Module M3.2 Polar representation of complex numbers

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## 1 Opening items

### 1.1 Module introduction

The symbol $i$ is defined to have the property that $i \times i=-1$. Expressions involving $i$, such as $3+2 i$, are known as complex numbers, and they are used extensively to simplify the mathematical treatment of many branches of physics, such as oscillations, waves, a.c. circuits, optics and quantum theory. This module is concerned with the representation of complex numbers in terms of polar coordinates, together with the related exponential representation. Both representations are particularly useful when considering the multiplication and division of complex numbers, and are widely used in physics.
In Subsection 2.1 we review the Cartesian representation of complex numbers and show how any complex number can be represented as a point on an Argand diagram (the complex plane). We also show how complex numbers can be interpreted as an ordered pair of real numbers. Points in a plane are often specified in terms of their Cartesian coordinates, $x$ and $y$, but they can equally well be defined in terms of polar coordinates $r$ and $\theta$. It will transpire that, while addition and subtraction of complex numbers is easy for complex numbers in Cartesian form, multiplication and division are usually simplest when the numbers are expressed in terms of polar coordinates. Subsection 2.2 and the subsequent two subsections are concerned with the polar representation of complex numbers, that is, complex numbers in the form $r(\cos \theta+i \sin \theta)$. Subsection 2.5 introduces the exponential representation, $\mathrm{r}^{i \theta}$. Section 3 is devoted to developing the arithmetic of complex numbers and the final subsection gives some applications of the polar and exponential representations which are particularly relevant to physics.

Study comment Having read the introduction you may feel that you are already familiar with the material covered by this module and that you do not need to study it. If so, try the Fast track questions given in Subsection 1.2. If not, proceed directly to Ready to study? in Subsection 1.3.

### 1.2 Fast track questions

Study comment Can you answer the following Fast track questions? If you answer the questions successfully you need only glance through the module before looking at the Module summary (Subsection 4.1) and the Achievements listed in Subsection 4.2. If you are sure that you can meet each of these achievements, try the Exit test in Subsection 4.3. If you have difficulty with only one or two of the questions you should follow the guidance given in the answers and read the relevant parts of the module. However, if you have difficulty with more than two of the Exit questions you are strongly advised to study the whole module.

## Question F1

The complex number $z$ is defined by $z=1+i$. Find the following in their simplest representations: $z,|z|, \arg (z), z^{*}$ and $z^{-1}$.


## Question F2

The complex numbers $z$ and $w$ are defined by $z=3 \mathrm{e}^{i \pi / 10}$ and $w=4 \mathrm{e}^{i \pi / 5}$. Find the simplest exponential representations of $z w$ and $z / w$.

## Question F3

Suppose that a complex quantity, $z$, is known to satisfy

$$
z=2+i+\mathrm{e}^{i \theta}
$$

where $\theta$ can take any real value. Sketch a curve on an Argand diagram giving the position of all possible points representing $z$.

## Study comment

Having seen the Fast track questions you may feel that it would be wiser to follow the normal route through the module and to proceed directly to Ready to study? in Subsection 1.3.

Alternatively, you may still be sufficiently comfortable with the material covered by the module to proceed directly to the Closing items.

### 1.3 Ready to study?

Study comment To begin the study of this module you need to be familiar with the following topics: the arithmetic of complex numbers in the form, $z=x+i y$, where $i^{2}=-1$. You should know how to add, subtract and multiply such numbers, be able to reduce the quotient of two complex numbers to rational form, to find the modulus, complex conjugate, real part and imaginary part of a complex number, and you should know how to plot a complex number on an Argand diagram. You should be familiar with the addition of two-dimensional vectors by means of a diagram and by adding their components. You should also be familiar with Pythagoras's theorem, the definition of sine, cosine and tangent, the measurement of angles in terms of radians and the following trigonometric identities

$$
\begin{align*}
& \sin (\alpha+\beta)=\cos \alpha \sin \beta+\sin \alpha \cos \beta  \tag{1}\\
& \cos (\alpha+\beta)=\cos \alpha \cos \beta-\sin \alpha \sin \beta \tag{2}
\end{align*}
$$

You should also know that $(n$ factorial $) n!=n(n-1) \ldots 3,2,1$, and $0!=1$. We will need to refer to the following power series for $\mathrm{e}^{x}, \sin x$ and $\cos x$ :

$$
\begin{align*}
& \mathrm{e}^{x}=\sum_{n=0}^{\infty} \frac{x^{n}}{n!}=1+\frac{x}{1!}+\frac{x^{2}}{2!}+\frac{x^{3}}{3!}+\ldots  \tag{3}\\
& \sin x=\sum_{n=0}^{\infty} \frac{(-1)^{n} x^{2 n+1}}{(2 n+1)!}=x-\frac{x^{3}}{3!}+\frac{x^{5}}{5!}-\frac{x^{7}}{7!}+\ldots  \tag{4}\\
& \cos x=\sum_{n=0}^{\infty} \frac{(-1)^{n} x^{2 n}}{(2 n)!}=1-\frac{x^{2}}{2!}+\frac{x^{4}}{4!}-\frac{x^{6}}{6!}+\ldots \tag{5}
\end{align*}
$$

You will need to be familiar with the following properties of powers (i.e. indices)

$$
\begin{equation*}
u^{a} u^{b}=u^{(a+b)},(u \mathrm{~V})^{a}=u^{a} \mathbf{V}^{a},\left(u^{a}\right)^{b}=u^{a b} \tag{6}
\end{equation*}
$$

also to know that the $n^{\text {th }}$ root of $u$ can be written as $u^{1 / n}$ and to be able to use inverse trigonometric functions to solve an equation such as $\sin \theta=0.5$ for $\theta$ (and to use the graph of $\sin \theta$, or otherwise, to find all the solutions). If you are unfamiliar with any of these topics you can review them by referring to the Glossary, which will indicate where in FLAP they are developed. The following Ready to study questions will help you to establish whether you need to review some of the above topics before embarking on this module.
Throughout this module $\sqrt{x}$ means the positive square root, so that $\sqrt{4}=2$.

## Question R1

Rationalize the expression $z=\frac{3+2 i}{1+2 i}$ (i.e. express $z$ in the form $x+i y$, finding the values of the real numbers $x$ and $y$ ). What are the real and imaginary parts of $z$ ? Also find the complex conjugate and the modulus of $z$.

## Question R2

Draw and label the points representing the complex numbers $-2+i,-2-i$ and $-3 i$ on an Argand diagram.

## Question R3



The equilateral triangle shown in Figure 1 has a perpendicular drawn from one vertex to the opposite side. Use the triangle in Figure 1 to find the values of $\cos (\pi / 3), \sin (\pi / 3), \tan (\pi / 3)$, $\cos (\pi / 6), \sin (\pi / 6)$ and $\tan (\pi / 6)$.

## Question R4

 19898Use the right-angled triangle with two sides equal, shown in Figure 2 to find the values of $\cos (\pi / 4), \sin (\pi / 4)$ and $\tan (\pi / 4)$.


Figure 2 See Question R4.



Figure 1 See Question R3.

## Question R5

Two two-dimensional vectors, $\boldsymbol{u}$ and $\mathbf{V}$, are specified in component form as $(2,3)$ and $(1,-4)$, respectively. Find $\boldsymbol{u}+\mathbf{v}$ by (a) drawing a suitable diagram and (b) adding the components directly.


## Question R6

Solve the equation $\tan \theta=1$.

## Question R7

Express $\sqrt{\mathrm{e}^{3 x}} / \mathrm{e}^{x}$ in the form $\mathrm{e}^{k x}$ (for some value of $k$ ) and hence write down the first three terms of its power series expansion.


