $|\mathbf{a}|$. So, it makes sense to write

$$|\mathbf{a}| = 5 \text{ m} \quad (\text{In a manuscript: } |\underline{a}| = 5 \text{ m})$$

even though it was meaningless to write $\mathbf{a} = 5 \text{ m}$.

Sometimes, when there is no risk of confusion, the magnitude of a vector is simply represented by the symbol used for the vector itself, but without the emboldening. Thus, you might see


$$a = |\mathbf{a}| = 5 \text{ m}, \text{ or simply } a = 5 \text{ m}$$

This is obviously much less bother to write, but it is easy to forget that a , being a magnitude, must be non-negative, whereas you are unlikely to forget that something as complicated as $|\mathbf{a}|$ *cannot* be negative.

Question T7

Using your copy of Figure 2 having completed Question T6 and working to three significant figures, evaluate the following quantities:

(a) $|\vec{RS}|$ (b) $|\mathbf{b}|$ and $|\mathbf{c}|$ (c) $|\mathbf{b}| + |\mathbf{c}|$ and $|\mathbf{b}||\mathbf{c}|$

(d) $|\mathbf{a}| + |\mathbf{b}| + |\mathbf{c}| + |\mathbf{d}| + |\mathbf{e}|$ (e) $2|\mathbf{a}| + \frac{|\mathbf{b}||\mathbf{c}|}{|\mathbf{d}|}$ 



The main result of this subsection may be summarized as follows:

By convention, bold-face symbols such as \mathbf{a} are used to represent vector quantities. The magnitude of such a vector is represented by $|\mathbf{a}|$ and is a non-negative scalar quantity. In diagrams vectors are represented by directed line segments. In handwritten work vectors are distinguished by a wavy underline ($\underline{\sim}a$) and the magnitude of a vector is denoted $|\underline{\sim}a|$.

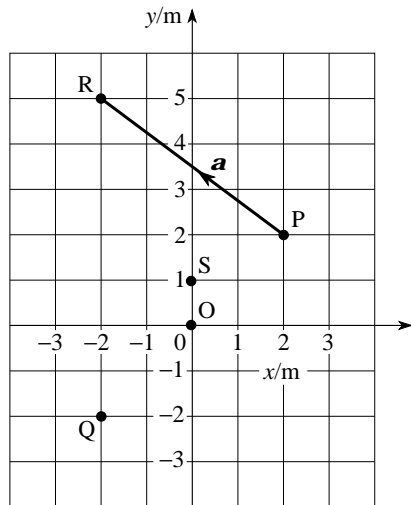


Figure 2 A representation of the locations of the five points O, P, Q, R and S in a two-dimensional plane.

3 Introducing vector algebra

Scalar quantities can be added, subtracted, multiplied or divided. So, it is quite possible to form equations and carry out all the normal processes of algebra with scalar quantities. (Finding unknown scalar quantities was probably the first use you ever made of algebra, even if the word ‘scalar’ wasn’t used at the time!) Working with vectors is more difficult because it involves directions as well as magnitudes. Nonetheless, it is possible to make sense of algebraic expressions such as $\mathbf{a} + \mathbf{b}$ or $3\mathbf{a} - 2\mathbf{b}$ and to formulate and solve vector equations. This section introduces the basic concepts of [vector algebra](#) — a subject that is more fully developed in other modules.

3.1 Scaling vectors

The simplest operation of vector algebra is the *scaling* of a vector, that is the multiplication of a vector by a scalar. For example, if \mathbf{v} is a velocity of 15 m s^{-1} due north then $2\mathbf{v}$ is a velocity of 30 m s^{-1} in the same direction or, to give another example, if \mathbf{a} is a displacement of 50 m due east then $-3\mathbf{a}$ is a displacement of 150 m due west. Note that in this last example the inclusion of the minus sign has had the effect of *reversing* the direction. Some more examples of scaling are shown in Figure 3.

It should be noted that $-1\mathbf{a}$ is usually written $-\mathbf{a}$ and that $1.5\mathbf{a}$ could equally well be written $(1.5)\mathbf{a}$ or even $\mathbf{a}(1.5)$, so the order of the multiplication is unimportant. Nonetheless, it is more conventional to use the form $1.5\mathbf{a}$ since this is more suggestive of ‘taking 1.5 lots of \mathbf{a} ’.

