Module M3.3 Demoivre's theorem and complex algebra

Exit module

1 Opening items

1.1 Module introduction

Section 2 of this module is concerned with *Demoivre's theorem* and its applications. We start in Subsection 2.1 by proving the theorem which states that

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$$

(where $i^2 = -1$), and then use it to derive trigonometric identities, in Subsection 2.2, and to find all solutions to the equation $z^n - 1 = 0$ (the *roots of unity*) in Subsection 2.3.

The remainder of this module is concerned with complex algebra; that is the manipulation of expressions involving complex variables. In Subsections 3.1 and 3.2 we solve some algebraic equations and consider the related problem of factorization. In Subsection 3.3 we point out techniques for simplifying complex algebraic expressions. Subsection 3.4 is concerned with the complex *binomial expansion*; that is, expanding $(a + b)^n$ in terms of powers of the variables a and b. Proofs of this theorem do not usually distinguish between real and complex variables, but there are applications which are specific to the complex case. Finally in Subsection 3.5 we mention the complex form of the geometric series and use it to obtain more trigonometric identities. Don't worry if you are unfamiliar with the physics used in the examples in this module.

Study comment Having read the introduction you may feel that you are already familiar with the material covered by this module and that you do not need to study it. If so, try the *Fast track questions* given in Subsection 1.2. If not, proceed directly to *Ready to study?* in Subsection 1.3.

1.2 Fast track questions

Study comment Can you answer the following *Fast track questions*? If you answer the questions successfully you need only glance through the module before looking at the *Module summary* (Subsection 4.1) and the *Achievements* listed in Subsection 4.2. If you are sure that you can meet each of these achievements, try the *Exit test* in Subsection 4.3. If you have difficulty with only one or two of the questions you should follow the guidance given in the answers and read the relevant parts of the module. However, *if you have difficulty with more than two of the Exit questions you are strongly advised to study the whole module*.

Question F1

Use Demoivre's theorem to find all the roots of $z^n - 1 = 0$, where n is a positive integer. For $n = 3$, plot your results on an Argand diagram.

Question F2

Use Demoivre's theorem to find z^5 in its simplest form, where

$$z = 2[\cos(\pi/10) + i\sin(\pi/10)]$$

Question F3

Use Demoivre's theorem, together with the complex binomial theorem, to show that

$$\cos(4\theta) = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$$

$$\sin(4\theta) = 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta$$

Question F4

Z is given by

$$Z = R + i\left(\omega L - \frac{1}{\omega C}\right)$$ ☞

where R, ω, L and C are all real. Find the real and imaginary parts of Z^{-1}.

Study comment Having seen the *Fast track questions* you may feel that it would be wiser to follow the normal route through the module and to proceed directly to *Ready to study?* in Subsection 1.3.

Alternatively, you may still be sufficiently comfortable with the material covered by the module to proceed directly to the *Closing items*.

1.3 Ready to study?

Study comment To begin the study of this module you need to be familiar with the following: the representation of a *complex number* on an *Argand diagram*; the *modulus*, *complex conjugate* and *argument* (including the *principal value*) of a *complex number*, *Euler's formula*; the *rationalization* of *complex quotients*; *trigonometric identities*, such as $\cos(2\theta) = \cos^2\theta - \sin^2\theta$ and results such as $\sin(\pi/3) = \sqrt{3}/2$ and $\cos(\pi/3) = 1/2$. You will also need to be familiar with the *binomial expansion* (for real numbers), the sum of a *geometric series* of real numbers, the formula for the solution of a *quadratic equation* and the *fundamental theorem of algebra*. If you are uncertain about any of these terms, you can review them now by reference to the *Glossary*, which will indicate where in *FLAP* they are developed. The following *Ready to study questions* will help you to establish whether you need to review some of the above topics before embarking on this module.

Throughout this module $\sqrt{x}$ means the *positive* square root so that $\sqrt{4} = 2$, and $i^2 = -1$.

Question R1

A complex number, z, is such that its real part has the value, 1, and its imaginary part is $\sqrt{3}$.
(a) Express z in Cartesian, polar and exponential forms.
(b) Express the complex conjugate of z, z^*, and z^{-1} in exponential form.

Question R2

A complex number, w, is defined by $w = 1 + i$ and $z = 1 + i\sqrt{3}$. (a) Express w in polar form. Use this result, together with your answer to Question R1, to find the polar form of zw.
(b) Find $\operatorname{Re}(zw)$ and $\operatorname{Im}(zw)$ (that is, the real and imaginary parts of zw).

Question R3

Given that $z = 3 + 4i$, plot z and z^* on an Argand diagram. What is the value of $|z|$?

Question R4

Rationalize the expression $$\frac{2+3i}{(1+i)(1-2i)}$$

Question R5

(a) State the fundamental theorem of algebra. (b) How many roots would you expect the following equation to have

$$z^5 + z^4 + z^3 + z^2 + z + 1 = 0$$

(c) If z is restricted to real values, what does the fundamental theorem of algebra tell us about the *number* of real roots of this equation?

Question R6

(a) Sum the series $1 + x + x^2 + \dots + x^{10} \quad (\text{for } x \neq 1)$

(b) Expand $(1 + x)^7$ in powers of x.

