Module M4.2 Basic differentiation

Exit module

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1 Opening items

1.1 Module introduction

Being able to *differentiate* accurately and confidently is a necessary skill for any physicist. Differentiation is such a powerful tool with so many applications that it is worth some effort to ensure that you have mastered the techniques explained in this module.

The main objective of this module is to answer the following question:

> 'If we know how to differentiate two functions $f(x)$ and $g(x)$, how do we differentiate their sum, difference, product and quotient?'

To answer this question we discuss first the idea of a *function* and how functions can be combined to produce new ones. Section 2 examines the mathematical basis of *differentiation*, giving first a graphical interpretation and then a formal definition.

Section 3 answers our question and is the core of the module. It provides a list of derivatives of various standard functions (such as power functions (x^n) and trigonometric functions) and introduces the various rules that will enable you to differentiate a wide range of combinations of these standard functions. In particular it will explain how to differentiate *sums*, *constant multiples*, *products* and *quotients* of the standard functions. Sometimes the method is straightforward, as with the derivative of a *sum*, but it may also be quite complicated, as with the derivative of a *quotient*.

We also discuss the *logarithmic* and *exponential* functions, and, while $\log_e x$ and e^x are the most important of such functions, it is sometimes necessary to be able to differentiate $\log_a x$ and a^x where a is some positive number other than e. In Section 4 we consider why e^x is so important and then add a^x and $\log_a x$ to the list of functions that can be differentiated.

Study comment Having read the introduction you may feel that you are already familiar with the material covered by this module and that you do not need to study it. If so, try the *Fast track questions* given in Subsection 1.2. If not, proceed directly to *Ready to study?* in Subsection 1.3.

1.2 Fast track questions

Study comment Can you answer the following *Fast track questions*?. If you answer the questions successfully you need only glance through the module before looking at the *Module summary* (Subsection 5.1) and the *Achievements* listed in Subsection 5.2. If you are sure that you can meet each of these achievements, try the *Exit test* in Subsection 5.3. If you have difficulty with only one or two of the questions you should follow the guidance given in the answers and read the relevant parts of the module. However, *if you have difficulty with more than two of the Exit questions you are strongly advised to study the whole module*.

Question F1

Differentiate each of the following functions (it is not necessary to simplify your answers).

(a) $f(x) = x^2 \sin x + \log_e x$

(b) $f(x) = (x^3 - 6x)(2x^2 + 5x - 1)$ (use the *product rule*)

(c) $f(x) = \dfrac{1}{\sin x}$

(d) $f(x) = \dfrac{x^2 + 4x - 1}{x^2 + 9}$

(e) $f(x) = \dfrac{x \tan x}{3x^2 - 2x + 1}$

(f) $f(x) = 2^x + x^2 + e^2$

(g) $f(x) = (e^x + e^{-x})^2$

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Question F2

If at time t the displacement, $s_x(t)$, of a particle along the x-axis from some particular reference point is given by

$$s_x(t) = e^{-\alpha t}\,[A\cos(\omega t) + B\sin(\omega t)]$$

where A, B, α and ω are constants, find expressions for $v_x(t)$, the velocity at time t, and $a_x(t)$ the acceleration at time t, and show that

$$a_x(t) + 2\alpha\, v_x(t) + (\omega^2 + \alpha^2)s_x(t) = 0$$

Study comment

Having seen the *Fast track questions* you may feel that it would be wiser to follow the normal route through the module and to proceed directly to *Ready to study?* in Subsection 1.3.

Alternatively, you may still be sufficiently comfortable with the material covered by the module to proceed directly to the *Closing items*.

1.3 Ready to study?

Study comment To begin the study of this module you will need to be familiar with the following terms: *base* (of a *logarithm*), *exponential function*, *inequality* (in particular *greater than* (>) and *less than* (<)), *integer*, *inverse function*, *logarithmic function*, *modulus* (or *absolute value*, $|x|$), *product*, *quotient*, *radian*, *real number*, *reciprocal*, *set*, *square root*, *sum*, *trigonometric function* and *trigonometric identities*. (The *trigonometric identities* that you require are repeated in this module, as is the definition of the *modulus* function.) In addition you will need to have some familiarity with the concept of a *function*, and with related terms such as *argument*, *codomain*, *domain* and *variable* (both *dependent* and *independent*), but the precise meaning of these terms is briefly reviewed in the module. Similar comments apply to physics concepts used in this module, such as *acceleration*, *displacement*, *force*, *speed*, *velocity* and *Newton's second law.* If you are uncertain of any of these items you can review them now by referring to the *Glossary*, which will indicate where in *FLAP* they are developed. The following *Ready to study questions* will allow you to establish whether you need to review some of the topics, or to improve your general algebraic skills, before embarking on this module.

Question R1

Write the following sums, products and quotients of logarithms and exponentials as compactly as you can:

(a) $\log_{10} x + \log_{10} y$, (b) $(\log_2 4 \times \log_2 4)^3$, (c) $\log_e x - \log_e y$, (d) e^x/e^y,

(e) $(e^x/e^y)^2$, (f) $\log_e x + \log_e x + \log_e x$.

Question R2

If $f(x) = \cos x$, $g(x) = \sin x$, and $h(x) = \tan x$, rewrite the following expressions in terms of trigonometric functions and simplify them where possible ☞ :

(a) $[f(x) + g(x)]^2$

(b) $f(x)h(x) + g(x)/h(x)$

(c) $[f(x) - g(x)]/[1 + h(x)]$

Question R3

Given that $f(x) = \dfrac{x^2}{4} - 1$, find the following values, $f(1)$, $f(0)$, $f(-2)$, $|f(0)|$, $|f(0) - f(-2)|$ and sketch the graph of $y = f(x)$

Question R4

Given that a^x and $\log_a x$ are inverse functions, simplify the following ☞ :

(a) $\log_e[e^{2\log_{10}(x)}]$, (b) $\exp(\log_e(10^x))$, (c) $\log_2(2^2)$.

2 Variables, functions and derivatives

2.1 Functions and variables

Study comment An understanding of *functions* is crucial to an understanding of differentiation, and it is vital that the notation and terminology used to describe functions should be clear and unambiguous. For that reason, this subsection reviews the definitions of terms such as *function* and *variable* even though it is assumed that you have met these ideas before. If you are completely unfamiliar with these concepts you should consult the entry on *functions* in the *Glossary*.

A *function* f is a rule that assigns a single value $f(x)$ in a set called the *codomain* to each value x in a set called the *domain*.

Functions are very often defined by formulae, for example $f(x) = x^2$, and in such cases we assume, unless we are told otherwise, that the domain is the largest set of real values for which the formula makes sense. In the case of $f(x) = x^2$ the domain of the function is the set of all real numbers. The function $g(x) = \dfrac{1}{1-x}$ is not defined when $x = 1$, since 1/0 has no meaning, and so we take the set of all real numbers x with $x \neq 1$ as its domain.

The function
$$f(x) = 1 + x + x^2 \tag{1a}$$

is another example of a function that is defined for all values of x.

One may think of the function as a sort of ‘machine’ with x as the input and $1 + x + x^2$ as the output. The input x is known as the *independent variable* and, if we write $y = f(x)$, the output y is known as the *dependent variable* since the function $f(x)$ determines the way in which y depends on x.

It is important to note that the same function f could equally well be defined using some other symbol, such as t, to represent the independent variable:

$$f(t) = 1 + t + t^2 \tag{1b}$$

This freedom to relabel the independent variable is often of great use, though it is vital that such changes are made consistently throughout an equation.