In general, if \mathbf{a} is a vector and α is a scalar (which may be positive or negative) then $\alpha\mathbf{a}$, the result of **scaling** \mathbf{a} by α , is a vector of magnitude $|\alpha||\mathbf{a}|$ that points in the same direction as \mathbf{a} if $\alpha > 0$ and points in the opposite direction to \mathbf{a} if $\alpha < 0$.

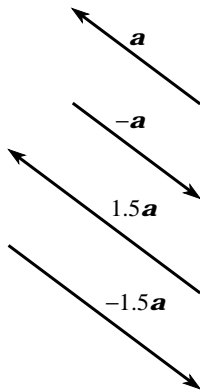


Figure 3 A vector \mathbf{a} and three of its scaled counterparts $-1\mathbf{a}$, $1.5\mathbf{a}$ and $-1.5\mathbf{a}$.

When dealing with vectors, some authors use the term [parallel](#) to mean ‘in the same direction’ and the term [antiparallel](#) to mean ‘in the opposite direction’. Thus, the vectors \mathbf{a} and $1.5\mathbf{a}$ maybe described as *parallel*, while \mathbf{a} and $(-1.5)\mathbf{a}$ are *antiparallel*. Unfortunately the use of antiparallel, though widespread, is not universal. So, whenever you see the term ‘parallel’ you should take it to mean ‘parallel *or* antiparallel’ unless the more specialized interpretation is clearly indicated.

Question T8

If a body of mass $m = 6\text{ kg}$ has velocity \mathbf{v} of magnitude 5 m s^{-1} in the direction north-west, what is the magnitude and direction of $m\mathbf{v}$? (In Newtonian mechanics this vector quantity is known as the [momentum](#) of the body.) ☐



Question T9

If \mathbf{a} is a vector and α is a non-zero scalar, classify each of the following statements as always true, always false or sometimes true and sometimes false. Explain your answers.

- (a) $|\alpha \mathbf{a}|$ cannot be less than zero.
- (b) $\alpha^2 \mathbf{a}$ is in the same direction as \mathbf{a} .
- (c) $-\alpha \mathbf{a}$ is antiparallel to \mathbf{a} .
- (d) Provided $|\mathbf{a}|$ is not zero, $\hat{\mathbf{a}} = \frac{\mathbf{a}}{|\mathbf{a}|}$ is a vector of unit magnitude in the same direction as \mathbf{a} .
- (e) $\left| \frac{\mathbf{a}}{\alpha} \right| = \frac{|\mathbf{a}|}{|\alpha|}$
- (f) $|\alpha \mathbf{a}| = |\alpha| |\mathbf{a}|$ \square



3.2 Adding and subtracting vectors

Two vector quantities of the same type (two displacements say, or two velocities) may be added together to produce another vector quantity of the same type. This operation is called [vector addition](#). It is straightforward, though quite different from the everyday operation of (scalar) addition with which you are familiar.

Before examining the general rules for vector addition let's look at an example to see how the process works in practice. Figure 4 shows the points O, P, Q, R and S that we considered earlier — we shall use these points to provide the example we need. Imagine yourself to be located at Q, the lowest of the five points, and suppose that you undergo a displacement \vec{QO} . Where would you find yourself? Obviously, at O.

Now suppose you undergo another displacement, \vec{OS} . What would your new location be after this second displacement? You would find yourself at point S. So, the overall result of the two successive displacements \vec{QO} and \vec{OS} would be to move you from Q to S. But, of course, such an effect could also have been produced by the single displacement \vec{QS} . Thus, it makes sense to write

$$\vec{QO} + \vec{OS} = \vec{QS} \quad (8)$$

The left-hand side of Equation 8 contains a new kind of quantity — *the sum of two vectors*. Thus, Equation 8 is an example of vector addition and the vector \vec{QS} on the right-hand side is said to be the **vector sum** or **resultant** of \vec{QO} and \vec{OS} .