2 Demoivre's theorem

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2.1 Introduction

If we have an arbitrary complex number, z, then we can choose to write it in polar form as

$$z = r(\cos\theta + i\sin\theta)$$

where r and θ are real (and $i^2 = -1$). Furthermore, if we have another complex number

$$w = \rho(\cos\phi + i\sin\phi)$$

then the product of z and w can be written as

$$zw = r\rho[\cos(\theta + \phi) + i\sin(\theta + \phi)] \tag{1}$$

In other words, the moduli of the numbers are multiplied together and the arguments are summed. It is easy to generalize this result to n complex numbers in the following way

$$\begin{aligned} z_1 &= r_1[\cos(\theta_1) + i\sin(\theta_1)] \\ z_2 &= r_2[\cos(\theta_2) + i\sin(\theta_2)] \\ &\vdots \qquad\quad \vdots \qquad\quad \vdots \\ z_n &= r_n[\cos(\theta_n) + i\sin(\theta_n)] \end{aligned}$$

to obtain the product

$$z_1 z_2 \times \ldots \times z_n = (r_1 r_2 \times \ldots \times r_n)[\cos(\theta_1 + \theta_2 + \ldots + \theta_n) + i\sin(\theta_1 + \theta_2 + \ldots + \theta_n)]$$

Setting $r_1 = r_2 = \ldots = r_n = 1$ and $\theta_1 = \theta_2 = \ldots = \theta_n = \theta$ we obtain **Demoivre’s theorem**

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta) \qquad \textit{Demoivre's theorem} \qquad (2)$$

Question T1

Evaluate both $(\cos\theta + i\sin\theta)^2$ and $[\cos(2\theta) + i\sin(2\theta)]$ for $\theta = 0$, $(\pi/4)$ and $(\pi/2)$ rad. Show that your results are consistent with Demoivre's theorem for $n = 2$. ❏

The proof we have given for Demoivre's theorem is only valid if n is a positive integer, but it is possible to show that the theorem is true for any real n and we will make this assumption for the remainder of this module.

✦ Use Demoivre's theorem to show that one of the square roots of $i - 1$ is $2^{1/4}[\cos(3\pi/8) + i\sin(3\pi/8)]$.

Question T2

Use Demoivre's theorem to show that one of the square roots of $1 + i$ is

$$2^{1/4}[\cos(\pi/8) + i\sin(\pi/8)]$$

(*Hint*: First write $1 + i$ in polar form.) ❏

The significance of Demoivre's theorem is that instead of calculating expressions such as $(\cos\theta + i\sin\theta)^n$ by writing out the $n + 1$ individual terms in its binomial expansion, we know that the answer must be $\cos(n\theta) + i\sin(n\theta)$. To emphasize the advantage of Demoivre's theorem, consider the evaluation of z^8 where

$$z = 2[\cos(\pi/8) + i\sin(\pi/8)]$$

Without using Demoivre's theorem, we could write $z = x + iy$ and use a calculator to discover that $x \approx 1.847\,759$ and $y \approx 0.765\,367$, so that

$$z^8 = (x + iy)^8$$

$$= x^8 + 8ix^7y - 28x^6y^2 - 56ix^5y^3 + 70x^4y^4 + 56ix^3y^5 - 28x^2y^6 - 8ixy^7 + y^8$$

and, after a considerable amount of arithmetic, we would obtain the approximate answer

$$z^8 \approx -256.0 - 1.53 \times 10^{-4}i$$

Compare this brute force approach with the elegance of Demoivre's theorem which gives the exact answer

$$z^8 = \{2[\cos(\pi/8) + i\sin(\pi/8)]\}^8 = 2^8[\cos(\pi) + i\sin(\pi)] = -2^8 = -256$$

2.2 Trigonometric identities

Demoivre's theorem can be used to obtain a variety of useful identities involving $\cos^n\theta$, $\sin^n\theta$, $\cos(n\theta)$ and $\sin(n\theta)$. The trick is to let $z = e^{i\theta}$, from which we obtain $z^{-1} = e^{-i\theta}$ and therefore from Euler's formula

$$z = \cos\theta + i\sin\theta \tag{3}$$

$$\frac{1}{z} = \cos\theta - i\sin\theta \tag{4}$$

Adding and subtracting these two equations gives the useful relations

$$\boxed{z + \frac{1}{z} = 2\cos\theta \tag{5}}$$

$$\boxed{z - \frac{1}{z} = 2i\sin\theta \tag{6}}$$

More generally, we have $z^n = e^{in\theta}$ and $z^{-n} = e^{-in\theta}$ and in this case Demoivre's theorem gives

$$z^n = \cos(n\theta) + i\sin(n\theta) \tag{7}$$

$$\frac{1}{z^n} = \cos(n\theta) - i\sin(n\theta) \tag{8}$$

Again, we can either add or subtract these two equations to obtain

$$z^n + \frac{1}{z^n} = 2\cos(n\theta) \tag{9}$$

$$z^n - \frac{1}{z^n} = 2i\sin(n\theta) \tag{10}$$

These results can be used in two ways; that is, to write

$\cos^n\theta$ or $\sin^n\theta$ in terms of $\cos(m\theta)$ or $\sin(m\theta)$ for various values of m

or

$\cos(n\theta)$ or $\sin(n\theta)$ in terms of $\cos^m\theta$ or $\sin^m\theta$ for various values of m

Although it is possible to obtain general identities, they are quite complicated and the technique is better illustrated by considering specific examples.