## 2 Representing complex numbers

### 2.1 Complex numbers and Cartesian coordinates

A complex number, $z$, can be written as $z=x+i y$ where $x$ and $y$ are real numbers and $i^{2}=-1$. Some examples of complex numbers are $2+3 i, 7 i$ and 2.4.

Such numbers satisfy straightforward rules for addition and subtraction, which essentially mean that the real and imaginary parts are treated separately, so that, for example,

$$
(3+4 i)+(2-i)-(-2-3 i)=(3+2+2)+(4 i-i+3 i)=7+6 i
$$

Multiplication is quite simple provided that we remember to replace every occurrence of $i \times i$ by -1 , although a mathematician would probably prefer a formal statement that the product of two complex numbers $(a+i b)$ and $(x+i y)$ is given by

$$
\begin{equation*}
(a+i b)(x+i y)=(a x-b y)+i(a y+b x) \tag{7}
\end{equation*}
$$

For a long time the meaning of the symbol $i$ gave many famous mathematicians cause for concern. However, in 1833 Sir William Rowan Hamilton (1805-1865) realized that the $i$ and $+\operatorname{sign}$ in $z=x+i y$ are both unnecessary sources of confusion. The role of the $i$ is really to keep the $x$ and $y$ separate, while the $+\operatorname{sign}$ is there to tell us that $x$ and $y$ are part of a single entity; it does not mean addition in the sense that we might, for example, add 2 apples to 3 apples to get 5 apples. In fact, the $x$ and $y$ in $z=x+i y$ are very much like the (ordered) pairs of numbers used as Cartesian coordinates. Hamilton's ideas are closely linked to the those of Robert Argand (1768-1822) and Karl Friedrich Gauss (1777-1855) who both suggested representing a complex number by a point in a plane. As an example, the complex number $a+i b$ is shown on an $(x, y)$ coordinate system in Figure 3; notice that it is conventional that the number multiplying the $i$ corresponds to the $y$-value. A figure in which the real and imaginary parts of complex numbers are used as Cartesian coordinates is known as an Argand diagram or the complex plane. An expression such as $a+i b$, where $a$ and $b$ are real numbers, is said to be the Cartesian form (or Cartesian representation) of a complex number.


Figure 3 An Argand diagram showing the point corresponding to a complex number, $z=a+i b$.

In some mathematics textbooks the authors avoid the problem of the meaning of the symbol $i$ entirely - by not mentioning it - and they introduce the complex numbers as a set of ordered pairs of real numbers $(x, y)$ with certain operations defined on them. Such a treatment has the advantage that complex numbers can immediately be seen to have much in common with vectors. The addition of two complex numbers is then defined by

$$
\begin{equation*}
(x, y)+(a, b)=(x+a, y+b) \tag{8}
\end{equation*}
$$

which is just the same as the rule that defines the addition of two vectors. An example is shown in Figure 4 where the addition of $z=3+2 i$ and $w=1+3 i$ is performed graphically on an Argand diagram or, equivalently

$$
z+w=(3,2)+(1,3)=(4,5)
$$



Figure 4 An Argand diagram showing the addition of $3+2 i$ and $1+3 i$.

## Question T1

If $z=4+8 i$ and $w=15-12 i$, use an Argand diagram to find the sum, $z+w$. Check your answer by means of vector addition using the $(x, y)$ notation.

Although complex numbers behave like vectors as far as addition is concerned, when it comes to multiplication and division the two topics diverge. In terms of ordered pairs of real numbers, multiplication of complex numbers can be defined by 稢

$$
\begin{equation*}
(a, b) \times(x, y)=[(a x-b y),(a y+b x)] \tag{9}
\end{equation*}
$$

Although one can introduce complex numbers by this route, which is entirely independent of the symbol $i$, it must be admitted that Equation 9 looks as though it came out of thin air. In practice $i$ is a very useful notational convenience which makes Equation 9 look much more natural. The $i$ notation is used throughout science and engineering, and even by the purest of pure mathematicians. It is a practice which we follow in FLAP.
(1)

### 2.2 Polar coordinates

We have seen how it is straightforward to interpret complex addition as vector addition on an Argand diagram. In order to investigate the effect of complex multiplication, try the following question.

## Question T2

(a) Plot the numbers, $1,2 i,-3-i$ and $2-i$ on an Argand diagram.
(b) Multiply each of the numbers in part (a) by 2 and plot the resulting points on the same diagram. Suggest a geometric interpretation of multiplication by 2 and check your conjecture by finding the effect of multiplying $-1-i$ by 2 .
(c) Repeat parts (a) and (b), but multiply by $i$ instead of 2.
(d) Repeat parts (a) and (b), but multiply by $2 i$.

The solution to this problem suggests that the geometric interpretation of complex multiplication may involve both a rotation and a change in the distance from the origin, but this is not easy to see if we write complex numbers in the Cartesian form $x+i y$.

However, the geometric properties of complex multiplication are quite evident when the complex numbers are expressed in terms of polar coordinates. Figure 5 shows a point specified by means of polar coordinates; we can see that the 'distance' of the point from the origin is called $r$ and $\theta$ is the angle between the line from the point to the origin and the $x$-axis. Notice that $r$ is (by definition) non-negative and that $\theta$ is conventionally measured anticlockwise.



Figure 5 Polar coordinates, $r, \theta$.

Standard results from trigonometry enable us to express the Cartesian coordinates ( $x$ and $y$ ) in terms of polar coordinates ( $r$ and $\theta$ )

$$
\begin{align*}
& x=r \cos \theta  \tag{10}\\
& y=r \sin \theta \tag{11}
\end{align*}
$$

This means that we can write a complex number, $z=x+i y$, in the form

$$
\begin{equation*}
z=r(\cos \theta+i \sin \theta) \tag{12}
\end{equation*}
$$

which is known as the polar representation or polar form of the complex number, $z$. Examples of complex numbers in polar form are

$$
2[\cos (\pi / 4)+i \sin (\pi / 4)]
$$

$$
3.5[\cos (\pi / 16)+i \sin (\pi / 16)]
$$

and $\quad 0.025[\cos (1.05)+i \sin (1.05)]$
To convert a complex number from polar to Cartesian form we can again use Equations 10 and 11.
(1)

For example, $z=3[\cos (\pi / 4)+i \sin (\pi / 4)]$ has $r=3$ and $\theta=\pi / 4$. If we substitute these values into Equations 10 and 11, we find

$$
\begin{aligned}
& x \\
&=r \cos \theta=3 \cos (\pi / 4)=\frac{3}{\sqrt{2}} \quad \text { and } \quad y=r \sin \theta=3 \sin (\pi / 4)=\frac{3}{\sqrt{2}} \\
& \text { i.e. } \quad z=\frac{3}{\sqrt{2}}+\frac{3}{\sqrt{2}} i=\frac{3}{\sqrt{2}}(1+i)
\end{aligned}
$$

It is also straightforward to convert from Cartesian to polar form since the length, $r$, is given by

$$
\begin{equation*}
r=\sqrt{x^{2}+y^{2}} \tag{13}
\end{equation*}
$$

and the angle, $\theta$, is such that

$$
\begin{align*}
& \cos \theta=\frac{x}{\sqrt{x^{2}+y^{2}}}  \tag{14}\\
& \sin \theta=\frac{y}{\sqrt{x^{2}+y^{2}}} \tag{15}
\end{align*}
$$

For example, the complex number, $z=1+\sqrt{3} i$ which has $x=1$ and $y=\sqrt{3}$, can be represented in terms of polar coordinates by

$$
r=\sqrt{x^{2}+y^{2}}=\sqrt{1+3}=2
$$

and since $\sin (\theta)=\sqrt{3} / 2$ and $\cos (\theta)=1 / 2$ we have $\theta=(\pi / 3)$ rad. The position of the number $1+\sqrt{3} i$ on an Argand diagram is shown in terms of polar coordinates in Figure 6.