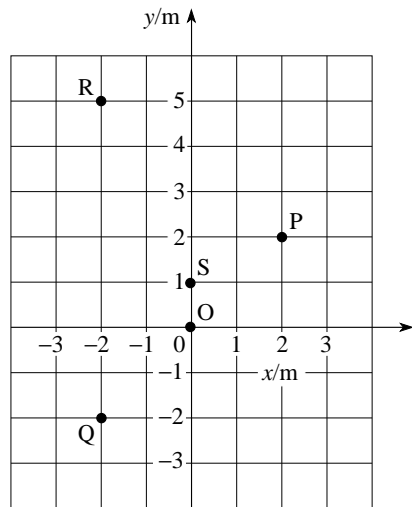



Figure 4 The points O, P, Q, R, S and their locations in the (x, y) plane.

Question T10

Complete the following (vector) sums. 

(a) $\vec{OS} + \vec{SP} =$ (b) $\vec{PS} + \vec{SR} =$

(c) $(\vec{PS} + \vec{SR}) + \vec{RQ} =$ (d) $\vec{PS} + (\vec{SR} + \vec{RQ}) =$ 



So far, all the examples of vector addition that we have considered have involved vectors that are ‘nose-to-tail’ like \overrightarrow{OS} and \overrightarrow{SP} , but remember, a vector is completely specified by its magnitude and direction; the end-points are immaterial. Points such as O, P, Q, R and S merely provide a convenient way of specifying vectors, the vectors themselves could just as easily be represented by bold-face letters, such as the \mathbf{u} and \mathbf{v} shown in Figure 5 together with their sum \mathbf{w} . Moreover, any vector that has the same magnitude and direction as \mathbf{v} is equal to \mathbf{v} , irrespective of its endpoints, and could be used in place of \mathbf{v} in the equation

$$\mathbf{u} + \mathbf{v} = \mathbf{w} \quad (9)$$

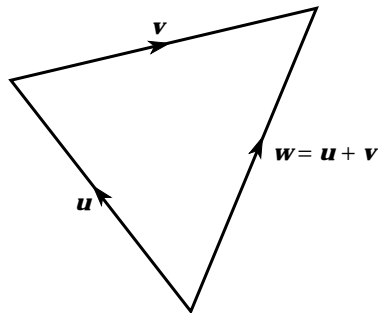


Figure 5 Two vectors, \mathbf{u} and \mathbf{v} , together with their sum $\mathbf{w} = \mathbf{u} + \mathbf{v}$.

Figure 6 shows just such a vector, \mathbf{t} . Even though \mathbf{t} is shown in a different location from \mathbf{v} , the fact that $\mathbf{t} = \mathbf{v}$ means that it is correct to write:

$$\mathbf{u} + \mathbf{t} = \mathbf{w} \quad (10)$$

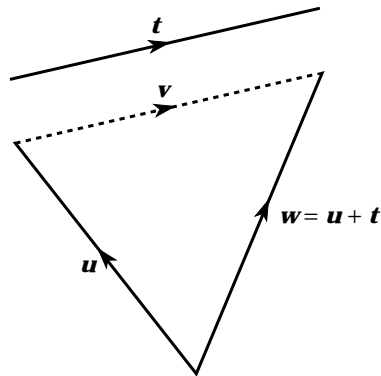


Figure 6 The vectors \mathbf{t} and \mathbf{v} are equal, so $\mathbf{w} = \mathbf{u} + \mathbf{t}$.

So, what is the *general* rule for adding two vectors? The answer is illustrated graphically in Figure 7 and may be summarized in the following *triangle rule for vector addition*.

The **triangle rule** for adding vectors:

Let vectors ***a*** and ***b*** be represented by appropriate arrows (or directed line segments). If the arrow representing ***b*** is drawn from the head of the arrow representing ***a***, then an arrow from the tail of ***a*** to the head of ***b*** represents the vector sum ***a* + *b***, marked ***c*** in Figure 7.

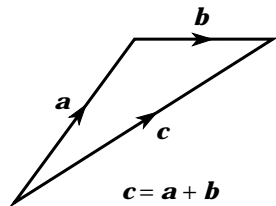


Figure 7 The triangle rule for adding vectors. Note the directions of the arrows; the diagram would be incorrect if any one, or any two of the arrows were reversed.