To express a power of cos θ in terms of cos $(m\theta)$

Example 1 Express $\cos^2\theta$ in terms of the cosines of multiples of θ.

Solution From Equation 5

$$2^2\cos^2\theta = \left(z + \frac{1}{z}\right)^2 = z^2 + 2 + \frac{1}{z^2} = \left(z^2 + \frac{1}{z^2}\right) + 2$$

The term in parentheses has precisely the correct form for Equation 9 (with $n = 2$), we have

$$\left(z^2 + \frac{1}{z^2}\right) + 2 = 2\cos(2\theta) + 2$$

and therefore $$\cos^2\theta = \frac{1 + \cos(2\theta)}{2}$$

which is the desired result. ❏

Question T3

Use Demoivre's theorem to show that

$$\cos^3\theta = \frac{\cos(3\theta) + 3\cos\theta}{4} \tag{11}$$

(*Hint*: Let $z = e^{i\theta}$ and then expand $(z + 1/z)^3$.) ☞ ❑

To express an odd power of sin θ in terms of sin ($m\theta$)

Similar techniques can be used to express an odd power of sin θ in terms of the sines of multiples of θ.

Example 2 Express $\sin^7\theta$ in terms of the sines of multiples of θ.

Solution Starting from the identity for sin θ in terms of z and $1/z$ we proceed as follows

$$(2i)^7\sin^7\theta = \left(z - \frac{1}{z}\right)^7$$

$$= \left(z^7 - \frac{1}{z^7}\right) - 7\left(z^5 - \frac{1}{z^5}\right) + 21\left(z^3 - \frac{1}{z^3}\right) - 35\left(z - \frac{1}{z}\right)$$

$$= 2i\sin(7\theta) - 14i\sin(5\theta) + 42i\sin(3\theta) - 70i\sin\theta$$

So the required result is

$$\sin^7\theta = \frac{35\sin\theta - 21\sin(3\theta) + 7\sin(5\theta) - \sin(7\theta)}{2^6} \tag{12}$$ ❏

✦ Express $\sin^5\theta$ in terms of the sines of multiples of θ.

Question T4

Use Demoivre's theorem to show that

$$\sin^3\theta = \frac{3\sin\theta - \sin(3\theta)}{4} \tag{13}$$ ❏

To express $\cos(n\theta)$ and $\sin(n\theta)$ in terms of $\cos\theta$ and $\sin\theta$

Again it is best to consider an example rather than the general case, so let us suppose that we want to express $\cos(2\theta)$ or $\sin(2\theta)$ in terms of $\cos\theta$ and $\sin\theta$. (In fact, we can derive two identities at the same time.) We start by using Demoivre's theorem with $n = 2$

$$\cos(2\theta) + i\sin(2\theta) = (\cos\theta + i\sin\theta)^2$$

and then expand the right-hand side to give

$$\cos(2\theta) + i\sin(2\theta) = \cos^2\theta + 2i\sin\theta\cos\theta - \sin^2\theta$$

We can now equate the real and imaginary parts to obtain two identities

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta \tag{14}$$

$$\sin(2\theta) = 2\sin\theta\cos\theta \tag{15}$$

which are the required results.

✦ Use Demoivre's theorem to obtain identities for $\cos(5\theta)$ and $\sin(5\theta)$ in terms of $\cos\theta$ and $\sin\theta$.

Question T5

Use Demoivre's theorem to obtain identities for $\cos(3\theta)$ and $\sin(3\theta)$ in terms of $\cos\theta$ and $\sin\theta$. ❏

2.3 Roots of unity

When we multiply two complex numbers we multiply their moduli and add their arguments; so to square a complex number we square the modulus and double the argument.

We know how to find the square root of a positive real number, but how can we find the square root of a complex number? Obviously we reverse the process of squaring, and find the square root of the modulus and halve the argument. However, a complex number has many different arguments, for example

$$1 = e^{0i} \quad \text{or} \quad e^{2\pi i} \quad \text{or} \quad e^{4\pi i} \quad \text{or} \quad e^{6\pi i} \quad \text{and so on}$$

so it follows that

$$1^{1/2} = e^{0i} \quad \text{or} \quad e^{\pi i} \quad \text{or} \quad e^{2\pi i} \quad \text{or} \quad e^{3\pi i} \quad \text{and so on}$$

From this it would at first sight appear that we have found an embarrassingly large number of square roots of 1, but in fact $1 = e^{0i} = e^{2\pi i} = e^{4\pi i} \ldots$, whereas $-1 = e^{\pi i} = e^{3\pi i} = e^{5\pi i} \ldots$, so that we have actually found just the two square roots that we expect.

The method can clearly be extended to cube roots. To find the cube root of a given complex number, we first write it in exponential form, then find the cube root of the modulus (a positive real number), and divide the modulus by three.