We may *evaluate* the function in Equation 1, whether we call it $f(x)$ or $f(t)$, for any value of the independent variable; for example, if we choose to use x to denote the independent variable, and set $x = 1$, we have

$$f(1) = 1 + 1 + 1^2 = 3$$

Similarly, if $x = \pi$ $\qquad f(\pi) = 1 + \pi + \pi^2$

and, if $x = 2a$ $\qquad f(2a) = 1 + (2a) + (2a)^2 = 1 + 2a + 4a^2$

We may even apply the function to $2x$, $-x$, or $1 + x$ to obtain

$$f(2x) = 1 + (2x) + (2x)^2 = 1 + 2x + 4x^2$$

$$f(-x) = 1 + (-x) + (-x)^2 = 1 - x + x^2$$

$$f(1 + x) = 1 + (1 + x) + (1 + x)^2 = 3 + 3x + x^2$$

When we write expressions such as $f(\pi)$ or $f(2a)$, whatever appears within the brackets is called the *argument* of the function. The value of $f(x)$ is determined by the *value* of its argument, irrespective of what we call the argument.

A special note about $\sqrt{x}$ Generally in *FLAP* we follow the convention that $\sqrt{x}$ may be positive or negative. Thus $\sqrt{4} = \pm 2$. A consequence of this convention is that $f(x) = \sqrt{x}$ does not define a function since it does not associate a unique value of $f(x)$ with each value of x. This would be an exceptionally inconvenient convention to follow in this module so instead we adopt the convention that $\sqrt{x}$ is positive. Of course, it remains true that the square roots of x may be positive or negative, $\sqrt{x}$ or $-\sqrt{x}$, so within this convention we should call $\sqrt{x}$ the positive square root of x, and not simply the square root. In a similar spirit $x^{1/2} = \sqrt{x}$ will also be a positive quantity.

✦

If $f(x) = \cos^2 x + \sin x$ find $f(2x)$ and $f\left(\dfrac{\pi}{4}\right)$ ☞

Question T1

If $f(x) = x^2 + 1$ and $g(x) = 2x$,

(a) write down expressions for $f(\sqrt{x})$ and $g\left(\dfrac{x}{2}\right)$.

(b) For which values of the independent variable x is $f(x) = g(x)$?

(c) For which value of the independent variable x is the following true?

$$\frac{f(x+0.1) - f(x)}{0.1} = 0.2$$ ❏

2.2 Rates of change, gradients and derivatives

The graph of the function $g(t) = t^2$ is shown in Figure 1. It is clear from the graph that as the value of t increases from $t = 0$, the value of $g(t)$ also increases from 0. In particular, note that as t increases from 0 to 1 the value of $g(t)$ changes from 0 to 1, and that as t increases from 1 to 2 the value of $g(t)$ changes from 1 to 4. Thus, the change in the value of $g(t)$ that corresponds to a change of one unit in the value of t depends on the initial value of t. For this particular function, if the initial value of t is large, then the change in $g(t)$ will also be large; for example, the change in $g(t)$ from $t = 100$ to $t = 101$ is

$$g(101) - g(100) = 101^2 - 100^2 = 201$$

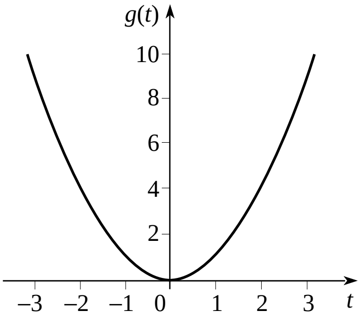

Figure 1 The graph of $g(t) = t^2$.

Clearly, as t increases the *rate* at which $g(t)$ is changing is itself changing, becoming greater and greater. How are we to measure the rate at which a function changes? If you imagine yourself walking up a hill in the shape of the graph shown in Figure 1, starting at $t = 0$ and travelling to the right, your path would become steeper and steeper. It would become increasingly difficult to make progress, because, with each step you took, the slope would become greater and your walk would become more and more of a climb. The *slope* of the graph is the key; once we can describe the slope precisely for any value of t, we will be able to measure the *rate of change* of the function $g(t)$ — or of any other function.

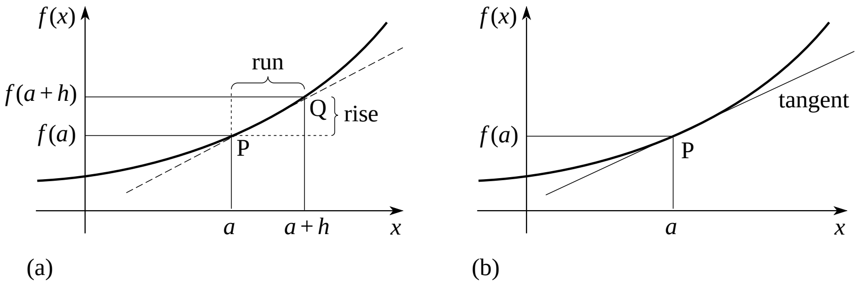

Figure 2 The graph of an arbitrary function.

The graph of an arbitrary function $f(x)$ is shown in Figure 2a. Roughly speaking, the slope of this graph at the point P, where $x = a$ and $f(x) = f(a)$, is the same as the slope of the dashed straight line PQ where Q is a nearby point corresponding to $x = a + h$ and $f(x) = f(a + h)$. Of course, the two slopes are not exactly the same; the point P is fixed and its location determines the slope of $f(x)$ at P, whereas the slope of the line PQ will depend on the location of Q as well as that of P. Nonetheless, the two slopes are similar, so we can roughly describe the slope of the curve in terms of the **gradient** of the line PQ which is defined by

$$\text{gradient of the line PQ} = \frac{\text{rise}}{\text{run}} = \frac{f(a+h) - f(a)}{(a+h) - a} = \frac{f(a+h) - f(a)}{h} \qquad (2)$$

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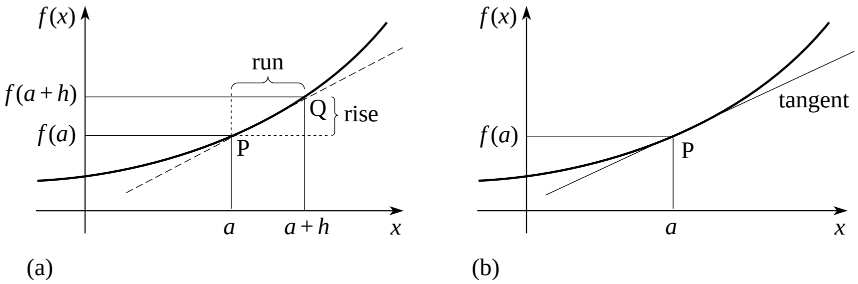

Figure 2 The graph of an arbitrary function.

Now, the line PQ (the dashed line in Figure 2a) that cuts the graph of $f(x)$ at P and Q and is called a **chord**, but if we let the point Q get closer and closer to P then this chord becomes more and more like the **tangent** in Figure 2b that just touches the graph at P. It is the gradient of this tangent that really represents the slope of the curve at P, this is clear from the figure and from the fact that the gradient of the tangent (unlike that of the chord) is determined by the location of P alone. Fortunately, the gradient of the tangent is fairly easy to work out, all we have to do is to consider what happens to the gradient of the chord PQ as Q gets closer and closer to P. As Q approaches P, h gets smaller and the gradient of the chord PQ approaches the gradient of the tangent at P ever more closely.