- The polar coordinates of three points A, B and C are, respectively,

$$
\begin{aligned}
r=2 & \theta=(\pi / 4) \mathrm{rad} \\
r=3 & \theta=(-\pi / 3) \mathrm{rad} \\
r=4 & \theta=(5 \pi / 6) \mathrm{rad}
\end{aligned}
$$

Express the points in Cartesian coordinates.

Figure 6 The complex number, $z=1+\sqrt{3} i$, in terms of polar coordinates.


One big advantage of the polar representation is that the multiplication of complex numbers is easy when they are expressed in this form. To see this, consider two complex numbers, $z=x+i y$ and $w=a+i b$ for which

$$
\begin{aligned}
& x=r \cos \theta \quad \text { and } \quad y=r \sin \theta \\
& a=\rho \cos \phi \quad \text { and } \quad b=\rho \sin \phi
\end{aligned}
$$

Recalling two results from trigonometry:

$$
\begin{aligned}
& \sin (\alpha+\beta)=\cos \alpha \sin \beta+\sin \alpha \cos \beta \\
& \cos (\alpha+\beta)=\cos \alpha \cos \beta-\sin \alpha \sin \beta
\end{aligned}
$$

(Eqn 1) 挥罗
(Eqn 2)
we see that the real part of the product $z w$ (see Equation 7) is given by

$$
\begin{aligned}
\operatorname{Re}(z w) & =a x-b y=\rho \cos \phi \times r \cos \theta-\rho \sin \phi \times r \sin \theta \\
& =\rho r(\cos \theta \cos \phi-\sin \theta \sin \phi) \\
& =\rho r \cos (\theta+\phi)
\end{aligned}
$$

(1)
and the imaginary part is given by

$$
\begin{aligned}
\operatorname{Im}(z w) & =a y+b x=\rho \cos \phi \times r \sin \theta+\rho \sin \phi \times r \cos \theta \\
& =\rho r(\cos \phi \sin \theta+\sin \phi \cos \theta) \\
& =\rho r \sin (\theta+\phi)
\end{aligned}
$$

We can summarize these two results by the following rule for multiplying complex numbers in polar form:

$$
\begin{equation*}
r(\cos \theta+i \sin \theta) \times \rho(\cos \phi+i \sin \phi)=\rho r[\cos (\theta+\phi)+i \sin (\theta+\phi)] \tag{16}
\end{equation*}
$$

So, multiplying a complex number, $w$ say, by a complex number with a polar representation $r(\cos \theta+i \sin \theta)$, produces a new complex number which corresponds to the line from the origin to the point representing $w$ being first scaled by a factor $r$, then the resulting line being rotated anticlockwise about the origin through an angle $\theta$.

As an illustration we will start with the complex number
$w=2[\cos (\pi / 4)+i \sin (\pi / 4)]$
and first multiply it by 2 in Figure 7,then by
$[\cos (\pi / 8)+i \sin (\pi / 8)]$
in Figure 8,


Figure 7 Complex multiplication resulting in a change in distance from the origin.


Figure 8 Complex multiplication causing a rotation.
and finally by $2[\cos (\pi / 8)+i \sin (\pi / 8)]$ in Figure 9.
Notice that in Figure 9 the line from the origin to the original point is rotated through an angle of $(\pi / 8)$ and the distance from the origin is doubled.


Figure 9 Complex multiplication causing a rotation and a change in distance from the origin.

### 2.3 The modulus of a complex number

Given a complex number, $z=x+i y$, the modulus of $z$ is defined by

$$
\begin{equation*}
|z|=\sqrt{x^{2}+y^{2}} \tag{17}
\end{equation*}
$$

In polar coordinates, where $x=r \cos \theta$ and $y=r \sin \theta$, we have

$$
x^{2}+y^{2}=r^{2} \cos ^{2} \theta+r^{2} \sin ^{2} \theta=r^{2} \quad\left(\text { because } \cos ^{2} \theta+\sin ^{2} \theta=1\right)
$$

So in polar coordinates, the modulus of a complex number is simply the distance from the origin of a point on an Argand diagram. This distance is clearly a non-negative real number.

## Question 13

(a) Figures 10 a and 10 b show two complex numbers $z_{1}$ and $z_{2}$, respectively. Write $z_{1}$ and $z_{2}$ in the $x+$ iy form.
(b) Figures 10 c and 10 d show two complex numbers $z_{3}$ and $z_{4}$, respectively. Calculate the modulus of the complex numbers $z_{3} z_{4}$.

Figure 10 See Question T3.


(b)


### 2.4 The argument of a complex number

If a complex number is written in polar form as
$z=r(\cos \theta+i \sin \theta)$, then $\theta$ is known as the argument of $z$ and is denoted by $\arg (z)$. For example, if
$z=4[\cos (\pi / 15)+i \sin (\pi / 15)]$ then $\arg (z)$ is $(\pi / 15) \mathrm{rad}$ which can be interpreted geometrically as follows. Consider a typical point representing a complex number, $z$, on an Argand diagram such as in Figure 5. Then the angle $\theta$ made by the line joining the point to the origin with the positive real axis is the argument. Notice that by convention the angle is measured anticlockwise (so that a negative angle would be measured in a clockwise direction). If $z=1+\sqrt{3} i$ then $\arg (z)$ is $(\pi / 3) \mathrm{rad}$, as shown in Figure 6.


Figure 6 The complex number,

There is a slight complication in the definition of the argument since a point, $z$, on an Argand diagram does not correspond to a single value of $\arg (z)$ because we can always add any integer multiple of $2 \pi$ to the value of $\theta$; in other words, if we rotate the point about the origin through any number of complete turns we always get back to the same point (see Figure 11). $\underline{\underline{29} 9}$

However, if we impose the condition that $-\pi<\theta \leq \pi \xrightarrow{\text { asg }}$ then $\theta$ is uniquely determined, and with this condition $\theta$ is known as the principal value of the argument of $z$. Notice that whereas the lower limit is greater than $-\pi$, the upper limit is less than or equal to $\pi$. [1088

Figure 11 The non-uniqueness of $\arg (z)$ (shown here as $\theta$ ).

If we have a complex number, $z=x+i y$, where $x$ and $y$ are not both zero, then the argument, $\theta$, is given by the solution to the following pair of equations

$$
\begin{align*}
& \cos \theta=\frac{x}{\sqrt{x^{2}+y^{2}}}  \tag{18}\\
& \sin \theta=\frac{y}{\sqrt{x^{2}+y^{2}}} \tag{19}
\end{align*}
$$

From Equations 18 and 19 we obtain

$$
\begin{equation*}
\tan \theta=y / x \tag{20}
\end{equation*}
$$

and while it is possible to use Equation 20 to find the angle $\theta$, this must be done with some care as the following example shows. (Alternatively we can use the Equations 18 and 19, see Solution B to Example 1.)

Example 1 Find the principal value of the argument of the complex number $z=-1+i$.

Solution A In this case $x=-1$ and $y=1$, and from Equation 20

$$
\begin{equation*}
\tan \theta=y / x \tag{Eqn20}
\end{equation*}
$$

we have $\tan \theta=-1$. In order to find $\theta$ your first thought might be to set your calculator to radian mode and to evaluate $\arctan (-1)$, which will give you the approximate value -0.785398 rad for $\theta$.

However, this is not a value of the argument, and certainly not the principal value, as you can easily see if you plot the point $-1+i$ on an Argand diagram, see Figure 12. The correct answer is approximately 2.356194 rad (or more precisely ( $3 \pi / 4$ ) rad).