Question T11

Figure 8 shows three pairs of vectors; **A**, **B**; **C**, **D** and **E**, **F**. Sketch and label some simple diagrams showing how the triangle rule can be used to find the following vector sums:

- (a) **A** + **B**, (b) **C** + **D**, (c) **E** + **F**. □

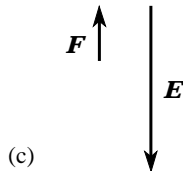
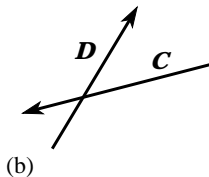
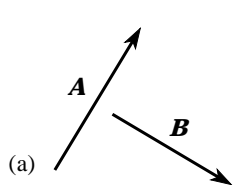


Figure 8 See Question T11.



An alternative but equivalent method of adding vectors graphically is provided by the *parallelogram rule*. This has no real advantages over the triangle rule, but it is preferred by some authors. It is illustrated in Figure 9 and may be stated as follows.

The **parallelogram rule** for adding vectors:

Let vectors ***a*** and ***b*** be represented by appropriate arrows (or directed line segments). If the arrows representing ***a*** and ***b*** are drawn from a common point O so that they form two sides of a parallelogram, when the parallelogram is completed an arrow from O along the diagonal of the parallelogram represents the vector sum ***a* + *b***.

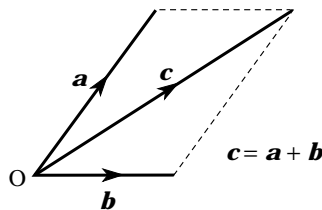


Figure 9 The parallelogram rule for adding vectors.

Despite a lengthy discussion of vector addition, nothing has yet been said about vector subtraction. The time has come to remedy that deficiency.

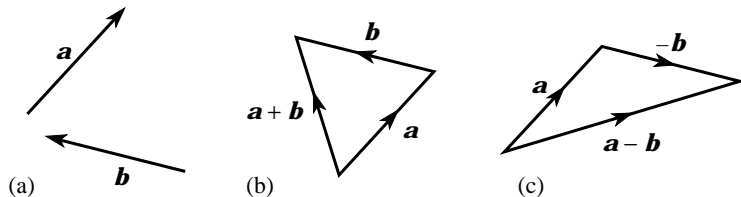


Figure 10 (a) Two vectors, (b) their sum, and (c) their difference.

Given two vectors of the same type, such as the vectors \mathbf{a} and \mathbf{b} shown in Figure 10a, you already know how to add them together to form their sum $\mathbf{a} + \mathbf{b}$ (Figure 10b). You also know how to scale the vector \mathbf{b} by the number -1 to produce the vector $-\mathbf{b}$ that has the same magnitude as \mathbf{b} but points in the opposite direction. You should therefore be able to convince yourself that Figure 10c correctly represents the following summation

$$\mathbf{a} + (-\mathbf{b})$$

This quantity is more usually written as

$$\mathbf{a} - \mathbf{b}$$

and is called the *vector difference* of \mathbf{a} and \mathbf{b} .

The vector difference $\mathbf{a} - \mathbf{b}$ of the vectors \mathbf{a} and \mathbf{b} is the vector sum of \mathbf{a} and $-\mathbf{b}$.

Question T12

In Question T11 you were asked to sketch diagrams showing the sums of various pairs of vectors (shown in Figure 8). Now sketch similar diagrams for the differences of the same pairs:

(a) $\mathbf{A} - \mathbf{B}$, (b) $\mathbf{C} - \mathbf{D}$, (c) $\mathbf{E} - \mathbf{F}$. \square

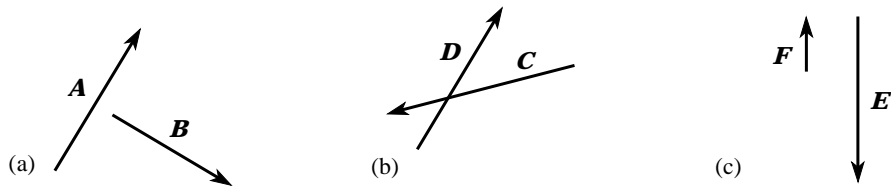


Figure 8 See Question T12.

◆ How does the vector $\mathbf{A} - \mathbf{B}$ in Question T12 differ from the vector $\mathbf{B} - \mathbf{A}$?