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Expressed more formally, the gradient of the tangent at P is given by Equation 2 in the **limit** as h tends to zero.

$$\text{gradient of the line PQ} = \frac{\text{rise}}{\text{run}} = \frac{f(a+h) - f(a)}{(a+h) - a} = \frac{f(a+h) - f(a)}{h} \qquad \text{(Eqn 2)}$$

Thus

$$\text{gradient of tangent at P} = \lim_{h\to 0}\left[\frac{f(a+h) - f(a)}{h}\right] \qquad (3)$$

Since this gradient represents the slope of the graph of $f(x)$ at P, it makes sense to *define* the gradient of the graph of $f(x)$ at P to be the gradient of its tangent at P. We call this gradient the *derivative* of $f(x)$ at $x = a$ and denote it by $f'(a)$ so that

$$\boxed{f'(a) = \lim_{h\to 0}\left[\frac{f(a+h) - f(a)}{h}\right] \qquad (4)}$$

The derivative at $x = a$ defined in this way also represents the **rate of change** of $f(x)$ with respect to x when $x = a$. This last point is even easier to appreciate if we introduce a dependent variable y such that $y = f(x)$, for we can the represent the vertical *rise* in Figure 2a by $\Delta y = f(a + h) - f(a)$, and the horizontal *run* by $\Delta x = h$. With these definitions it follows from Equation 4

$$f'(a) = \lim_{h \to 0}\left[\frac{f(a+h) - f(a)}{h}\right] \qquad \text{(Eqn 4)}$$

that the derivative at $x = a$ is given by

$$f'(a) = \lim_{\Delta x \to 0}\left(\frac{\Delta y}{\Delta x}\right)$$

Even greater emphasis can be given to the idea that the derivative is a rate of change with respect to x by replacing $f'(a)$ by the alternative symbol $\dfrac{dy}{dx}(a)$. ☞

$$\text{Thus,}\quad \frac{dy}{dx}(a) = \lim_{\Delta x \to 0}\left(\frac{\Delta y}{\Delta x}\right) = \lim_{h \to 0}\left[\frac{f(a+h) - f(a)}{h}\right] \qquad (5)$$

The definition of the derivative at $x = a$ represented by Equations 4 and 5

$$f'(a) = \lim_{h \to 0}\left[\frac{f(a+h) - f(a)}{h}\right] \qquad \text{(Eqn 4)}$$

Thus, $$\frac{dy}{dx}(a) = \lim_{\Delta x \to 0}\left(\frac{\Delta y}{\Delta x}\right) = \lim_{h \to 0}\left[\frac{f(a+h) - f(a)}{h}\right] \qquad \text{(Eqn 5)}$$

can be applied at any point on the graph provided that Δy and Δx can be defined at that point and that their quotient $\Delta y/\Delta x$ has a unique limit as $\Delta x \to 0$ at that point. Thus, provided the derivative of $f(x)$ exists at every point within some domain (i.e. some set of x values) it is possible to define a new function on that domain that associates any given value of x with the gradient of $f(x)$ at that point. This new function is called the **derived function** or **derivative** of $f(x)$ and is written $f'(x)$ or $\dfrac{df}{dx}(x)$. If $y = f(x)$, the derived function may also be written $\dfrac{dy}{dx}(x)$ or just $\dfrac{dy}{dx}$. ☞ Thus, provided unique limits exist

$$\frac{dy}{dx} = \lim_{h \to 0}\left[\frac{f(x+h) - f(x)}{h}\right] \qquad (6)$$

When using this formula

$$\frac{dy}{dx} = \lim_{h\to 0}\left[\frac{f(x+h) - f(x)}{h}\right] \qquad \text{(Eqn 6)}$$

it is important to remember that dy/dx is *not* a quotient of two quantities dy and dx even though it may look like one. At a given value of x, the derivative dy/dx represents the gradient of the graph of $y = f(x)$ at that value of x.

Although the graphical interpretation of the derivative is important, it does not lend itself to calculation since drawing tangents can only be approximate, and in any case it would be impossible to do it for every point on the graph. However, in simple cases it is not difficult to determine the derivative from the formula for the function and the above definition (Equation 6). Consider the case of $f(x) = x^2$, for which

$$f'(x) = \lim_{h\to 0}\left[\frac{f(x+h) - f(x)}{h}\right] = \lim_{h\to 0}\left[\frac{(x+h)^2 - x^2}{h}\right] = \lim_{h\to 0}\left(\frac{2xh + h^2}{h}\right)$$

i.e. $f'(x) = \lim\limits_{h\to 0}(2x + h) = 2x$

so that $f'(x) = 2x$

The function $f(x) = x^2$ has the set of all real numbers as its domain, and its derivative $f'(x) = 2x$ has the same domain.

As a further example, we can use the definition to obtain the derivative of the function $f(x) = 1/x$.

In this case $\quad f(x+h) = \dfrac{1}{x+h}$

giving us

$$\frac{f(x+h) - f(x)}{h} = \frac{1}{h}\left(\frac{1}{x+h} - \frac{1}{x}\right) = \frac{1}{h}\left(\frac{x-(x+h)}{(x+h)x}\right) = \frac{-1}{(x+h)x}$$

As h tends to zero the expression $\dfrac{-1}{(x+h)x}$ approaches $-\dfrac{1}{x^2}$.

So, $$f'(x) = \lim_{h\to 0}\left[\frac{-1}{(x+h)x}\right] = -\frac{1}{x^2} \qquad (7)$$

The function $f(x) = 1/x$ has the set of all non-zero real numbers as its domain, and its derivative $f'(x) = -1/x^2$ has the same domain.

Question T2

(a) If $f(x) = 1/(\omega x)$ where ω is a constant, use the definition of the derivative to find $f'(x)$.

(b) If $g(t) = 1/(\omega t)$ use the answer to part (a) to write $g'(t)$ and $g'(2t)$. ❏

Don't be misled into thinking that if a function exists then its derivative must also exist. The function $f(x) = |x|$ ('the *modulus* of x') illustrates the situation where the derivative has a smaller domain than the function (see Figure 3). The modulus of x is defined by

$$|x| = x \quad \text{if } x > 0$$

and $$|x| = -x \quad \text{if } x < 0$$

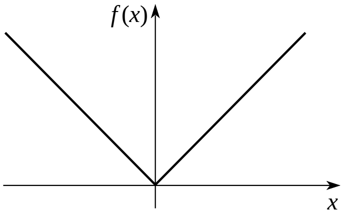

Figure 3 The graph of $f(x) = |x|$.

For positive x the graph of $f(x) = |x|$ is a straight line with gradient 1, so

$$f'(x) = 1 \text{ if } x > 0$$

while for negative x the graph is a straight line with gradient –1,

so $$f'(x) = -1 \text{ if } x < 0$$

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However, when $x = 0$ no unique limit exists since we get different values for the limit depending on how we calculate it. In particular, if we approach $x = 0$ from the positive side, where $x > 0$ and h is positive

$$\lim_{h\to 0+}\left[\frac{f(0+h)-f(0)}{h}\right] = \lim_{h\to 0+}\left(\frac{|h|-|0|}{h}\right) = \lim_{h\to 0+}\left(\frac{|h|}{h}\right) = 1$$

whereas, if we approach $x = 0$ from the negative side, where $x < 0$ and h is negative

$$\lim_{h\to 0-}\left[\frac{f(0+h)-f(0)}{h}\right] = \lim_{h\to 0-}\left(\frac{|h|-|0|}{h}\right) = \lim_{h\to 0-}\left(\frac{|h|}{h}\right) = -1$$

So, if h approaches zero through positive values the limit is 1, but if h approaches zero through negative values the limit is -1. A *unique* limit exists only if we obtain the *same* result no matter how h approaches 0. Since *different* answers have been obtained in this case there is no unique limit and hence no derivative at $x = 0$. Although the original function is defined for all real values of x the derivative is only defined for non-zero values of x.

On the whole, the domains of functions are of more concern to mathematicians than to physicists. Nonetheless, it is important that you should be aware of their significance, and prepared to investigate them if necessary.

Question T3

(a) Use your calculator to convince yourself that $\dfrac{\exp(h) - 1}{h}$ ☞ is very nearly equal to 1 when h is very small.

(b) Let $f(x) = \exp(x)$. What does part (a) tell you about $f'(0)$?