Figure 12 The complex number, $z=-1+i$, plotted on an Argand diagram.

The essential point to realize here is that Equation 20 does not determine the angle $\theta$ uniquely because there are generally two angles in the range $-\pi<\theta \leq \pi$ that correspond to a given value of the tangent, and these angles differ by $\pi$ radians. The difficulty is quite easy to resolve if we always draw a diagram (such as Figure 12) when calculating a value for $\arg (z)$. In this case we would obtain the value -0.785398 rad from the calculator as before, then we see from the diagram that we must add $\pi$ radians to obtain the principal value

$$
\arg (z) \approx-0.785398+3.141593=2.356194 \mathrm{rad}
$$

For this particular value of $z$ you may be able to avoid the use of the calculator if you can see that $\arg (z)=(3 \pi / 4)$ rad directly from the figure.

Solution B Alternatively we can use Equations 18 and 19

$$
\begin{align*}
& \cos \theta=\frac{x}{\sqrt{x^{2}+y^{2}}}  \tag{Eqn18}\\
& \sin \theta=\frac{y}{\sqrt{x^{2}+y^{2}}} \tag{Eqn19}
\end{align*}
$$

to obtain $\cos \theta=-1 / \sqrt{2}$ and $\sin \theta=1 / \sqrt{2}$. The only angles in the range $-\pi<\theta \leq \pi$ that satisfy the first equation are $\theta=(3 \pi / 4)$ rad or $\theta=(-3 \pi / 4) \mathrm{rad}$, while the only angles that satisfy the second equation are $\theta=(\pi / 4) \mathrm{rad}$ or $\theta$ $=(3 \pi / 4) \mathrm{rad}$. Thus the required angle is
$\theta=(3 \pi / 4) \mathrm{rad}$.
Notice that the argument of $z=x+i y$ is not defined when $x$ and $y$ are both zero; in other words, the argument of $z=0$ is not defined.

The polar representation of a complex number is not unique. For example, both $\sqrt{2}[\cos (\pi / 4)+i \sin (\pi / 4)]$ and $\sqrt{2}[\cos (9 \pi / 4)+i \sin (9 \pi / 4)]$ represent the same complex number $1+i$. We can, however, make the representation unique if we insist that the argument takes its principal value. Notice that $-2[\cos (\pi / 3)+i \sin (\pi / 3)]$ is not in polar form. In fact (from Equation 16),

$$
\begin{align*}
& r(\cos \theta+i \sin \theta) \times \rho(\cos \phi+i \sin \phi)=\rho r[\cos (\theta+\phi)+i \sin (\theta+\phi)]  \tag{Eqn16}\\
& -2[\cos (\pi / 3)+i \sin (\pi / 3)]=2(\cos \pi+i \sin \pi)[\cos (\pi / 3)+i \sin (\pi / 3)] \\
& \\
& =2[\cos (4 \pi / 3)+i \sin (4 \pi / 3)]
\end{align*}
$$

and this final result is in polar form.

## Question T4

For each of the following complex numbers, find the principal value of the argument:
(a) $-1+\sqrt{3} i$,
(b) $1-\sqrt{3} i$,
(c) $\sqrt{3}+i$,
(d) $3+2 i$.
(Hint: You may find Figure 1 helpful.)



Figure 1 See Question R3.

### 2.5 The exponential form

You should be familiar with the following functions, and their series expansions

$$
\begin{align*}
& \mathrm{e}^{x}=\sum_{n=0}^{\infty} \frac{x^{n}}{n!}=1+\frac{x}{1!}+\frac{x^{2}}{2!}+\frac{x^{3}}{3!}+\ldots  \tag{Eqn3}\\
& \sin x=\sum_{n=0}^{\infty} \frac{(-1)^{n} x^{2 n+1}}{(2 n+1)!}=x-\frac{x^{3}}{3!}+\frac{x^{5}}{5!}-\frac{x^{7}}{7!}+\ldots  \tag{Eqn4}\\
& \cos x=\sum_{n=0}^{\infty} \frac{(-1)^{n} x^{2 n}}{(2 n)!}=1-\frac{x^{2}}{2!}+\frac{x^{4}}{4!}-\frac{x^{6}}{6!}+\ldots \tag{Eqn5}
\end{align*}
$$

Each of these functions may be extended so that they apply to a complex variable $z$ (rather than the real variable $x$ ), but, since real numbers are just a special case of complex numbers, the new functions must agree with the old ones in the special case when $z$ is real. For example, in order to define $\mathrm{e}^{z}$ we simply replace $x$ by $z$ in Equation 3
so that $\mathrm{e}^{z}=\sum_{n=0}^{\infty} \frac{z^{n}}{n!}=1+\frac{z}{1!}+\frac{z^{2}}{2!}+\frac{z^{3}}{3!}+\ldots$

Now let us consider the special case of $z=i \theta$, where $\theta$ is real and therefore $z$ is imaginary

$$
\mathrm{e}^{i \theta}=1+\frac{i \theta}{1!}+\frac{(i \theta)^{2}}{2!}+\frac{(i \theta)^{3}}{3!}+\frac{(i \theta)^{4}}{4!}+\frac{(i \theta)^{5}}{5!}+\ldots
$$

This expression can be simplified by using $i^{2}=-1$ to give

$$
\mathrm{e}^{i \theta}=1+\frac{i \theta}{1!}-\frac{\theta^{2}}{2!}-\frac{i \theta^{3}}{3!}+\frac{\theta^{4}}{4!}+\frac{i \theta^{5}}{5!}-\ldots
$$

We can see that the terms are alternately real and imaginary, so it is useful to split the series into two

$$
\mathrm{e}^{i \theta}=\left(1-\frac{\theta^{2}}{2!}+\frac{\theta^{4}}{4!}-\frac{\theta^{6}}{6!}+\ldots\right)+i\left(\frac{\theta}{1!}-\frac{\theta^{3}}{3!}+\frac{\theta^{5}}{5!}-\frac{\theta^{7}}{7!}+\ldots\right)
$$

If we compare the right-hand side of this equation with the series for $\sin \theta$ and $\cos \theta$ we see that $\mathrm{e}^{i \theta}$ can be written as

$$
\begin{equation*}
\mathrm{e}^{i \theta}=\cos \theta+i \sin \theta \quad(\text { Euler's formula }) \tag{22}
\end{equation*}
$$

This important formula, known as Euler's formula, gives us the real and imaginary parts of $\mathrm{e}^{i \theta}$.

## Question T5

Plot $\mathrm{e}^{i n \pi / 8}$ on an Argand diagram for $n=0,1,2, \ldots 15$. What geometrical shape do you think would describe the position of the points representing $z=\mathrm{e}^{i \theta}$ where $\theta$ can take any real value?

If we put $\theta=\pi$ in Euler's formula then, since $\cos (\pi)=-1$ and $\sin (\pi)=0$, we find

$$
\begin{equation*}
\mathrm{e}^{i \pi}=-1 \tag{23}
\end{equation*}
$$

This identity is quite remarkable since it relates:

- the numerical constant e, which originates from problems of growth and decay;
- the numerical constant $\pi$, which originates from the ratio of the circumference to the diameter of a circle;
- the number 1 , which has the special property that $1 \times n=n$ for any number $n$;
- the symbol $i$, which has the property $i^{2}=-1$ and was originally introduced in order to solve equations such as $x^{2}+1=0$.