❖ Find a cube root of the complex number $1 + i$.

Although we have been able to find a cube root of a given complex number, there is one question we have not addressed. Is there more than one cube root; if so, what are the others? Demoivre’s theorem provides a complete answer to such questions.

According to *the fundamental theorem of algebra*, each polynomial with complex number coefficients and of degree n has, counting multiple roots an appropriate number of times, exactly n complex roots. More specifically, the theorem tells us that the equation

$$z^n - 1 = 0 \tag{16}$$

where n is a positive integer, has precisely n roots. These roots are known as the **n^{th} roots of unity** (because we can rewrite the equation as $z = 1^{1/n}$). To find these roots we use the fact that we can write the number 1 as ☞)

$$1 = e^{2\pi ki} = \cos(2\pi k) + i\sin(2\pi k) \tag{17}$$

where k is any integer (positive, negative or zero).

So z is given by

$$z = 1^{1/n} = [\cos(2\pi k) + i\sin(2\pi k)]^{1/n}$$

and Demoivre's theorem then allows us to rewrite the right-hand side, obtaining

$$z = \cos\left(\frac{2\pi k}{n}\right) + i\sin\left(\frac{2\pi k}{n}\right)$$

Don't forget that, whereas k can take any integer value, n is fixed (even if we don't specify what it is at the moment). However, the sine and cosine functions are periodic with period 2π, i.e.

$$\cos\left(\frac{2\pi k}{n} + 2\pi\right) = \cos\left(\frac{2\pi k}{n}\right) \tag{18}$$

$$\sin\left(\frac{2\pi k}{n} + 2\pi\right) = \sin\left(\frac{2\pi k}{n}\right) \tag{19}$$

So it follows that all the different integer values of k only give n distinct values of z and it is convenient to use $k = 0, 1, 2, \dots, (n-1)$ to generate these values.

In the case of $n = 2$, the two roots are given by

$$z = \cos(\pi k) + i\sin(\pi k)$$

where $k = 0, 1$ or, more explicitly

$$z_0 = \cos(0) + i\sin(0) = 1$$

$$z_1 = \cos(\pi) + i\sin(\pi) = -1$$

And once again we have the familiar result that $1^{1/2} = \pm 1$ which is plotted on an Argand diagram in Figure 1.

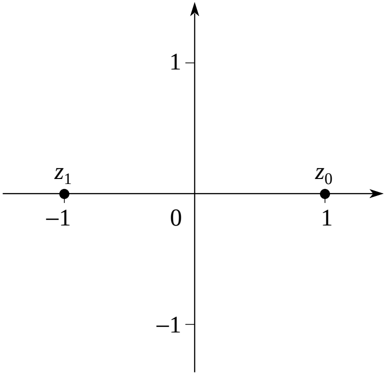

Figure 1 An Argand diagram showing the two roots of $z^2 - 1 = 0$.

Example 3 Find the three cube roots of 1.

Solution Arguing as before, we could express 1 in exponential form (using different values of the argument) then find the cube root of the modulus and divide the argument by three. More formally, using Demoivre's theorem, the three roots are given by

$$z = \cos(2\pi k/3) + i\sin(2\pi k/3)$$

where $k = 0, 1, 2$ or, more explicitly

$$z_0 = \cos(0) + i\sin(0) = 1$$

$$z_1 = \cos(2\pi/3) + i\sin(2\pi/3) = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$$

$$z_2 = \cos(4\pi/3) + i\sin(4\pi/3) = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$$

and these are plotted on an Argand diagram in Figure 2. ❏

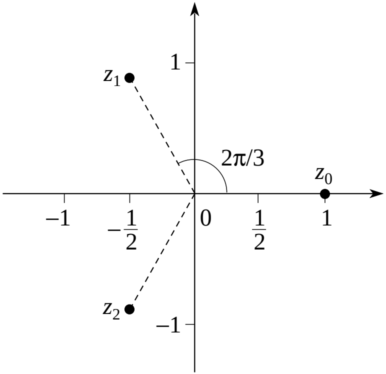

Figure 2 An Argand diagram showing the three roots of $z^3 - 1 = 0$.

Question T6

If z_1 and z_2 are as given above, find: (a) $\lvert z_1 \rvert$ and $\lvert z_2 \rvert$, (b) $\lvert z_1 \rvert^3$ and $\lvert z_2{}^3 \rvert$. ❑

Aside There is a slight problem here with notation (which can only be resolved properly by a discussion which is beyond the scope of *FLAP*). For a positive real variable x we know that $x^{1/2}$ and $\sqrt{x}$ are often used to denote the *positive* root of x, usually because we require the expression to define a *function* and so it must have just *one* value.

In this module we are following the convention that square roots of real numbers, such as $\sqrt{2}$ and $3^{1/2}$, are positive, because they usually arise as moduli of complex numbers, which *must* be positive.

For complex numbers we are often interested in *all* the n roots of unity, and we certainly do not wish to restrict the discussion to just one of them. So we follow the convention that for a complex number z, $z^{1/n}$ means all n values. This means that we will have to distinguish carefully between the square root of the real number 2, which takes only the positive value, and the square root of the complex number 2, which takes both positive and negative values. In practice the context would usually make the meaning clear, and this minor problem will cause us no great difficulty.