(c) Use the fact that $\lim\limits_{h \to 0}\left[\dfrac{\exp(h) - 1}{h}\right] = 1$, and the definition of the derivative to show that $f'(x) = \exp(x)$. ❏

2.3 Notation for derivatives

Various notations are in common use for the derivative of a given function. Two such notations have already been introduced in this module and you will probably meet a third elsewhere. Here we provide a brief summary.

Function notation

This is the notation we have mainly used so far, in which a function is represented by f or $f(x)$ or some similar symbol and its derivative (the derived function) is represented by f' or $f'(x)$. This is a neat and compact notation but care is needed in handwritten work to make sure that the all-important prime ($'$) is clearly visible.

Leibniz notation ☞

This is the dy/dx notation that is especially popular among physicists. We will make much use of it in what follows.

If we let $y = f(x)$, then in Leibniz notation the derivative is written $\dfrac{dy}{dx}(x)$ or $\dfrac{df}{dx}(x)$. ☞ Sometimes these are abbreviated to $\dfrac{dy}{dx}$ (in which case it is assumed that we are regarding the variable y as a function $y(x)$ of x), or $\dfrac{df}{dx}$ (in which case it is assumed that we are discussing a function $f(x)$).

An advantage of Leibniz notation is that it is also possible to write the derivative as

$$\frac{d}{dx}[f(x)]$$

so that for a specific function, $f(x) = x^2$ say, we could write

$$\frac{d}{dx}(x^2) = 2x$$

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Newtonian notation ☞

A different notation that is especially common in mechanics, but which will not be used in this module, uses a dot to denote derivatives. Thus, if $x(t)$ represents the x coordinate of a moving particle at time t, then $\dot{x}(t)$ represents the rate of change of x with respect to time, i.e. the x component of the particle's velocity.

❖ Given that $y = f(x) = 1/x$, and recalling Equation 7, determine the following:

(a) $\dfrac{dy}{dx}(2)$, (b) $\dfrac{df}{dx}(2a)$, (c) $\dfrac{d}{dt}\left(\dfrac{1}{t}\right)$, (d) $f'(2+3x)$.

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3 Derivatives of simple functions

Now that the definitions are out of the way we can get down to the real business of *doing* calculus. The procedure by which derivatives are determined is called **differentiation**. In practice everybody who uses differentiation regularly, knows the derivatives of various standard functions (x, $\sin x$, $\exp(x)$, etc.) and knows some simple rules for finding the derivatives of various combinations (sums, differences, products, quotients and reciprocals) of those standard derivatives. The formal definition of a derivative is rarely used in practice. This section introduces the standard derivatives and the basic rules for combining them.

Table 1 Some standard derivatives.

3.1 Derivatives of basic functions

Tables 1a and 1b list a number of functions with which you should already be familiar along with their derivatives. Each of these derivatives can be deduced from the definition given in the last section, though the proof is not always easy. If you are going to use calculus frequently you will need to know these derivatives or at least know where you can look them up quickly.

(a)

$f(x)$	$f'(x)$
k (constant)	0
kx^n	nkx^{n-1}
$\sin kx$	$k\cos kx$
$\cos kx$	$-k\sin kx$
$\tan kx$	$k\sec^2 kx$
$\operatorname{cosec} kx$	$-k\operatorname{cosec} kx\cot kx$
$\sec kx$	$k\sec kx\tan kx$
$\cot kx$	$-k\operatorname{cosec}^2 kx$
$\exp(kx)$	$k\exp(kx)$
$\log_e(kx)$	$1/x$

(b)

$f(x)$	$f'(x)$
1	0
x^n	nx^{n-1}
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\operatorname{cosec} x$	$-\operatorname{cosec} x\cot x$
$\sec x$	$\sec x\tan x$
$\cot x$	$-\operatorname{cosec}^2 x$
$\exp(x)$	$\exp(x)$
$\log_e(x)$	$1/x$

When using the tables it is important to remember the following points:

- n and k are constants.
- The functions in Table 1b are special cases of those in Table 1a, corresponding to $k = 1$.
- In each of the trigonometric functions x must be an angle in radians or a dimensionless real variable.

Table 1 Some standard derivatives.

(a)

$f(x)$	$f'(x)$
k (constant)	0
kx^n	nkx^{n-1}
$\sin kx$	$k\cos kx$
$\cos kx$	$-k\sin kx$
$\tan kx$	$k\sec^2 kx$
$\operatorname{cosec} kx$	$-k\operatorname{cosec} kx\cot kx$
$\sec kx$	$k\sec kx\tan kx$
$\cot kx$	$-k\operatorname{cosec}^2 kx$
$\exp(kx)$	$k\exp(kx)$
$\log_e(kx)$	$1/x$

(b)

$f(x)$	$f'(x)$
1	0
x^n	nx^{n-1}
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\operatorname{cosec} x$	$-\operatorname{cosec} x\cot x$
$\sec x$	$\sec x\tan x$
$\cot x$	$-\operatorname{cosec}^2 x$
$\exp(x)$	$\exp(x)$
$\log_e(x)$	$1/x$

Question T4

Use the definition of the derivative to show that if $f(x) = x^n$ and n is a positive integer, then $f'(x) = nx^{n-1}$.

The best way to get to know the standard derivatives is to use them frequently. Here are a few questions to start the process of familiarization.

Table 1 Some standard derivatives.

(a)

$f(x)$	$f'(x)$
k (constant)	0
kx^n	nkx^{n-1}
$\sin kx$	$k\cos kx$
$\cos kx$	$-k\sin kx$
$\tan kx$	$k\sec^2 kx$
$\operatorname{cosec} kx$	$-k\operatorname{cosec} kx\cot kx$
$\sec kx$	$k\sec kx\tan kx$
$\cot kx$	$-k\operatorname{cosec}^2 kx$
$\exp(kx)$	$k\exp(kx)$
$\log_e(kx)$	$1/x$

(b)

$f(x)$	$f'(x)$
1	0
x^n	nx^{n-1}
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\operatorname{cosec} x$	$-\operatorname{cosec} x\cot x$
$\sec x$	$\sec x\tan x$
$\cot x$	$-\operatorname{cosec}^2 x$
$\exp(x)$	$\exp(x)$
$\log_e(x)$	$1/x$

Table 1 Some standard derivatives.

(a)

$f(x)$	$f'(x)$
k (constant)	0
kx^n	nkx^{n-1}
$\sin kx$	$k\cos kx$
$\cos kx$	$-k\sin kx$
$\tan kx$	$k\sec^2 kx$
$\operatorname{cosec} kx$	$-k\operatorname{cosec} kx\cot kx$
$\sec kx$	$k\sec kx\tan kx$
$\cot kx$	$-k\operatorname{cosec}^2 kx$
$\exp(kx)$	$k\exp(kx)$
$\log_e(kx)$	$1/x$

(b)

$f(x)$	$f'(x)$
1	0
x^n	nx^{n-1}
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\operatorname{cosec} x$	$-\operatorname{cosec} x\cot x$
$\sec x$	$\sec x\tan x$
$\cot x$	$-\operatorname{cosec}^2 x$
$\exp(x)$	$\exp(x)$
$\log_e(x)$	$1/x$

❖

Find the derivatives (using Table 1) of the following functions:

(a) $f(x) = x^3$,

(b) $g(t) = 1/t^3$,

(c) $h(x) = \dfrac{1}{\sqrt[5]{x^2}}$.

❖ Which function in Table 1 is its own derivative?

Question T5

Find where the gradients of the tangents to the graph of $y = \sin x$ have the following values: (a) 0, (b) –1, (c) +1.