## Question T6

Use Euler's formula to prove the following important relations (Equations 24 and 25):

$$
\begin{align*}
& \sin \theta=\frac{\mathrm{e}^{i \theta}-\mathrm{e}^{-i \theta}}{2 i}  \tag{24}\\
& \cos \theta=\frac{\mathrm{e}^{i \theta}+\mathrm{e}^{-i \theta}}{2} \tag{25}
\end{align*}
$$

Now let us consider a general complex number, $z=x+i y$, and write it in terms of polar coordinates by substituting $x=r \cos \theta$ and $y=r \sin \theta$

$$
z=x+i y=r \cos \theta+i r \sin \theta=r(\cos \theta+i \sin \theta)
$$

which can be written more compactly, using Euler's equation, to obtain

$$
\begin{equation*}
z=r \mathrm{e}^{i \theta} \tag{26}
\end{equation*}
$$

which is known as the exponential form (or exponential representation) of a complex number. Notice that $r$ is (by definition) non-negative.

## Question T7

Find the exponential representation for the following complex numbers:
(a) $\sqrt{2}(1+i)$,
(b) $3(1+i \sqrt{3})$,
(c) $2(\sqrt{3}+i)$,
(d) $0.2+2.3 i$.
$\square$
$\leftrightarrow$ Express the complex numbers $i$ and 2 in exponential form.

## 3 Arithmetic of complex numbers

### 3.1 Products and quotients

Multiplication of complex numbers in exponential form is very easy, however in this form addition is more difficult.

## Products

One important property of real powers of any real quantity is the identity

$$
a^{s} a^{t}=a^{s+t}
$$

$$
\begin{equation*}
\underline{1089} \tag{27}
\end{equation*}
$$

which we will assume is also true if any (or all) of the quantities $a, s$ and $t$ are complex. This identity allows us to multiply complex numbers easily in exponential form since, if $z=r \mathrm{e}^{i \theta}$ and $w=\rho \mathrm{e}^{i \phi}$, then the product is given by $r \mathrm{e}^{i \theta} \times \rho \mathrm{e}^{i \phi}=r \rho \mathrm{e}^{i(\theta+\phi)}$
which is closely related to the result quoted in Equation 16.

$$
r(\cos \theta+i \sin \theta) \times \rho(\cos \phi+i \sin \phi)=\rho r[\cos (\theta+\phi)+i \sin (\theta+\phi)]
$$

Notice that in Equation 28 the moduli $\underline{\text { 畦 }}$ of the two complex numbers are multiplied whereas the arguments are added (as in Equation 16).
(1)

## Question T8

Find $z w$ in exponential form (choosing the principal value of the argument in each case) for each of the following:
(a) $z=2 \mathrm{e}^{i \pi / 4} \quad w=2 \mathrm{e}^{i \pi / 4}$
(b) $z=3 \mathrm{e}^{i \pi} \quad w=2 \mathrm{e}^{i \pi / 4}$
(c) $z=2 \mathrm{e}^{i \pi / 16} \quad w=\frac{1}{2} \mathrm{e}^{-i \pi / 16}$

## Quotients

For real numbers we have the general result that $\quad \frac{a^{m}}{b^{n}}=a^{m} b^{-n} \quad$ where $b$ is non-zero.
Extending this result to complex numbers, we find that simplifying a quotient of two complex numbers in exponential form is a straightforward variation of multiplication. So if $z=r \mathrm{e}^{i \theta}$ and $w=\rho \mathrm{e}^{i \phi}$, then $z / w$ is given
by $\quad \frac{z}{w}=\frac{r \mathrm{e}^{i \theta}}{\rho \mathrm{e}^{i \phi}}=\frac{r}{\rho} \mathrm{e}^{i(\theta-\phi)} \quad$ that is, we divide the moduli and subtract the arguments.

## Question T9

Find $z / w$ in exponential form (choosing the principal value of the argument in each case) for each of the following:
(a) $z=2 \mathrm{e}^{i \pi / 4} \quad w=4 \mathrm{e}^{-i \pi / 8}$
(b) $z=3 \mathrm{e}^{i \pi / 4} \quad w=2 \mathrm{e}^{i \pi / 2}$
(c) $z=2 \mathrm{e}^{-i \pi / 16} \quad w=2 \mathrm{e}^{i \pi / 16}$

### 3.2 Sums and differences

Suppose that we are given two complex numbers in exponential form, $a \mathrm{e}^{i \alpha}$ and $b \mathrm{e}^{i \beta}$, we can certainly write their sum in exponential form

$$
\begin{equation*}
a \mathrm{e}^{i \alpha}+b \mathrm{e}^{i \beta}=c \mathrm{e}^{i \gamma} \tag{29}
\end{equation*}
$$

$\underline{1088}$
but it is not particularly easy to find the values of $c$ and $\gamma$ directly. First we will show you a direct method of finding the sum, then an alternative method which is often easier.
(1)

## Method 1

To find $c$ we can write

$$
\begin{aligned}
c^{2} & =\left(c \mathrm{e}^{i \gamma}\right)\left(c \mathrm{e}^{-i \gamma}\right)=\left(a \mathrm{e}^{i \alpha}+b \mathrm{e}^{i \beta}\right)\left(a \mathrm{e}^{-i \alpha}+b \mathrm{e}^{-i \beta}\right) \\
& =a^{2}+b^{2}+a b\left[\mathrm{e}^{i(\alpha-\beta)}+\mathrm{e}^{-i(\alpha-\beta)}\right] \\
& =a^{2}+b^{2}+2 a b \cos (\alpha-\beta)
\end{aligned}
$$

So we have an equation for $c$

$$
\begin{equation*}
c=\sqrt{a^{2}+b^{2}+2 a b \cos (\alpha-\beta)} \tag{30}
\end{equation*}
$$

$\underline{1088}$
To find $\gamma$ we can use Euler's formula to write $\mathrm{e}^{i \alpha}, \mathrm{e}^{i \beta}$ and $\mathrm{e}^{i \gamma}$ in polar form, and Equation 29

$$
a \mathrm{e}^{i \alpha}+b \mathrm{e}^{i \beta}=c \mathrm{e}^{i \gamma}
$$

(Eqn 29
then becomes

$$
a(\cos \alpha+i \sin \alpha)+b(\cos \beta+i \sin \beta)=c(\cos \gamma+i \sin \gamma)
$$

and equating real and imaginary parts gives us

$$
\begin{aligned}
& \cos \gamma=\frac{a \cos \alpha+b \cos \beta}{c} \\
& \sin \gamma=\frac{a \sin \alpha+b \sin \beta}{c}
\end{aligned}
$$

So if we are given $a, b, \alpha$ and $\beta$ we can first find $c$, then $\cos \gamma$ and $\sin \gamma$. Then the condition, $-\pi<\gamma \leq \pi$ uniquely fixes $\gamma$.
$\leftrightarrow$ Given that $z=a \mathrm{e}^{i \alpha}=\mathrm{e}^{2 i \pi / 3}$ and $w=b \mathrm{e}^{i \beta}=\mathrm{e}^{i \pi / 3}$, find $z+w$ in exponential form.

## Method 2

The alternative method of finding the sum in exponential form relies on the fact that addition in Cartesian form is very straightforward. So if $z$ and $w$ are given in exponential form, we first convert them into Cartesian form, then find the sum, and finally convert the answer back into exponential form. We can use the previous example to illustrate the method.

Given that $z=\mathrm{e}^{2 i \pi / 3}$ and $w=\mathrm{e}^{i \pi / 3}$, we can use Euler's formula (Equation 22)

$$
\begin{equation*}
\mathrm{e}^{i \theta}=\cos \theta+i \sin \theta \tag{Eqn22}
\end{equation*}
$$

to write $z$ and $w$ in Cartesian form

$$
z=\frac{-1+i \sqrt{3}}{2} \quad \text { and } \quad w=\frac{1+i \sqrt{3}}{2}
$$

from which it is straightforward to find the sum

$$
z+w=\frac{-1+i \sqrt{3}}{2}+\frac{1+i \sqrt{3}}{2}=i \sqrt{3}
$$

Since $i \sqrt{3}=\sqrt{3} \mathrm{e}^{i \pi / 2}$ this result for $z+w$ agrees with our previous result, but you should now be convinced that addition (and subtraction) are much easier using the Cartesian rather than exponential form of complex numbers.