One final point needs to be discussed before leaving the subject of vector addition and subtraction — the *zero vector*. It has already been stressed that a vector (which has direction and magnitude) cannot be equal to a scalar (which does not have any direction). However, if we consider the result of a vector subtraction such as $\mathbf{a} - \mathbf{a}$ it's obvious that the result has zero magnitude and does not point in any specific direction. So, how should we symbolize this quantity and how should we refer to it? For the sake of consistency we certainly want to call it a vector so it makes sense to dub it the zero vector. As far as its symbolic representation is concerned, different authors take different attitudes. Many simply represent it by 0 as you might expect, but others prefer to use a bold-face zero, $\mathbf{0}$, to emphasize its vector nature. In *FLAP* we adopt this somewhat more formal convention, but don't be surprised if you see the other elsewhere.

◆ The above discussion of the zero vector has not provided a concise definition of $\mathbf{0}$. Write down an equation that *defines* the zero vector.




Now that you know about the zero vector you should be able to carry out algebraic manipulations with vectors. Try the following exercise.

- ◆ Rearrange the following vector equation to express \mathbf{c} in terms of \mathbf{a} and \mathbf{b} :

$$3\mathbf{a} - 4\mathbf{b} + 2\mathbf{c} = \mathbf{0}$$



Question T13


 An angler sitting on a riverbank is watching a duck paddling across the river. The water in the part of the river in which the duck is paddling is moving at velocity \mathbf{v} relative to the angler. The duck is moving at velocity \mathbf{u} relative to the water.

- (a) What is the velocity of the duck according to the angler?
(b) What is the velocity of the angler according to the duck? ☐



3.3 Resolving vectors

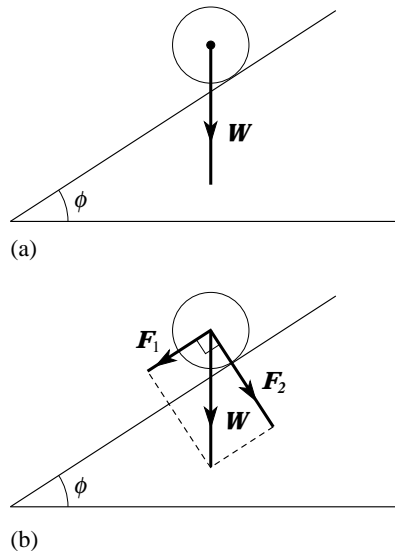
Imagine a ball released from rest on a perfectly smooth inclined plane, as shown in Figure 11a. What will happen to the ball immediately after its release? Obviously, the ball will start to move down the plane, accelerating as it does so. Anyone familiar with Newton's laws of motion would say that the acceleration of the ball must be caused by a force pointing down the plane. But what is the origin of the force causing that acceleration? The only 'downward' force that acts on the ball is its [weight](#), \mathbf{W} —the force that arises from the action of gravity on the ball's mass—and that force acts vertically downwards, *not* parallel to the plane. So, where does the accelerating force come from?

Happily, vector addition provides a simple answer. The weight \mathbf{W} of the ball can be regarded as the sum of two other forces as shown in Figure 11b, and we can write $\mathbf{W} = \mathbf{F}_1 + \mathbf{F}_2$. The force \mathbf{F}_1 that is parallel to the plane causes the acceleration, while the force \mathbf{F}_2 , that is normal (i.e. at right angles) to the plane stops the ball from leaving the plane and accounts for the difference between \mathbf{W} and \mathbf{F}_1 . 

This process of splitting a given vector into constituent parts at right angles to each other is called **(orthogonal) resolution**; we speak of *resolving* the vector into its **(orthogonal) component vectors** along the chosen directions. The process of orthogonal resolution is not restricted to the directions shown in Figure 11; a vector in a two-dimensional plane can be resolved into orthogonal component vectors along *any* two mutually perpendicular directions in that plane. (In three dimensions it is always possible to choose *three* mutually perpendicular directions, and any given vector in three dimensions can be correspondingly resolved into *three* orthogonal component vectors.

Problems in which a given vector has to be resolved into (orthogonal) component vectors are very common, so it is worth knowing how to perform such resolutions.

Figure 11 (a) A ball of weight W , released from rest on an inclined plane. (b) The component vectors of W , parallel and normal to the plane.