☞ ❑

Question T6

(a) Use your calculator (making sure that it is in radian mode) to investigate the limits

$$\lim_{h\to 0}\left[\frac{\cos(h)-1}{h}\right] \quad \text{and} \quad \lim_{h\to 0}\left[\frac{\sin h}{h}\right]$$

Using successively smaller values of h (e.g. $h = \pm 0.1$, ± 0.01, etc.) try to estimate the limit in each case.

(b) Use the limits obtained in part (a) to find $\dfrac{d}{dx}(\sin x)$ from the general definition of the derived function.

(*Hint*: Use the trigonometric identity $\sin(A + B) = \sin A \cos B + \cos A \sin B$.) ❑

Question T7

Which functions in Table 1b have derivatives which are:

(a) always positive;

(b) always negative;

(c) can be positive or negative? ❏

Table 1 Some standard derivatives.

(a)

$f(x)$	$f'(x)$
k (constant)	0
kx^n	nkx^{n-1}
$\sin kx$	$k\cos kx$
$\cos kx$	$-k\sin kx$
$\tan kx$	$k\sec^2 kx$
$\operatorname{cosec} kx$	$-k\operatorname{cosec} kx\cot kx$
$\sec kx$	$k\sec kx\tan kx$
$\cot kx$	$-k\operatorname{cosec}^2 kx$
$\exp(kx)$	$k\exp(kx)$
$\log_e(kx)$	$1/x$

(b)

$f(x)$	$f'(x)$
1	0
x^n	nx^{n-1}
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\operatorname{cosec} x$	$-\operatorname{cosec} x\cot x$
$\sec x$	$\sec x\tan x$
$\cot x$	$-\operatorname{cosec}^2 x$
$\exp(x)$	$\exp(x)$
$\log_e(x)$	$1/x$

3.2 Derivative of a sum of functions

While it is possible, in theory, to find the derivative of any given function from the definition, this would in fact be an arduous process. In practice, those using calculus employ a set of simple rules which can be applied to combinations of functions to find the derivatives of a wide variety of functions with relative ease. The first of these simple rules is called the **sum rule**, which states:

The derivative of a sum of functions is the sum of the derivatives of the individual functions.

If $f(x)$ and $g(x)$ are two functions, this can be written as:

sum rule $$\frac{d}{dx}[f(x)+g(x)] = \frac{d}{dx}[f(x)] + \frac{d}{dx}[g(x)] \qquad (8)$$

Alternatively we may write this rule in the form

$$[f(x)+g(x)]' = f'(x) + g'(x) \quad \text{or just} \quad (f+g)' = f' + g'$$

This rule enables us to differentiate functions such as $y = x^{1/2} + \log_e x$.

To do so we first note that it is the sum of two standard functions $f(x) = x^{1/2}$ and $g(x) = \log_e x$, we then differentiate each of these functions and finally add the derivatives to obtain the required answer.

So, if

$f(x) = x^{1/2}$ and $g(x) = \log_e x$

then, from Table 1

$$\frac{df}{dx} = \frac{1}{2}x^{-1/2} \quad \text{and} \quad \frac{dg}{dx} = \frac{1}{x}$$

so,

$$\frac{dy}{dx} = \frac{df}{dx} + \frac{dg}{dx} = \frac{1}{2}x^{-1/2} + \frac{1}{x}$$

and if $y = \sqrt{x} + \log_e x$

then $$\frac{dy}{dx} = \frac{1}{2\sqrt{x}} + \frac{1}{x}$$

It is usually a good idea to *think* before applying the rule, as in the following question.

Table 1 Some standard derivatives.

(a)

$f(x)$	$f'(x)$
k (constant)	0
kx^n	nkx^{n-1}
$\sin kx$	$k\cos kx$
$\cos kx$	$-k\sin kx$
$\tan kx$	$k\sec^2 kx$
$\operatorname{cosec} kx$	$-k\operatorname{cosec} kx\cot kx$
$\sec kx$	$k\sec kx\tan kx$
$\cot kx$	$-k\operatorname{cosec}^2 kx$
$\exp(kx)$	$k\exp(kx)$
$\log_e(kx)$	$1/x$

(b)

$f(x)$	$f'(x)$
1	0
x^n	nx^{n-1}
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\operatorname{cosec} x$	$-\operatorname{cosec} x\cot x$
$\sec x$	$\sec x\tan x$
$\cot x$	$-\operatorname{cosec}^2 x$
$\exp(x)$	$\exp(x)$
$\log_e(x)$	$1/x$

Find the derivative of

$$h(x) = \frac{\cos x}{\sin x} + \frac{\sin x}{\cos x}.$$

The rule 'derivative of a sum equals the sum of the derivatives' applies equally well if more than two functions are added together. This allows us to obtain the derivatives of functions such as those in the following question.

Table 1 Some standard derivatives.

(a)

$f(x)$	$f'(x)$
k (constant)	0
kx^n	nkx^{n-1}
$\sin kx$	$k\cos kx$
$\cos kx$	$-k\sin kx$
$\tan kx$	$k\sec^2 kx$
$\operatorname{cosec} kx$	$-k\operatorname{cosec} kx \cot kx$
$\sec kx$	$k\sec kx \tan kx$
$\cot kx$	$-k\operatorname{cosec}^2 kx$
$\exp(kx)$	$k\exp(kx)$
$\log_e(kx)$	$1/x$

(b)

$f(x)$	$f'(x)$
1	0
x^n	nx^{n-1}
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$
$\sec x$	$\sec x \tan x$
$\cot x$	$-\operatorname{cosec}^2 x$
$\exp(x)$	$\exp(x)$
$\log_e(x)$	$1/x$

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Question T8

Use the sum rule to find dy/dx in each of the following cases:

(a) $y = 2 + \sqrt[3]{x} + e^x$

(b) $y = (1 + \sqrt{x})^2$

(c) $y = \log_e (xe^x)$

(d) $y = \cot x \sin x + \tan x \cos x$.

(*Hint*: Write each function as a sum of functions appearing in Table 1.) ❑

Table 1 Some standard derivatives.

(a)

$f(x)$	$f'(x)$
k (constant)	0
kx^n	nkx^{n-1}
$\sin kx$	$k \cos kx$
$\cos kx$	$-k \sin kx$
$\tan kx$	$k \sec^2 kx$
$\operatorname{cosec} kx$	$-k \operatorname{cosec} kx \cot kx$
$\sec kx$	$k \sec kx \tan kx$
$\cot kx$	$-k \operatorname{cosec}^2 kx$
$\exp(kx)$	$k \exp(kx)$
$\log_e(kx)$	$1/x$

(b)

$f(x)$	$f'(x)$
1	0
x^n	nx^{n-1}
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$
$\sec x$	$\sec x \tan x$
$\cot x$	$-\operatorname{cosec}^2 x$
$\exp(x)$	$\exp(x)$
$\log_e(x)$	$1/x$

Question T9

If you were to use the definition of the derived function (Equation 6)

$$\frac{dy}{dx} = \lim_{h\to 0}\left[\frac{f(x+h) - f(x)}{h}\right] \qquad \text{(Eqn 6)}$$

to show that the ‘derivative of the sum is the sum of the derivatives’, what assumption must you make about the limit of a sum? ❏

3.3 Derivative of a constant multiple of a function

Table 1 gives the derivative of $\sin x$, for example, but what about $2\sin x$? In other words, what is the derivative of a constant times a function? The answer is given by the **constant multiple rule,** which states:

The derivative of a constant times a function is equal to the constant times the derivative of the function.