## Question T10

Express $\mathrm{e}^{i \pi}-\mathrm{e}^{i \pi / 2}$ in exponential form.

## Question T11

Use the exponential representation to show that for any complex numbers $z_{1}$ and $z_{2}$

$$
\left|z_{1}\right|+\left|z_{2}\right| \geq\left|z_{1}+z_{2}\right| \quad \underline{\text { Iq88 }}
$$

(Hint: Use Equation 30 and the fact that the cosine of any angle is less than or equal to one.)

$$
c=\sqrt{a^{2}+b^{2}+2 a b \cos (\alpha-\beta)}
$$

(Eqn 30)

### 3.3 Powers, roots and reciprocals

The meaning of an expression such as $(1+2 i)^{3}$ is clear enough, 10 however the meaning of an expression such as $(1+2 i)^{\pi}$ is by no means obvious. In this subsection we will attempt to attach a sensible meaning to such expressions.

The following results are certainly true for real numbers $a$ and $b$

$$
(u \vee)^{a}=u^{a} \mathbf{V}^{a} \quad \text { and } \quad\left(u^{a}\right)^{b}=u^{a b}
$$

and we assume that they also hold for complex numbers. So to raise the complex number $z=r \mathrm{e}^{i \theta}$ to the power $\alpha$ we have

$$
\begin{equation*}
z=\left(r \mathrm{e}^{i \theta}\right)^{\alpha}=r^{\alpha} \mathrm{e}^{i \alpha \theta} \tag{31}
\end{equation*}
$$

For example, if $z=2 \mathrm{e}^{i \pi / 4}$ then

$$
z^{2}=4 \mathrm{e}^{i \pi / 2}=4[\cos (\pi / 2)+i \sin (\pi / 2)]=4 i
$$

We can check this result by first putting $z$ into Cartesian form

$$
z=2 \mathrm{e}^{i \pi / 4}=2[\cos (\pi / 4)+i \sin (\pi / 4)]=\frac{2}{\sqrt{2}}(1+i)=\sqrt{2}(1+i)
$$

and then evaluating $z^{2}$ to give $z^{2}=2\left(1+2 i+i^{2}\right)=4 i$.

## Question T12

Find $z^{2}$ and $z^{3}$, where $z=2 \mathrm{e}^{i \pi / 3}$. Plot $z$ and $z^{2}$ and $z^{3}$ on an Argand diagram and use it to explain the movement of the point representing $z^{n}$ for successive integer values of $n$.

It is certainly possible to calculate powers of a complex number in Cartesian form, but it must be done with some care if we are to keep the algebra under control.

- Express $(1+i)^{8}$ in Cartesian form.

Very often a better method would be to express the complex number in exponential form, then raise it to the power, and finally convert the answer back into Cartesian form. Using this approach to evaluate $(1+i)^{8}$ we have

$$
(1+i)=\sqrt{2} \mathrm{e}^{i \pi / 4} \text { so that }(1+i)^{8}=\left(\sqrt{2} \mathrm{e}^{i \pi / 4}\right)^{8}=(\sqrt{2})^{8} \mathrm{e}^{2 \pi i}=16
$$

## Question T13

Evaluate $\underline{\square 128}(1+\sqrt{3} i)^{3}$ and hence express $(1+\sqrt{3} i)^{10}$ in Cartesian form. Also express $z=1+\sqrt{3} i$ in exponential form and use this result to evaluate $(1+\sqrt{3} i)^{10}$.

## Question T14

The complex number, $u$, is defined by

$$
u=(1.1) \mathrm{e}^{2 \pi i / 15}
$$

Use an Argand diagram to plot $u^{n}$ for $n=1,2,3, \ldots 15$. What sort of curve would you expect to get if larger (integer) values of $n$ were plotted and the points joined by a smooth curve?

The following example illustrates an area of physics that uses complex numbers.
$\uparrow$ The propagation constant of a cable carrying an alternating current of angular frequency $\omega$ is given by

$$
\sigma=\sqrt{(R+i \omega L)(G+i \omega C)}
$$

$$
\underline{1088}
$$

where $R, L, G$ and $C$ are, respectively, the resistance, inductance, conductance and capacitance of the cable $\underline{\text { nge }}$
Evaluate $\sigma$ if $R=90 \Omega, L=0.002 \mathrm{H}, G=5 \times 10^{-5} \mathrm{~S}, C=0.05 \times 10^{-6} \mathrm{~F}$ and $\omega=5000 \mathrm{~s}^{-1}$.

## Complex powers

The meaning of a complex power of a complex number becomes clear if we use the exponential, rather than Cartesian, form. For example

$$
z=\left(\mathrm{e}^{i \pi / 4}\right)^{i}=\mathrm{e}^{-\pi / 4}
$$

## Question T15

Use suitable exponential representations to reduce the following to simpler (or at least, more familiar) expressions
(a) $i^{i}$,
(b) $\left(\frac{1+\sqrt{3} i}{2}\right)^{i}$,
(c) $\left(\frac{1+\sqrt{3} i}{2}\right)^{1+i}$.

## Roots

The exponential form is also convenient for working out roots of complex numbers since roots are fractional powers. As an example, if we require $z^{1 / 2}$, where $z=2 \mathrm{e}^{i \pi / 2}$, then we can write

$$
z^{1 / 2}=2^{1 / 2} \mathrm{e}^{i \pi / 4}=\frac{\sqrt{2}(1+i)}{\sqrt{2}}=1+i
$$

$$
\underline{\underline{1 q 8}}
$$

This result can be checked by realizing that

$$
z=2 \mathrm{e}^{i \pi / 2}=2[\cos (\pi / 2)+i \sin (\pi / 2)]=2 i
$$

and that $\quad\left(z^{1 / 2}\right)^{2}=(1+i)^{2}=1+2 i+i^{2}=2 i$

## Question T16

Use a suitable exponential representation to express $i^{1 / 2}$ in Cartesian form. Check your answer by means of explicit multiplication.

## Reciprocals

The reciprocal $\underline{\text { 吗 }}$ of a complex number is a special case of a power, so,
if $z=r \mathrm{e}^{i \theta}$ with $r$ non-zero, we have

$$
\begin{equation*}
z^{-1}=\left(r \mathrm{e}^{i \theta}\right)^{-1}=\frac{\mathrm{e}^{-i \theta}}{r} \tag{32}
\end{equation*}
$$

For example, if $z=\mathrm{e}^{i \pi / 3}$ then $z^{-1}=\mathrm{e}^{-i \pi / 3}$. We can compare this result with the equivalent calculation using the Cartesian form for $z$.
Since $z=\mathrm{e}^{i \pi / 3}=\frac{1+\sqrt{3} i}{2}$

$$
z^{-1}=\frac{2}{1+i \sqrt{3}}=\frac{2(1-i \sqrt{3})}{(1+i \sqrt{3})(1-i \sqrt{3})}=\frac{2(1-i \sqrt{3})}{1+3}=\frac{1-i \sqrt{3}}{2}=\mathrm{e}^{-i \pi / 3}
$$

As you can see, the exponential form provides a somewhat easier method of calculating reciprocals than does the Cartesian form.

## Question T17

If $z=2 \mathrm{e}^{(i+1) \pi / 4}$ what are the exponential and Cartesian forms of $z^{-1}$ ?