✧ Find the three values of $(1 + i)^{1/3}$ in exponential form.

Question T7

Find all roots of the equation $z^6 - 1 = 0$ and plot your results on an Argand diagram. ❑

Question T8

Show that all roots of the equation $z^n - 1 = 0$, where n is a positive integer, satisfy $|z| = 1$. Describe the geometric figure on which all such points lie, when plotted on an Argand diagram. ❑

3 Complex algebra

In this section we consider complex algebra; that is the manipulation of expressions involving complex variables.

3.1 Solving equations

Finding the n^{th} root of unity has already provided us with experience of using complex algebra to solve equations. In this subsection we consider other examples of solving equations.

Example 4 Solve the following equation

$$-z^2 + izγ + ω^2 = 0 \tag{20}$$

where $γ$ and $ω$ are real constants, which occurs in the theory of damped oscillations.

Solution We can solve for z by using the well-known formula for the roots of the quadratic equation

$$az^2 + bz + c = 0$$

namely

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

In our particular example we have $a = -1$, $b = i\gamma$ and $c = \omega^2$ and therefore

$$z = \frac{-i\gamma \pm \sqrt{(i\gamma)^2 - 4(-1)\omega^2}}{-2}$$

$$= \frac{i\gamma \pm \sqrt{4\omega^2 - \gamma^2}}{2} = \frac{i\gamma}{2} \pm \sqrt{\omega^2 - \gamma^2/4} \quad ❏$$

✦ Find the roots of the equation $z^2 + z + i = 0$ in the form $x + iy$.

Question T9

Find the roots of the equation $2z^2 - 11zi - 5 = 0$. ❑

Many of the algebraic operations for complex variables are almost identical to those for real variables; for example, the solution of simultaneous equations.

✧ Solve the pair of simultaneous equations

$$2z + iw = 5 + i \tag{21}$$

$$iz - 3w = i \tag{22}$$

Example 5 Suppose we want to solve the pair of equations

$$I_1(Z_1 + Z_2) + I_2 Z_2 = \varepsilon_1$$

$$I_1 Z_2 + I_2 Z_2 = \varepsilon_2$$

for I_1 and I_2 where all the variables are complex.

Solution Equations such as these (but sometimes having many more variables) often arise in the mesh analysis of a.c. circuits. We can subtract the second equation from the first to find

$$I_1 Z_1 = \varepsilon_1 - \varepsilon_2$$

so that $$I_1 = \frac{\varepsilon_1 - \varepsilon_2}{Z_1}$$

Substituting this result back in the second equation gives us

$$\frac{\varepsilon_1 - \varepsilon_2}{Z_1} Z_2 + I_2 Z_2 = \varepsilon_2$$

and hence $$I_2 Z_1 Z_2 = \varepsilon_2 Z_1 - (\varepsilon_1 - \varepsilon_2) Z_2 = \varepsilon_2 (Z_1 + Z_2) - \varepsilon_1 Z_2$$

so that $$I_2 = \frac{\varepsilon_2 (Z_1 + Z_2) - \varepsilon_1 Z_2}{Z_1 Z_2}$$

So far the algebra would have been no different for real variables, but now suppose we want to find the real and imaginary parts of I_1 and we are told that

$$Z_1 = Z_2 = R + iX, \quad \varepsilon_1 = \varepsilon \quad \text{and} \quad \varepsilon_2 = \varepsilon e^{i\phi}$$

where R, X, ε and ϕ are all real. I_1 is given by

$$I_1 = \frac{\varepsilon_1 - \varepsilon_2}{R + iX} = \frac{[\overbrace{\varepsilon}^{\varepsilon_1} - \overbrace{\varepsilon(\cos\phi + i\sin\phi)}^{\varepsilon_2}]}{R + iX} \times \frac{R - iX}{R - iX}$$

$$= \frac{\varepsilon(1 - \cos\phi - i\sin\phi)}{(R^2 + X^2)}(R - iX)$$

$$= \frac{\varepsilon}{R^2 + X^2}\{[R(1 - \cos\phi) - X\sin\phi] + i[-R\sin\phi - X(1 - \cos\phi)]\}$$

and therefore

$$\mathrm{Re}(I_1) = \frac{\varepsilon}{R^2 + X^2}[R(1 - \cos\phi) - X\sin\phi]$$

$$\mathrm{Im}(I_1) = \frac{\varepsilon}{R^2 + X^2}[-R\sin\phi - X(1 - \cos\phi)] \quad ❏$$

Question T10

Find $\mathrm{Re}(I_2)$ and $\mathrm{Im}(I_2)$ in Example 5.

i.e. $I_2 = \dfrac{\varepsilon_2(Z_1 + Z_2) - \varepsilon_1 Z_2}{Z_1 Z_2}$

and $Z_1 = Z_2 = R + iX$ ❏

3.2 Factorizing

Complex numbers enable us to **factorize** expressions with real coefficients that are impossible to factorize in terms of real numbers. For example, expanding the right-hand side of the equation will verify the equation

$$z^2 + 1 = (z - i)(z + i)$$

and so $z^2 + 1$ (which is a polynomial with real coefficients) has complex factors. Complex numbers are also involved in the factorization of expressions, which have complex coefficients, as in

$$z^2 + (1 + i)z + i = (z + i)(z + 1)$$

In simple cases, factorization can be achieved by spotting values which make the expression zero (in this case noticing that $z = -i$ and $z = -1$ are values for which $z^2 + (1 + i)z + i = 0$). More complicated cases may involve finding the roots by numerical means, and there are actually computer programs designed to do precisely this.