Using k for the constant and $f(x)$ for the function, this result can be written as:

constant multiple rule $$\frac{d}{dx}[kf(x)] = k\frac{d}{dx}[f(x)] \tag{9}$$ ☞

i.e. $[kf(x)]' = kf'(x)$ or just $(kf)' = kf'$

Therefore, for the above example,

$$\frac{d}{dx}(2\sin x) = 2\frac{d}{dx}(\sin x) = 2\cos x$$

similarly, $$\frac{d}{dx}(\pi e^x) = \pi\frac{d}{dx}(e^x) = \pi \times e^x = \pi e^x$$

✦

Differentiate $h(x) = \left(x + \dfrac{1}{x}\right)^3$

✦

Use the constant multiple rule to show that $[f(x) - g(x)]' = f'(x) - g'(x)$.

This last result shows that *the derivative of the difference of two functions is the difference of the derivatives*, i.e.

$$\frac{d}{dx}[f(x) - g(x)] = \frac{d}{dx}[f(x)] - \frac{d}{dx}[g(x)] \tag{10}$$

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Question T10

Find the derivatives of the following functions:

(a) $\left(\sqrt{x} - \dfrac{1}{\sqrt{x}}\right)^4$ (b) $\log_e \sqrt[3]{x}$

(c) $\left(\sin \dfrac{x}{2} + \cos \dfrac{x}{2}\right)^2$ (d) $(\pi - x)^2$

(e) $\sin(x + 2)$ (f) $\left[\tan\left(x + \dfrac{\pi}{4}\right)\right](1 - \tan x)$

$$\left[\mathit{Hint}\colon \tan(A + B) = \left(\frac{\tan A + \tan B}{1 - \tan A \tan B}\right)\right]$$ ❏

Question T11

If k is a constant and $f(x)$ a function, write down the definition of $f'(x)$ and $[kf(x)]'$. What assumption must you make about limits to justify the conclusion $[kf(x)]' = kf'(x)$? ❏

3.4 Derivative of a product of functions

This subsection introduces yet another way of combining functions and obtaining the derivative of the result. This is the *product* of two functions, $f(x)g(x)$. Our aim is to express the derivative of such a product in terms of the derivatives of the functions $f(x)$ and $g(x)$ that are multiplied together to form the product, but the answer is not as obvious as it was for the sum or difference of functions. The 'obvious' answer that the derivative of a product is the product of the derivatives is *wrong*.

You cannot simply multiply derivatives.

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To convince yourself of this consider $f(x) = x^2$ and $g(x) = x^3$. Then $f(x)g(x) = x^2x^3 = x^5$.
We can differentiate each of these functions using Table 1:

thus, $\quad f(x) = x^2$

implies $\quad f'(x) = 2x$

$\quad g(x) = x^3$

implies $\quad g'(x) = 3x^2$

and $\quad f(x)g(x) = x^5$

implies $\quad [f(x)g(x)]' = 5x^4$

Clearly, $[f(x)g(x)]' \neq f'(x) \times g'(x)$

Table 1 Some standard derivatives.

(a)

$f(x)$	$f'(x)$
k (constant)	0
kx^n	nkx^{n-1}
$\sin kx$	$k\cos kx$
$\cos kx$	$-k\sin kx$
$\tan kx$	$k\sec^2 kx$
$\operatorname{cosec} kx$	$-k\operatorname{cosec} kx\cot kx$
$\sec kx$	$k\sec kx\tan kx$
$\cot kx$	$-k\operatorname{cosec}^2 kx$
$\exp(kx)$	$k\exp(kx)$
$\log_e(kx)$	$1/x$

(b)

$f(x)$	$f'(x)$
1	0
x^n	nx^{n-1}
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\operatorname{cosec} x$	$-\operatorname{cosec} x\cot x$
$\sec x$	$\sec x\tan x$
$\cot x$	$-\operatorname{cosec}^2 x$
$\exp(x)$	$\exp(x)$
$\log_e(x)$	$1/x$

Let $f(x) = 2x$ and $g(x) = \dfrac{1}{4x}$

Find $f'(x)$, $g'(x)$, $[f(x)g(x)]'$ and show that $[f(x)g(x)]' \neq f'(x) \times g'(x)$.

The correct determination of $[f(x)g(x)]'$ involves not only $f'(x)$ and $g'(x)$ *but also* $f(x)$ and $g(x)$. It is given by the **product rule** and is easier to express in symbols than in words:

product rule $$\frac{d}{dx}[f(x)g(x)] = g(x)\frac{df}{dx} + f(x)\frac{dg}{dx} \qquad (11)$$

i.e. $[f(x)g(x)]' = f'(x)g(x) + f(x)g'(x)$ or just $(fg)' = f'g + fg'$

The expression of this rule in terms of words is probably not very helpful:

The derivative of f times g equals the derivative of f times g plus f times the derivative of g.

You may find Figure 4 a more useful memory aid.

In the case of $f(x) = x^2$ and $g(x) = x^3$ (discussed earlier) where

$$f'(x) = 2x \quad \text{and} \quad g'(x) = 3x^2$$

the application of the product rule to $f(x)g(x)$

$$\frac{d}{dx}[f(x)g(x)] = g(x)\frac{df}{dx} + f(x)\frac{dg}{dx} \qquad \text{(Eqn 11)}$$

gives

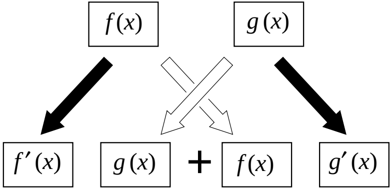

Figure 4 Differentiating a product of functions.

$$[f(x)g(x)]' = f'(x)g(x) + f(x)g'(x) = 2x \times x^3 + x^2 \times 3x^2 = 2x^4 + 3x^4 = 5x^4$$

and $5x^4$ is what we expect for the derivative of $f(x)g(x) = x^2x^3 = x^5$.

✧

If $f(x) = 2x$ and $g(x) = \dfrac{1}{4x}$ apply the product rule to find the derivative of $f(x)g(x)$.

Illustrations of the product rule

With the aid of the product rule we can now differentiate many more functions. The important step is to express the function that is to be differentiated in terms of the sum, difference or product of functions with known derivatives. Here are some examples.

Example 1 Differentiate $2\sin x\cos x$.

Solution $2\sin x\cos x$ may be written as a product of the functions

$$f(x) = 2\sin x \quad \text{and} \quad g(x) = \cos x$$

for which $\quad f'(x) = 2\cos x \quad \text{and} \quad g'(x) = -\sin x$

Using the product rule

$$\frac{d}{dx}[f(x)g(x)] = g(x)\frac{df}{dx} + f(x)\frac{dg}{dx} \tag{Eqn 11}$$

we can therefore write

$$\frac{d}{dx}(2\sin x\cos x) = (2\cos x)(\cos x) + (2\sin x)(-\sin x) \quad = 2(\cos^2 x - \sin^2 x) \quad ❏$$

☞

Example 2 Differentiate $xe^x + \sec^2 x$.

Solution $xe^x + \sec^2 x$ is the sum of two functions xe^x and $\sec^2 x$, each of which is a product. So we differentiate xe^x first and then $\sec^2 x$ (using the product rule for each) and then add the answers using the sum rule.