### 3.4 Complex conjugates

The rule for finding the complex conjugate is to change $i$ to $-i$, as in

$$
(2+3 i)^{*}=2-3 i
$$

so for any complex number in exponential form, $z=r \mathrm{e}^{i \theta}$, we have

$$
z^{*}=r \mathrm{e}^{-i \theta}
$$

From this result we see that the product of any complex number with its complex conjugate is real

$$
z z^{*}=r \mathrm{e}^{i \theta} \times r \mathrm{e}^{-i \theta}=r^{2}
$$

But from Subsection 2.3 we know that $r$ is actually the modulus of $z$, i.e. $|z|$ and so we have the identity

$$
\begin{equation*}
z z^{*}=|z|^{2} \tag{34}
\end{equation*}
$$

and therefore, for non-zero values of $r$, we have

$$
z^{-1}=r^{-1} \mathrm{e}^{-i \theta}=\frac{r \mathrm{e}^{-i \theta}}{r^{2}}
$$

so that

$$
\begin{equation*}
z^{-1}=\frac{z^{*}}{|z|^{2}} \tag{35}
\end{equation*}
$$

Equation 34 can often be useful if you wish to check that a complicated expression is real and positive (which is often the case in quantum mechanics, for example), for you simply need to recognize that the complicated expression is a product of a complex number with its conjugate. It can also be useful if you wish to establish a result involving moduli (as in the following example).
©

- Show that for any two complex numbers $z$ and $w$

$$
|z+w|^{2}+|z-w|^{2}=2|z|^{2}+2|w|^{2}
$$

The next question establishes some results which you have probably already assumed to be true.

## Question 118 <br> Quetion

For any complex numbers $z$ and $w$ show that:

$$
\text { (a) }(z w)^{*}=z^{*} w^{*}, \quad \text { (b) }\left|z^{-1}\right|=|z|^{-1}, \quad \text { (c) }|z w|=|z||w|
$$

### 3.5 Applications

In this subsection we consider some applications of the polar and exponential representations which are of particular relevance to physics. You do not need to be familiar with the physics involved.

## Example 2

Two complex numbers, $Z_{1}$ and $Z_{2}$ are

$$
\begin{aligned}
& z_{1}=2+2 i \\
& z_{2}=1+2 i
\end{aligned}
$$

If $Z=Z_{1}+Z_{2}$ find $|Z|$ and the principal value of $\arg (Z)$.


Solution

$$
\begin{aligned}
& Z=(2+2 i)+(1+2 i)=3+4 i \\
& |Z|=\sqrt{3^{2}+4^{2}}=5
\end{aligned}
$$

If $\theta=\arg (Z)$, then $\sin \theta=4 / 5$ and $\cos \theta=3 / 5$ which implies that $\theta \approx 0.927 \mathrm{rad}$.

Example 3 Repeat Example 2 with $Z_{1}$ and $Z_{2}$ as before, but this time suppose that $Z$ is given by

$$
\frac{1}{z}=\frac{1}{z_{1}}+\frac{1}{z_{2}} \quad \underline{\text { 苞 }}
$$

## Solution

$$
\begin{aligned}
Z & =\frac{z_{1} z_{2}}{z_{1}+z_{2}}=\frac{(2+2 i)(1+2 i)}{3+4 i}=\frac{(2-4)+i(2+4)}{3+4 i} \\
& =\frac{(-2+6 i)(3-4 i)}{25}=\frac{18+26 i}{25} \\
|z| & =\frac{\sqrt{(18)^{2}+(26)^{2}}}{25} \approx 1.26
\end{aligned}
$$

If $\theta=\arg (Z)$, then

$$
\sin \theta=\frac{26}{\sqrt{(18)^{2}+(26)^{2}}} \quad \text { and } \quad \cos \theta=\frac{18}{\sqrt{(18)^{2}+(26)^{2}}}
$$

which implies that $\theta \approx 0.965 \mathrm{rad}$.

Example 4 The function

$$
\begin{equation*}
f(\phi)=\mathrm{e}^{i m \phi} \tag{36}
\end{equation*}
$$

where $\phi$ is an angle (and therefore real) and $m$ is a real number, occurs in quantum theory. What are the possible values of $m$, if we require that for every value of $\phi$ :

$$
f(\phi)=f(\phi+2 \pi)
$$

Solution If $f(\phi)=f(\phi+2 \pi)$ then

$$
\mathrm{e}^{i m \phi}=\mathrm{e}^{i m(\phi+2 \pi)}=\mathrm{e}^{i m \phi} \mathrm{e}^{2 m \pi i}
$$

We can cancel the factor $\mathrm{e}^{i m \phi}$ (since it is non-zero) giving

$$
\mathrm{e}^{2 m \pi i}=1
$$

Using Euler's formula, this can be written as

$$
\cos (2 \pi m)+i \sin (2 \pi m)=1
$$

so that $\quad \cos (2 \pi m)=1 \quad$ and $\quad \sin (2 \pi m)=0$
We recall that $\quad \cos \theta=1$ if $\theta=0, \pm 2 \pi, \pm 4 \pi, \ldots$
and $\quad \sin \theta=0$ if $\theta=0, \pm \pi, \pm 2 \pi, \ldots$
and (since both conditions must apply simultaneously) we have

$$
2 \pi m=0, \pm 2 \pi, \pm 4 \pi, \ldots
$$

therefore it follows that

$$
m=0, \pm 1, \pm 2, \pm 3, \ldots
$$

In other words, the condition $f(\phi)=f(\phi+2 \pi)$ ensures that only integer values of $m$ are acceptable.

## Example 5

Given that $a=\frac{\mathrm{e}^{-i \phi t}-1}{\phi}$, with $\phi$ and $t$ real, show that

$$
|a|^{2}=\frac{4}{\phi^{2}} \sin ^{2}\left(\frac{\phi t}{2}\right) \quad \underline{\| 28}
$$

Solution We could certainly calculate $|a|^{2}$ by writing

$$
\begin{equation*}
|a|^{2}=\frac{\left(\mathrm{e}^{-i \phi t}-1\right)\left(\mathrm{e}^{i \phi t}-1\right)}{\phi^{2}} \tag{Eqn35}
\end{equation*}
$$

(using Equation 35) $\quad z^{-1}=\frac{z^{*}}{|z|^{2}}$
and then simplifying the result (using various trigonometric identities); however if you notice that the required expression for $|a|^{2}$ involves only ( $\phi t / 2$ ) then you might think of the following (more elegant) method.

We force the expression for $a$ to involve terms in ( $\phi t / 2$ ) by extracting a factor $\mathrm{e}^{-i \phi t / 2}$, so that

$$
a=\frac{\mathrm{e}^{-i \phi t / 2}}{\phi}\left(\mathrm{e}^{-i \phi t / 2}-\mathrm{e}^{i \phi t / 2}\right)=\frac{\mathrm{e}^{-i \phi t / 2}}{\phi}\left[-2 i \sin \left(\frac{\phi t}{2}\right)\right]=(-2 i)\left[\frac{\sin (\phi t / 2)}{\phi} \mathrm{e}^{-i \phi t / 2}\right]
$$

In the right-hand expression, $-2 i$ has a modulus of 2 , while the term in square brackets is in exponential form $r \mathrm{e}^{i \theta}$ with $r=\frac{\sin (\phi t / 2)}{\phi}$, so it follows that

$$
|a|^{2}=\left[\frac{2}{\phi} \sin \left(\frac{\phi t}{2}\right)\right]^{2}=\frac{4}{\phi^{2}} \sin ^{2}\left(\frac{\phi t}{2}\right)
$$

## 4 Closing items

### 4.1 Module summary

1 A complex number is equivalent to an ordered pair of real numbers, $(x, y)$. The addition and subtraction of complex numbers obey the same rules as of two-dimensional vectors

$$
\begin{equation*}
(x, y)+(a, b)=(x+a, y+b) \tag{Eqn8}
\end{equation*}
$$

Multiplication of two complex numbers obeys the rule

$$
\begin{equation*}
(a, b) \times(x, y)=[(a x-b y),(a y+b x)] \tag{Eqn9}
\end{equation*}
$$

In practice, a complex number is more usually written as $x+i y$ with $i$ having the property that $i^{2}=-1$.
2 A complex number, $z=x+i y$, is said tg be in Cartesian form or a Cartesian representation.
3 A complex number, $z=r(\cos \theta+i \sin )$ is said to be in polar form or a polar representation.
The polar form $r(\cos \theta+i \sin )$ of a given complex number $z$ is not unique. However, we can make it so by chosing the principal value of the argument of $z$, i.e. if $-\pi<\theta \leq \pi$. (Note that $r=|z| \geq 0$.)