If we need to factorize an expression of the form

$$z^n + a_{n-1}z^{n-1} + a_{n-2}z^{n-2} + \dots + a_0$$

for some (possibly quite large) positive integer, n, then the n roots, $z_1, z_2, \dots, z_n$, of the equation

$$z^n + a_{n-1}z^{n-1} + a_{n-2}z^{n-2} + \dots + a_0 = 0$$

correspond to the factorization

$$z^n + a_{n-1}z^{n-1} + a_{n-2}z^{n-2} + \dots + a_0 = (z - z_1)(z - z_2) \dots (z - z_n)$$

In practice, it is unlikely that you will need to perform factorizations for large values of n. However, it is important that you should know how many roots (and therefore factors) to look for.

Example 6 Factorize $3z^2 + 27$.

Solution Notice that the expression is zero for $z = \pm 3i$ and therefore $(z - 3i)$ and $(z + 3i)$ are factors. Since the highest power of z in the polynomial $3z^2 + 27$ is z^2, the fundamental theorem of algebra tells us that these are the only roots, and therefore

$$3z^2 + 27 = k(z - 3i)(z + 3i)$$

for some constant k. Comparing the coefficients on each side of this expression for any particular power of z (z^2 is the most convenient in this case) we obtain $k = 3$, so that

$$3z^2 + 27 = 3(z - 3i)(z + 3i) \quad ❑$$

Question T11

Factorize the expression $2z^2 + 32$. ❑

Example 7 Factorize the expression

$$-z^2 + iz\gamma + w^2$$

where z is regarded as the 'unknown variable'.

Solution We need to find the two roots of Equation 20. From Example 4 we know that

$$z = \frac{i\gamma}{2} \pm \sqrt{\omega^2 - \gamma^2/4}$$

and therefore, for some constant k,

$$-z^2 + iz\gamma + w^2 = k\left(z - \frac{i\gamma}{2} - \sqrt{\omega^2 - \gamma^2/4}\right)\left(z - \frac{i\gamma}{2} + \sqrt{\omega^2 - \gamma^2/4}\right)$$

Comparing the coefficients of z^2 (or any other convenient power of z) tells us that $k = -1$, and so we have

$$-z^2 + iz\gamma + w^2 = -\left(z - \frac{i\gamma}{2} - \sqrt{\omega^2 - \gamma^2/4}\right)\left(z - \frac{i\gamma}{2} + \sqrt{\omega^2 - \gamma^2/4}\right)$$ ❏

Question T12

Factorize the expression $z^2 + iz + 2$. ❑

3.3 Simplifying

We simplify expressions involving complex numbers in much the same way that we simplify expressions involving real numbers, except that every occurrence of i^2 may be replaced by -1. It may also be necessary to rationalize any complex quotients, in other words to convert such quotients into the form $x + iy$ where x and y are real, in order to arrive at the simplest form. Since there is not really any general prescription for simplifying complex expressions, the best approach is to consider some typical examples.

Example 8 Simplify the following expression

$$Z = (3R + 2iX) - (R + iX) + (7R + 3iX)$$

Solution To simplify such an expression we treat the real and imaginary parts separately to obtain ☞

$$Z = 9R + 4iX \quad ❏$$

Example 9 Find the real and imaginary parts of a complex number Z defined by

$$Z = R_1 + \cfrac{1}{\cfrac{1}{R_2 + i\omega L} + \cfrac{1}{1/(i\omega C)}} \tag{25}$$

where R, ω, L and C are real.

Solution To find the real and imaginary parts of Z we rationalize the complex quotients

$$Z = R_1 + \frac{R_2 + i\omega L}{1 + i\omega C(R_2 + i\omega L)}$$

$$= R_1 + \frac{R_2 + i\omega L}{1 + i\omega C(R_2 + i\omega L)} \times \frac{1 - i\omega C(R_2 - i\omega L)}{1 - i\omega C(R_2 - i\omega L)}$$

$$= R_1 + \frac{(R_2 + i\omega L)[(1 - \omega^2 LC) - i\omega CR_2]}{(1 - \omega^2 LC)^2 + \omega^2 C^2 R_2^2}$$

$$= R_1 + \frac{R_2 + i[\omega L(1 - \omega^2 LC) - \omega CR_2^2]}{(1 - \omega^2 LC)^2 + \omega^2 C^2 R_2^2}$$

which gives the results

$$\mathrm{Re}(Z) = R_1 + \frac{R_2}{(1 - \omega^2 LC)^2 + \omega^2 C^2 R_2^2} \qquad \mathrm{Im}(Z) = \frac{\omega L(1 - \omega^2 LC) - \omega CR_2^2}{(1 - \omega^2 LC)^2 + \omega^2 C^2 R_2^2}$$ ❏

✦

Simplify the expression $\dfrac{3+2i}{1+i} - \dfrac{1}{1+2i}$.