First, xe^x may be written as the product of the functions

$$f(x) = x \quad \text{and} \quad g(x) = e^x$$

for which $\quad f'(x) = 1 \quad \text{and} \quad g'(x) = e^x$

then, using the product rule

$$\frac{d}{dx}[f(x)g(x)] = g(x)\frac{df}{dx} + f(x)\frac{dg}{dx} \qquad \text{(Eqn 11)}$$

we can write

$$\frac{d}{dx}(xe^x) = e^x \times \frac{d}{dx}(x) + x \times \frac{d}{dx}(e^x) = e^x \times 1 + x \times e^x = (1+x)e^x$$

Second, $\sec^2 x = \sec x \times \sec x$,

therefore $\quad f(x) = \sec x \quad \text{and} \quad g(x) = \sec x$

giving us $\quad f'(x) = \sec x \tan x \quad \text{and} \quad g'(x) = \sec x \tan x$

Then, again using the product rule

$$\frac{d}{dx}[f(x)g(x)] = g(x)\frac{df}{dx} + f(x)\frac{dg}{dx} \qquad \text{(Eqn 11)}$$

we find

$$\frac{d}{dx}(\sec^2 x) = (\sec x \tan x) \times (\sec x) + (\sec x) \times (\sec x \tan x) = 2\sec^2 x \tan x$$

Thus, combining these results, and using the sum rule we find

$$\frac{d}{dx}(xe^x + \sec^2 x) = \frac{d}{dx}(xe^x) + \frac{d}{dx}(\sec^2 x) = (1+x)e^x + 2\sec^2 x \tan x \quad ❏$$

✥ The *kinetic energy* of a particle of mass m moving with a speed $v(t)$ that varies with time t, is $m[v(t)]^2/2$. Use the product rule

$$\frac{d}{dx}[f(x)g(x)] = g(x)\frac{df}{dx} + f(x)\frac{dg}{dx} \qquad \text{(Eqn 11)}$$

to show that the rate of change of $m[v(t)]^2/2$ with time is $mv(t)\dfrac{dv}{dt}(t)$.

Question T12

Find the derivative of each of the following functions:

(a) $x^2 \log_e x$, (b) $(\sin x - 2\cos x)^2$, (c) $(1 - \sqrt{x})(1 + \sqrt[3]{x})$,

(d) e^{2x}, (e) $\sec x \cot x$, (f) $x^3 F(x)$, where $F(x)$ is an arbitrary function. ❏

Question T13

The instantaneous motion of a particle moving along a straight line (call it the x-axis) can be described, at time t, in terms of its *displacement* $s_x(t)$ from a fixed reference position, its *velocity* $v_x(t)$ and its *acceleration* $a_x(t)$. These quantities are defined in such a way that

$$a_x(t) = \frac{dv_x}{dt}(t) \quad \text{and} \quad v_x(t) = \frac{ds_x}{dt}(t)$$

If the displacement is given as a function of time by

$$s_x(t) = [\cos(\omega t) - 2\sin(\omega t)]\exp(\alpha t) \quad \text{where } \omega \text{ and } \alpha \text{ are constants}$$

show that $a_x(t) = 2\alpha v_x(t) - (\omega^2 + \alpha^2)s_x(t)$ ❏

Question T14

This question extends the product rule to three functions.

$$\frac{d}{dx}[f(x)g(x)] = g(x)\frac{df}{dx} + f(x)\frac{dg}{dx} \qquad \text{(Eqn 11)}$$

If $f(x)$, $g(x)$, $h(x)$ are any three functions, show that

$$\frac{d}{dx}\big[f(x)g(x)h(x)\big] = f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x)$$ ❏

Question T15

Use the result developed in Question T14 to find

$$\frac{d}{dx}(e^x \log_e x \sin x)$$ ❏

3.5 Derivative of a quotient of functions

It is frequently the case that we need to find the derivative of the quotient $\dfrac{f(x)}{g(x)}$ of two functions $f(x)$ and $g(x)$.

☞ The derivative of a quotient, like that of a product, depends not only on the values of $f'(x)$ and $g'(x)$, but also on $f(x)$ and $g(x)$. The **quotient rule** states:

quotient rule $$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\dfrac{df}{dx} - f(x)\dfrac{dg}{dx}}{[g(x)]^2} \tag{12}$$ ☞

i.e. $$\left[\frac{f(x)}{g(x)}\right]' = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$ or just $$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

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You may find Figure 5 a useful memory aid.

To apply this result

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\dfrac{df}{dx} - f(x)\dfrac{dg}{dx}}{[g(x)]^2} \qquad \text{(Eqn 12)}$$

to the function $y = \dfrac{\sin x}{x}$

take $\quad f(x) = \sin x \quad$ and $\quad g(x) = x$

so that $\quad f'(x) = \cos x \quad$ and $\quad g'(x) = 1$

Then $$\frac{dy}{dx} = \frac{(\cos x)x - (\sin x) \times 1}{x^2} = \frac{x\cos x - \sin x}{x^2}$$ ☞

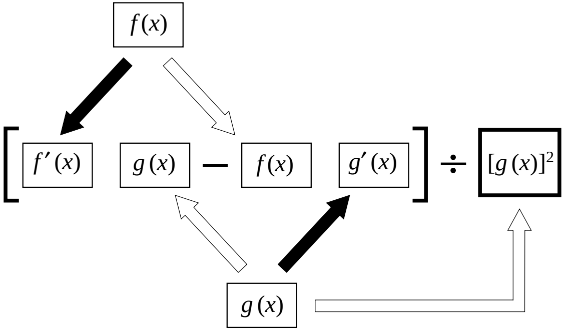

Figure 5 The quotient rule.

✥ Write $y = x^3$ as x^5/x^2 and use the quotient rule

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\dfrac{df}{dx} - f(x)\dfrac{dg}{dx}}{[g(x)]^2} \qquad \text{(Eqn 12)}$$

to find dy/dx.

✥ Use the quotient rule

to find $\dfrac{d}{dx}\left[\dfrac{1}{F(x)}\right]$ in terms of $F'(x)$ and $F(x)$.

You may find this useful, but there is no point in remembering it if you know the quotient rule since it is just a special case of that more general rule.

Question T16

Find dy/dx in each of the following cases:

(a) $y = \dfrac{x}{e^x}$ (b) $y = \dfrac{1 - x^2}{\cos x}$ (c) $y = \dfrac{x^5 - x^4 + x^3 - x^2}{(x + 1)^2}$ (d) $y = \dfrac{x \log_e x}{\tan x}$ ❏

Question T17

If $f(x) = (x - 1)^2$ and $g(x) = xe^x$, find the following:

(a) $\dfrac{d}{dx}[f(x) - g(x)]$ (b) $\dfrac{d}{dx}[f(x)g(x)]$ (c) $\dfrac{d}{dx}\left[\dfrac{f(x)}{g(x)}\right]$ (d) $\dfrac{d}{dx}\left[\dfrac{g(x)}{f(x)}\right]$ ❏ ☞

The aim of the next question is to derive the quotient rule from the product rule. It begins by showing how to obtain the reciprocal rule *without* using the quotient rule.

Question T18

(a) Let $f(x)$ be an arbitrary function and let $h(x) = 1/f(x)$ so that

$$f(x)h(x) = 1$$

Differentiate both sides of this equation and hence obtain a formula for $\dfrac{d}{dx}\left[\dfrac{1}{f(x)}\right]$

(similar to that given in Equation 13).

the reciprocal rule $$\frac{d}{dx}\left[\frac{1}{F(x)}\right] = -\frac{F'(x)}{[F(x)]^2} \qquad \text{(Eqn 13)}$$

(b) Let $f(x)$ and $g(x)$ be any two functions, and write $\dfrac{f(x)}{g(x)} = f(x) \times \dfrac{1}{g(x)}$, then use the answer to part (a) and

the product rule $$\frac{d}{dx}[f(x)g(x)] = g(x)\frac{df}{dx} + f(x)\frac{dg}{dx} \qquad \text{(Eqn 11)}$$

to derive the quotient rule $$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\dfrac{df}{dx} - f(x)\dfrac{dg}{dx}}{[g(x)]^2} \qquad \text{(Eqn 12)}$$ ❏

Question T19

Use $\dfrac{d}{dx}(\sin x) = \cos x$, $\dfrac{d}{dx}(\cos x) = -\sin x$, and the quotient rule

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\dfrac{df}{dx} - f(x)\dfrac{dg}{dx}}{[g(x)]^2} \qquad \text{(Eqn 12)}$$

to differentiate (a) $\tan x$, and (b) $\cot x$. ❏

Question T20

(a) Use the reciprocal rule

$$\frac{d}{dx}\left[\frac{1}{F(x)}\right] = -\frac{F'(x)}{[F(x)]^2} \qquad \text{(Eqn 13)}$$

to find $\dfrac{d}{dx}(e^{-x})$.