The sort of problem you might be asked to solve is illustrated in Figure 12. A vector \mathbf{a} is given, and a line AB, inclined at an angle θ to \mathbf{a} , is specified. The problem is to resolve \mathbf{a} into two component vectors, one parallel to AB, the other normal to AB. To solve the problem, just use the parallelogram rule (in the case of orthogonal component vectors it's really a 'rectangle rule'). Construct a rectangle like the one shown in Figure 12, with \mathbf{a} as its diagonal and one side parallel to AB. Call the component vectors parallel and normal to AB, respectively, \mathbf{a}_p and \mathbf{a}_n . Applying basic trigonometry to the rectangle, you should then be able to see that the magnitudes of the two orthogonal component vectors are:



$$|\mathbf{a}_p| = |\mathbf{a}| \cos \theta \quad (11)$$

$$\text{and, } |\mathbf{a}_n| = |\mathbf{a}| \sin \theta \quad (12)$$

With this information you should be able to find orthogonal component vectors for yourself, though you may need to think carefully about their directions.

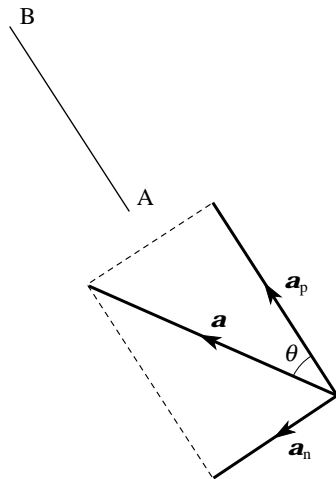
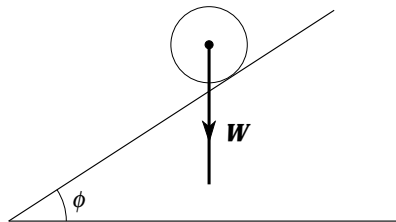


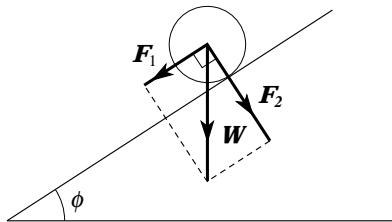
Figure 12 A given vector \mathbf{a} and a line AB. The vector is to be resolved into component vectors parallel (\mathbf{a}_p) and normal (\mathbf{a}_n) to the line AB.

Question T14

If the weight of the ball in Figure 11 is of magnitude 5 N and the angle of inclination of the plane (ϕ) is 30° , what are the magnitudes and directions of the orthogonal component vectors \mathbf{F}_1 and \mathbf{F}_2 ? (*Hint: Pay attention to the definition of ϕ .*) \square



(a)



(b)

Figure 11 (a) A ball of weight \mathbf{W} , released from rest on an inclined plane. (b) The component vectors of \mathbf{W} , parallel and normal to the plane.



Question T15

An aeroplane is flying along a northwesterly course at 500 km h^{-1} . What are the component vectors of the plane's velocity parallel and normal to a line running north-south? □



In principle, the process of resolution is not restricted to situations in which the chosen directions are perpendicular and the component vectors are at right angles to each other. In two dimensions a vector may be resolved into component vectors along *any* two directions that are not parallel, and in three dimensions a vector may be resolved into component vectors along any three directions that do not all lie in the same plane. The process of carrying out such resolutions is similar to that outlined above, though the mathematics is a little more complicated. Fortunately, such non-orthogonal resolutions are relatively rare in elementary work. It was in order to emphasize this restriction to mutually perpendicular directions that we made such widespread use of the word 'orthogonal' in this subsection. However, since orthogonal resolution is so common we do not follow this practice elsewhere in *FLAP*. Generally speaking, if you are given a vector and asked to find its component vector along some line you should assume that it is the *orthogonal* component vector along that line which is required unless you are given a clear indication to the contrary.

Important note Pay close attention to the term component vector. As you pursue the study of vectors you will soon encounter scalar quantities referred to as the components of a vector. Like component vectors, these (scalar) components of a given vector can be used to specify that vector. Nonetheless, (scalar) components are different from (though related to) the component vectors we have just been discussing. Take care not to confuse the (scalar) components of a vector with its component vectors. In *FLAP* the two quantities will always be clearly distinguished, other authors may not be so careful.