Question T13

Simplify the expression $\dfrac{1}{1-i} + \dfrac{1+i}{2}$. ❏

3.4 Complex binomial expansion

You probably recall that the *binomial coefficient*, ${}^nC_r$, is defined by

$$ {}^nC_r = \frac{n!}{r!(n-r)!} \tag{26}$$

where $n!$ is known as n factorial, and is defined by

$$ n! = n(n-1)(n-2)(n-3)\ \dots\ 2 \times 1 \text{ for } n \geq 1 \text{ and } 0! = 1 \tag{27}$$

The expression $(a + b)^n$, where a and b are real and n is a positive integer, can be written in terms of powers of a and b by means of the binomial expansion ☞

$$ (a+b)^n = {}^nC_n a^n + {}^nC_{n-1} a^{n-1} b + \dots + {}^nC_{n-r} a^{n-r} b^r + \dots + {}^nC_1 ab^{n-1} + {}^nC_0 b^n $$

$$ = \sum_{r=0}^{n} {}^nC_{n-r} a^{n-r} b^r \tag{28}$$

Question T14

(a) Evaluate ${}^3C_r$ for $r = 0, 1, 2, 3$. (b) Prove the general result ${}^nC_r = {}^nC_{n-r}$. ❏

There is nothing in the proof of the binomial theorem ☞ which restricts it to real quantities a and b, and for this module our interest in the theorem lies in its complex applications. The following examples are relevant to Subsection 2.2.

✥

Expand and simplify $\left(z - \dfrac{1}{z}\right)^3$.

✥

Expand and simplify $\left(z - \dfrac{1}{z}\right)^7$.

Question T15

Use the binomial theorem to show that

$$\left(z+\frac{1}{z}\right)^4=\left(z^4+\frac{1}{z^4}\right)+4\left(z^2+\frac{1}{z^2}\right)+6$$

(*Hint*: What is the relationship between ${}^nC_r$ and ${}^nC_{n-r}$?) ❑

In Subsection 2.2 we showed how to write powers of trigonometric functions in terms of functions of multiple angles by setting $z = e^{i\theta}$ and consequently

$$z^n+\frac{1}{z^n}=2\cos(n\theta) \qquad \text{(Eqn 9)}$$

$$z^n-\frac{1}{z^n}=2i\sin(n\theta) \qquad \text{(Eqn 10)}$$

The binomial theorem is a considerable help in using these results to derive identities for powers of trigonometric functions. For example, the answer to Question T15 enables us to write

$$2^4\cos^4(\theta) = \left(z + \frac{1}{z}\right)^4 = \left(z^4 + \frac{1}{z^4}\right) + 4\left(z^2 + \frac{1}{z^2}\right) + 6$$

$$= 2\cos(4\theta) + 4 \times 2\cos(2\theta) + 6$$

and therefore

$$\cos^4(\theta) = \frac{1}{8}[\cos(4\theta) + 4\cos(2\theta) + 3]$$

Question T16

(a) Use the binomial theorem to show that

$$\left(z + \frac{1}{z}\right)^5 = \left(z^5 + \frac{1}{z^5}\right) + 5\left(z^3 + \frac{1}{z^3}\right) + 10\left(z + \frac{1}{z}\right)$$

(b) Use the first part of this question, together with Demoivre’s theorem, to express $\cos^5\theta$ in terms of cosines of multiples of θ. ❏

3.5 Complex geometric series

For real numbers a and x, the series $a + ax + ax^2 + \dots + ax^n$ is known as a *geometric series*, and its sum is $a\dfrac{1-x^{n+1}}{1-x}$, provided that $x \neq 1$.

The **geometric series for complex numbers** a and z is identical in form, and ☞

$$a + az + az^2 + \dots + az^n = a\frac{1-z^{n+1}}{1-z} \quad \text{if } z \neq 1 \tag{29}$$

❖ Write down the sum of the series $1 + e^{i\theta} + e^{2i\theta} + e^{3i\theta}$.

Sums of this kind can be used to simplify certain trigonometric expressions, as, for example, in the following case.

❖ Sum the series $\sin(\theta) + \sin(2\theta) + \sin(3\theta) + \dots + \sin(9\theta)$.

Question T17

Sum the series $\cos(\theta) + \cos(2\theta) + \cos(3\theta) + \ldots + \cos(9\theta)$. ❏

4 Closing items

4.1 Module summary

1 The identity

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta) \qquad \text{(Eqn 2)}$$

is known as *Demoivre's theorem*. It is valid for any real value of n.

2 Defining $z = e^{i\theta}$ and using Demoivre's theorem gives us

$$z^n + \frac{1}{z^n} = 2\cos(n\theta) \qquad \text{(Eqn 9)}$$

$$z^n - \frac{1}{z^n} = 2i\sin(n\theta) \qquad \text{(Eqn 10)}$$

These results are useful for deriving trigonometric identities such as those which give

(a) $\cos^n\theta$ or $\sin^n\theta$ in terms of $\cos(m\theta)$ or $\sin(m\theta)$;

(b) $\cos(n\theta)$ or $\sin(n\theta)$ in terms of $\cos^m\theta$ and $\sin^m\theta$.