4 We can convert from Cartesian into polar form by using

$$
\begin{align*}
& r=\sqrt{x^{2}+y^{2}}  \tag{Eqn13}\\
& \cos \theta=\frac{x}{\sqrt{x^{2}+y^{2}}}  \tag{Eqn14}\\
& \sin \theta=\frac{y}{\sqrt{x^{2}+y^{2}}} \tag{Eqn15}
\end{align*}
$$

and from polar to Cartesian form by means of

$$
\begin{align*}
& x=r \cos \theta  \tag{Eqn10}\\
& y=r \sin \theta \tag{Eqn11}
\end{align*}
$$

5 A complex number can be represented by a point on an Argand diagram by using $(x, y)$ as the Cartesian coordinates or $(r, \theta)$ as the polar coordinates of the point. By convention, $\theta$ is measured anticlockwise from the positive real axis.

6 If a complex number is represented by a point on an Argand diagram, then multiplication by $z=r(\cos \theta+i \sin \theta)$ corresponds to a change in the distance of the point from the origin by a factor of $r$, together with an anticlockwise rotation through an angle $\theta$.
7 If a complex number, $z$, is represented in polar form by $z=r(\cos \theta+i \sin \theta)$ then the modulus of $z$ is given by $|z|=r$, and $\theta$ is known as the argument of $z$ (written as $\arg (z)$ ). If $z$ is represented by a point on an Argand diagram then $r$ is the 'distance' of the point from the origin and $\theta$ is the angle made by a line from the point to the origin with the positive real axis. If $\theta$ satisfies $-\pi<\theta \leq \pi$, then $\theta$ is known as the principal value of the argument of $z$.
8 Euler's formula states that

$$
\begin{equation*}
\mathrm{e}^{i \theta}=\cos \theta+i \sin \theta \tag{Eqn22}
\end{equation*}
$$

9 If a complex number, $z$, is written as $z=r \mathrm{e}^{i \theta}$, with $r \geq 0$, then $z$ is said to be in exponential form or exponential representation. Euler's formula provides a direct connection between the polar and exponential forms since

$$
r \mathrm{e}^{i \theta}=r(\cos \theta+i \sin \theta)
$$

(1)

10 The following operations can be carried out on complex numbers in exponential form: multiplication

$$
r \mathrm{e}^{i \theta} \times \rho \mathrm{e}^{i \phi}=r \rho \mathrm{e}^{i(\theta+\phi)}
$$

simplifying quotients

$$
\frac{r \mathrm{e}^{i \theta}}{\rho \mathrm{e}^{i \phi}}=\frac{r}{\rho} \mathrm{e}^{i(\theta-\phi)}
$$

finding real powers

$$
\left(r \mathrm{e}^{i \theta}\right)^{\alpha}=r^{\alpha} \mathrm{e}^{i \alpha \theta}
$$

finding reciprocals

$$
\left(r \mathrm{e}^{i \theta}\right)^{-1}=r^{-1} \mathrm{e}^{-i \theta}
$$

finding complex conjugates

$$
\left(r \mathrm{e}^{i \theta}\right)^{*}=r \mathrm{e}^{-i \theta}
$$

We may also calculate complex powers of a complex number. The same values apply as for real powers, i.e.

$$
z^{a} z^{b}=z^{a b} \quad \text { and } \quad\left(z^{a}\right)=z^{a b}
$$

for complex numbers $a, b$ and $z$.
11 The addition and subtraction of complex numbers in either exponential or polar form are complicated in comparison to the same operations on complex numbers in the Cartesian form. Specifically, the sum of $a \mathrm{e}^{i \theta}$ and $b \mathrm{e}^{i \beta}$ can be written as

$$
\begin{equation*}
a \mathrm{e}^{i \alpha}+b \mathrm{e}^{i \beta}=c \mathrm{e}^{i \gamma} \tag{Eqn29}
\end{equation*}
$$

where

$$
\begin{align*}
& c=\sqrt{a^{2}+b^{2}+2 a b \cos (\alpha-\beta)}  \tag{Eqn30}\\
& \cos \gamma=\frac{a \cos \alpha+b \cos \beta}{c} \\
& \sin \gamma=\frac{a \sin \alpha+b \sin \beta}{c}
\end{align*}
$$

Where possible, it is better to use the Cartesian form for addition and subtraction.

### 4.2 Achievements

Having completed this module, you should be able to:
A1 Define the terms that are emboldened and flagged in the margins of the module.
A2 Convert a complex number between the Cartesian, polar and exponential forms.
A3 Represent a complex number (in either Cartesian, exponential or polar form) by means of a point on an Argand diagram, and describe geometrically the effect of complex addition or multiplication.

A4 Find the modulus, argument and complex conjugate of complex numbers.
A5 Find powers and reciprocals of complex numbers.

Study comment You may now wish to take the Exit test for this module which tests these Achievements. If you prefer to study the module further before taking this test then return to the Module contents to review some of the topics.
(1)

### 4.3 Exit test

Study comment Having completed this module, you should be able to answer the following questions, each of which tests one or more of the Achievements.

## Question E1

(A2) If $\operatorname{Re}(z)=1$ and $\operatorname{Im}(z)=-\sqrt{3}$, express $z$ in (a) Cartesian form, (b) polar form using the principal value of the argument, (c) exponential form.

## Question E2

(A2, A4 and A5) Express $z=1-i \sqrt{3}$ and $w=1+i$ in exponential form and use your results to express $R$ in exponential form, where $R$ is given by

$$
R=\frac{1}{w}\left(\frac{z}{2}\right)^{12}
$$

Make sure that you give the principal value of $\arg (R)$.
(Hint: You may find Answers F1 and E1 helpful.)
©

## Question E3

(A3) If $z$ is given by

$$
z=\frac{\mathrm{e}^{i \theta}}{1+\theta}
$$

where $\theta$ can take any non-negative real value, sketch the curve on which any point corresponding to $z$ must lie on an Argand diagram. Justify the main features of the curve.
$\square$

## Question E4

(A3) (a) Show how the addition of complex numbers $z=1+i$ and $w=2+4 i$ can be considered as vector addition on an Argand diagram.
(b) A complex number $z$ is defined by $z=\mathrm{e}^{2 \pi i / n}$ for some fixed positive integer value of $n$. Show (giving a sketch) how the series

$$
S=\sum_{k=1}^{n} z^{k}
$$

can be considered as the sum of $n$ vectors on an Argand diagram. Hence determine the value of $S$.

## Question E5

(A2) Show that $(-1)^{n}=\mathrm{e}^{i n \pi}$, where $n$ is any integer.

Study comment This is the final Exit test question. When you have completed the Exit test go back to Subsection 1.2 and try the Fast track questions if you have not already done so.

If you have completed both the Fast track questions and the Exit test, then you have finished the module and may leave it here.