4 Working with vectors

The methods that have been used in this module have been largely graphical. Such methods provide a good basis for the introduction of fundamental concepts but they are not well suited to the everyday business of working with vectors. Methods of a more algebraic nature are really needed for that purpose. Such methods exist and are fully developed in other modules of *FLAP*.

Apart from simplifying calculations, the algebraic methods also have the advantage of making it almost as easy to treat three-dimensional problems involving directions in space as it is to treat two-dimensional problems confined to a plane. The use of vectors in describing and analysing three-dimensional situations makes them one of the most valuable mathematical tools at the disposal of the physicist.

5 Closing items

5.1 Module summary

- 1 Scalar quantities (or scalars) can be completely specified by a single number together with an appropriate unit of measurement.
- 2 Vector quantities (or vectors) can be completely specified by a magnitude and a direction. In diagrams, vectors are usually represented by arrows or directed line segments. In print, vectors are distinguished by bold-face type, and in a manuscript, vectors are indicated by a wavy underline.
- 3 The magnitude of a vector is a non-negative scalar that represents the ‘length’ or ‘size’ of that vector. The magnitude of \mathbf{a} is shown as $|\mathbf{a}|$ in print or $|\underline{a}|$ in a manuscript.
- 4 Any vector \mathbf{a} may be multiplied by a scalar α to produce a scaled vector $\alpha \mathbf{a}$ which points in the same direction as \mathbf{a} if $\alpha > 0$ and in the opposite direction if $\alpha < 0$. The magnitude of $\alpha \mathbf{a}$ is $|\alpha \mathbf{a}| = |\alpha| |\mathbf{a}|$.
- 5 Vectors may be added graphically using either the triangle rule or the parallelogram rule (Figures 7 and 9).
- 6 The zero vector, $\mathbf{0}$, has the property that $\mathbf{a} + \mathbf{0} = \mathbf{a}$ for any vector \mathbf{a} .

- 7 A vector may be *resolved* into *component vectors* along appropriately chosen lines. Given a vector \mathbf{a} , its *orthogonal component vectors* parallel and normal to a line inclined at an angle θ ($0^\circ \leq \theta \leq 90^\circ$) to \mathbf{a} are of magnitude

$$|\mathbf{a}_p| = |\mathbf{a}| \cos \theta$$

and,

$$|\mathbf{a}_n| = |\mathbf{a}| \sin \theta$$

- 8 [Displacement](#), [velocity](#), [acceleration](#), [force](#), [weight](#) and [momentum](#) are all examples of vector quantities.

5.2 Achievements

Having completed this module you should be able to:

- A1 Define the glossary terms introduced in the module — as shown in blue, emboldened and underlined.
- A2 Identify unfamiliar quantities as scalars or vectors given the definitions of those quantities and recognize displacement, velocity, acceleration, momentum, force and weight as vector quantities.
- A3 Carry out algebraic manipulations with scalar quantities.
- A4 Draw diagrams to represent specified vectors in two dimensions using arrows or directed line segments, and determine the magnitude and direction of any vector represented in that way.
- A5 Recognize, use and interpret the notations ($\underline{\mathbf{a}}$, \underline{a} , $|\mathbf{a}|$, $|\underline{a}|$) used to represent vectors and their magnitudes in print and in a manuscript.
- A6 Identify violations of vector notation such as equating vectors and scalars or equating vector magnitudes with negative quantities.
- A7 Understand and represent graphically the operations of scaling a vector, adding and subtracting vectors and resolving a vector into component vectors in various directions.

- A8 Evaluate the magnitudes of the orthogonal component vectors of a given vector parallel and normal to a given line.
- A9 Formulate and rearrange simple vector equations including those which involve the zero vector $\mathbf{0}$.

Study comment You may now wish to take the [Exit test](#) for this module which tests these Achievements. If you prefer to study the module further before taking this test then return to the [Module contents](#) to review some of the topics.

5.3 Exit test

Study comment Having completed this module, you should be able to answer the following questions each of which tests one or more of the Achievements.

Question E1

(A2) The magnetic field at a point in space (at any particular time) is completely specified by two items of information: (i) A direction. (ii) A non-negative quantity called the strength of the field that can be expressed as some multiple of an SI unit called the tesla (T). Justify the claim that the magnetic field at a point is a vector quantity.