3 The n solutions of the equation $z^n - 1 = 0$ are known as the n^{th} *roots of unity* and are given by

$$z = \cos\left(\frac{2\pi k}{n}\right) + i\sin\left(\frac{2\pi k}{n}\right)$$

where $k = 0, 1, 2, \dots, (n-1)$. If plotted on an Argand diagram, the roots correspond to n equally spaced points lying on a circle of unit radius, centred at the origin.

4 The fundamental theorem of algebra states that each polynomial with complex number coefficients and of degree n has, counting multiple roots an appropriate number of times, exactly n complex roots.

5 If we determine the roots, $z_1, z_2, \dots, z_n$, of the equation

$$z^n + a_{n-1}z^{n-1} + a_{n-2}z^{n-2} + \dots + a_0 = 0$$

then we have the following *factorization*

$$z^n + a_{n-1}z^{n-1} + a_{n-2}z^{n-2} + \dots + a_0 = (z - z_1)(z - z_2)\dots(z - z_n)$$

6 The complex binomial expansion is

$$(a+b)^n = {}^nC_n a^n + {}^nC_{n-1} a^{n-1} b + \dots + {}^nC_{n-r} a^{n-r} b^r + \dots + {}^nC_1 a b^{n-1} + {}^nC_0 b^n$$

$$= \sum_{r=0}^{n} {}^nC_{n-r} a^{n-r} b^r \qquad \text{(Eqn 28)}$$

where $${}^nC_r = \frac{n!}{r!(n-r)!} \qquad \text{(Eqn 26)}$$

and $n! = n(n-1)(n-2)(n-3) \dots 2 \times 1$, for $n \ge 1$, with $0! = 1$

7 The complex form of the *geometric series* is

$$a + az + az^2 + \dots + az^n = a\frac{1-z^{n+1}}{1-z} \quad \text{if } z \ne 1 \qquad \text{(Eqn 29)}$$

The complex binomial expansion and the complex form of the geometric series may be used to derive certain trigonometric identities.

4.2 Achievements

Having completed this module, you should be able to:

A1 Define the terms that are emboldened and flagged in the margins of the module.

A2 State and derive Demoivre’s theorem.

A3 Use Demoivre’s theorem to derive trigonometric identities.

A4 Explain what is meant by the n^{th} roots of unity and find all such roots.

A5 Describe the relevance of the fundamental theorem of algebra to the solution of equations and the factorization of polynomials.

A6 Solve quadratic equations and linear equations involving complex variables.

A7 Factorize simple polynomials.

A8 Simplify complex expressions.

A9 State and apply the complex binomial expansion.

A10 Apply the complex form of the geometric series.

Study comment You may now wish to take the *Exit test* for this module which tests these Achievements. If you prefer to study the module further before taking this test then return to the *Module contents* to review some of the topics.

4.3 Exit test

Study comment Having completed this module, you should be able to answer the following questions, each of which tests one or more of the *Achievements*.

Question E1

(*A2*) State Demoivre's theorem and use it to simplify z^n where

$$z = 3[\cos(\pi/n) + i\sin(\pi/n)]$$

for any integer, n.

Question E2

(*A2* and *A4*) Use Demoivre's theorem to find an expression for the roots of the equation $z^n - 1 = 0$. Hence find the simple form of the solutions of this equation when $n = 4$.

Question E3

(*A2*, *A3* and *A9*)   Expand the expression $(\cos\theta + i\sin\theta)^2$ by removing the brackets, and then rewrite it using Demoivre's theorem. Hence show that $\cos(2\theta) = \cos^2\theta - \sin^2\theta$ and $\sin(2\theta) = 2\sin\theta\cos\theta$. Use a similar method and the complex binomial theorem, together with Demoivre's theorem, to express $\cos(9\theta)$ and $\sin(9\theta)$ in terms of powers of $\cos\theta$ and $\sin\theta$.

Question E4

(*A8*)   The following equation occurs when solving the damped driven harmonic oscillator:

$$(-\omega^2 - i\omega\beta_0 + \omega_0{}^2)z_0 e^{i\omega t} = \alpha_0\, e^{i\omega t}$$

where t, α_0, ω_0, β_0, and ω are real and positive, and z_0 is complex. Find expressions for $\mathrm{Re}(z_0)$, $\mathrm{Im}(z_0)$ and $\lvert z_0 \rvert$.

Question E5

(*A6* and *A7*) Factorize the following expression:

$$2z^2 - 11izx - 5x^2$$

Question E6

(*A4* and *A10*) Write down the sum of the following geometric series

$$1 + z + z^2 + \ldots + z^8$$

and hence show that the roots of the equation $1 + z + z^2 + \ldots + z^8 = 0$ lie on a circle with its centre at the origin.

Study comment This is the final *Exit test* question. When you have completed the *Exit test* go back to Subsection 1.2 and try the *Fast track questions* if you have not already done so.

If you have completed **both** the *Fast track questions* and the *Exit test*, then you have finished the module and may leave it here.