Question E2

(A2 and A7) If a number of electrically charged particles are at rest relative to one another in some isolated region of space, and if one of those particles with charge q experiences a force \mathbf{F} due to the attraction or repulsion of the others, then we say that the particle is subject to an *electric field* given by \mathbf{F}/q . Is the electric field at the location of the particle a vector quantity? Explain your answer.



Question E3

(A4 and A7) The vectors \mathbf{a} and \mathbf{b} point north and east, respectively. $|\mathbf{a}| = 2$ cm and $|\mathbf{b}| = 1$ cm. Draw rough sketches to show the following vectors:

(a) $-\frac{1}{2}\mathbf{a}$, (b) $\mathbf{a} - \mathbf{b}$, (c) $2\mathbf{a} + 4\mathbf{b}$.



Question E4

(A4, A7 and A9) Figure 13 shows five vectors ***a***, ***b***, ***c***, ***d*** and ***e***. Express each of the vectors ***c***, ***d*** and ***e*** in terms of the vectors ***a*** and ***b*** and hence determine which of the following equations involving those vectors are correct.

- (a) $\mathbf{c} = 2\mathbf{a} + 2\mathbf{b}$
- (b) $-\mathbf{a} + 2\mathbf{b} + 2\mathbf{c} + \mathbf{e} = \mathbf{0}$
- (c) $-2\mathbf{a} + \frac{1}{2}\mathbf{c} + \frac{1}{2}\mathbf{d} = \mathbf{0}$

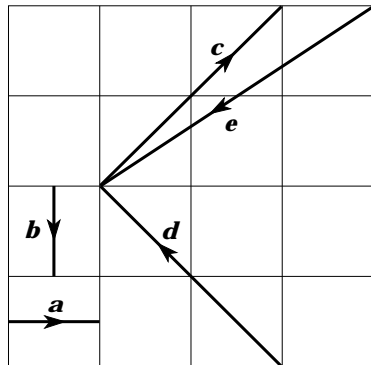


Figure 13 See Question E4.

Question E5

(A4 and A7) An aircraft has a speed relative to still air of 150 m s^{-1} . The pilot wishes to fly due north when the wind velocity is 20 m s^{-1} towards the north-east. Draw a sketch showing the compass bearing on which the pilot should fly. Determine the compass bearing to the nearest degree and calculate the speed of the aircraft relative to the ground, correct to one decimal place. (*Hint:* You might find it useful to recall Question T13.)



Question E6

(A3, A5 and A6) If \mathbf{a} , \mathbf{b} and \mathbf{c} are non-zero displacement vectors and α and β are non-zero numbers, classify the following statements as; always true, sometimes true and sometimes false, or always false. Explain your answers.

- (a) $|\alpha \mathbf{a}| = |\alpha| |\mathbf{a}|$
- (b) $|\mathbf{a} + \mathbf{b}| < |\mathbf{a}| + |\mathbf{b}|$
- (c) $\alpha \mathbf{a} = |\alpha \mathbf{a}|$
- (d) $\alpha(\mathbf{a} + \mathbf{b}) = \alpha \mathbf{a} + \alpha \mathbf{b}$
- (e) $(\alpha \beta)(\mathbf{a} + \mathbf{b}) = \alpha(\beta \mathbf{a} + \beta \mathbf{b})$
- (f) $(\alpha + \beta)(\mathbf{a} + \mathbf{b}) = \alpha \mathbf{a} + \beta \mathbf{b}$



Question E7

(A1 and A8) **a** is a displacement of 30 m, due north. **b** is a velocity of 20 m s^{-1} , 25° west of north.

- (a) What are the orthogonal component vectors of **a** parallel and normal to **b**?
- (b) What are the orthogonal component vectors of **b** parallel and normal to **a**?
- (c) Why are you unable to work out the orthogonal component vectors of **a + b**, parallel and normal to **a - b**?



Study comment This is the final *Exit test* question. When you have completed the *Exit test* go back to Subsection 1.2 and try the [Fast track questions](#) if you have not already done so.

If you have completed **both** the *Fast track questions* and the *Exit test*, then you have finished the module and may leave it here.