(b) The function $f(x) = \dfrac{1}{2}(e^x + e^{-x})$ is sometimes known as the *hyperbolic cosine* of x (and is denoted by $\cosh x$).

Find $\dfrac{df}{dx}$ and show that $[f(x)]^2 - \left(\dfrac{df}{dx}\right)^2 = 1$. ❏

Question T21

Let $F(x) = G\dfrac{Mm}{(x-a)^2}$ where G, M, m and a are constants. Find $F'(x)$. ❏

Question T22

Using $\dfrac{d}{dx}(\sin x) = \cos x$ and $\dfrac{d}{dx}(\cos x) = -\sin x$, along with the reciprocal rule,

$$\frac{d}{dx}\left[\frac{1}{F(x)}\right] = -\frac{F'(x)}{[F(x)]^2} \qquad \text{(Eqn 13)}$$

find the derivatives of (a) $\sec x$ and (b) $\operatorname{cosec} x$. ❏

4 More about logarithmic and exponential functions

4.1 Why exp(x) is considered to be special

Of all the functions listed in Table 1, $f(x) = \exp(x) = e^x$ is the only one which is its own derivative. Far from being a technical curiosity this fact turns out to be of great significance and ultimately explains why the exponential function plays such an important role in physics. ☞

It also has an interesting interpretation in terms of the graph of $y = e^x$, shown in Figure 6. Since $f'(x) = e^x$ the gradient of the tangent to the graph at the point A which has coordinates $x = a$ and $y = e^a$ is $f'(a) = e^a$ which is the same as the height of A above C.

So the rate of change of $f(x)$ when $x = a$ is simply $f(a)$, that is, the rate of change of the exponential function at any point is equal to the value of the function itself at that point. In Figure 6 the point B has coordinates $(a - 1, 0)$. Since A is the point (a, e^a), the line AB has a gradient of

$$\frac{\text{CA}}{\text{BC}} = \frac{e^a}{1} = e^a$$

and is therefore the tangent to the graph at A. So, to draw a tangent to $y = e^x$ at (a, e^a), simply join (a, e^a) to $(a - 1, 0)$ and the resulting line is bound to have a gradient of e^a.

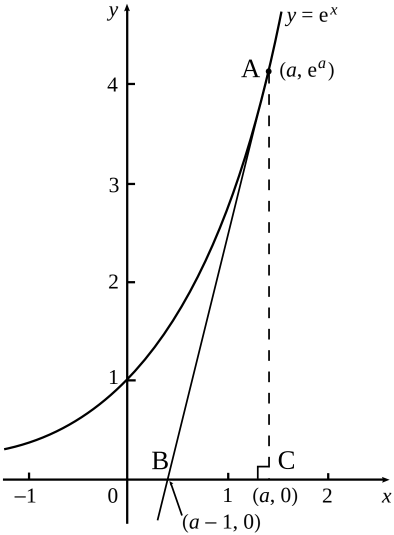

Figure 6 The graph of $y = e^x$.

✦ Is $f(x) = e^x$ the only function which is its own derivative?
(Think about the rules of differentiation that you already know.)

✦ Using the product rule

$$\frac{d}{dx}[f(x)g(x)] = g(x)\frac{df}{dx} + f(x)\frac{dg}{dx} \qquad \text{(Eqn 11)}$$

find the derivatives of (a) e^{2x}, (b) e^{3x}, (c) e^{4x}.

Deduce the formula for the derivative of e^{nx} where n is any positive integer.

A similar result holds true even when n is not a positive integer. This was given in Table 1, but it deserves to be emphasized again here.

$$\boxed{\frac{d}{dx}(e^{kx}) = ke^{kx} \quad \text{for any constant } k \qquad (14)}$$

Question T23

If $y = \dfrac{Ce^x}{1 + Ce^x}$ for some constant C, show that $\dfrac{dy}{dx} = y(1 - y)$. ❏

Question T24

Differentiate $f(x) = (e^x + e^{-x})(e^x - e^{-x})$. ❏

4.2 Derivative of a^x

The previous subsection considered the derivative of e^x. However, functions such as 2^x and π^x sometimes arise in physical applications, and so it is necessary to know how to differentiate any function of the form a^x, where the positive number a being raised to the power x is not necessarily e. As an example we will consider the differentiation of 2^x, but the method is of general applicability. First note that exp is the *inverse function* of $\log_e$, this means that exp reverses the effect of $\log_e$ so that

$$e^{\log_e 2} = \exp(\log_e 2) = 2$$

since $2 = \exp(\log_e 2)$ $\quad 2^x = [\exp(\log_e 2)]^x = \exp(x \log_e 2)$

So, $$\frac{d}{dx}(2^x) = \frac{d}{dx}[\exp(x \log_e 2)]$$

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Now, from Equation 14 (or Table 1)

$$\frac{d}{dx}(e^{kx}) = ke^{kx} \quad \text{for any constant } k \tag{Eqn 14}$$

we know that

$$\frac{d}{dx}(e^{kx}) = ke^{kx} \quad \text{for any constant } k$$

and, if we choose k to be $\log_e 2$, this gives us

$$\frac{d}{dx}(2^x) = \frac{d}{dx}[\exp(x\log_e 2)] = (\log_e 2)[\exp(x\log_e 2)] = (\log_e 2)2^x$$

A similar argument applies with any positive number, a say, in place of 2. So we have the result:

$$\boxed{\frac{d}{dx}(a^x) = (\log_e a)a^x \quad (a > 0) \tag{15}}$$

Rather than trying to remember this result you would probably be wiser to try to remember the method that was used to obtain it. Notice the way that the properties of exponentials and logarithms have been used to express a function of the form a^x in terms of e^{kx}, and how the simple properties of e^{kx} have then been exploited.

❖

Find $\dfrac{d}{dx}\,(\pi^x)$.

❖

Apply the formula for differentiating a^x (Equation 14)

$$\frac{d}{dx}\,(\mathrm{e}^{kx}) = k\mathrm{e}^{kx} \quad \text{for any constant } k \qquad \text{(Eqn 14)}$$

when $a = \mathrm{e}$.

Question T25

Differentiate each of the following functions:

(a) $f(x) = 3^x$, (b) $f(x) = 3^{x-1}$, (c) $f(x) = (2 + 3)^x$, (d) $f(x) = 2^x\mathrm{e}^x$. ❑

4.3 Derivative of $\log_a x$

As stated in Table 1:

$$\frac{d}{dx}(\log_e x) = \frac{1}{x}$$

From this it is easy to deduce the derivative of $\log_e kx$, where k is a constant

$$\log_e(kx) = \log_e k + \log_e x$$

so $$\frac{d}{dx}(\log_e(kx)) = \frac{d}{dx}(\log_e k) + \frac{d}{dx}(\log_e x) = \frac{1}{x}$$ ☞

If we had been compelled to differentiate $\log_{10} x$ or $\log_2 x$ the problem would have been a little more difficult, but still not intractable. The general method for differentiating logs to an arbitrary base, such as $\log_a x$ is remarkably similar to that for dealing with a^x: it involves expressing $\log_a x$ in terms of $\log_e x$ which can be differentiated easily. As an example, let us consider $\log_{10} x$. To express it in terms of $\log_e x$ we write

$$y = \log_{10} x$$

then $\quad x = 10^y$

and, taking logarithms to the base e of both sides of this equation, we obtain

$$\log_e x = \log_e (10^y) = y \log_e 10$$

and recalling that $y = \log_{10} x$ this can be rearranged to give

$$y = \log_{10} x = \frac{\log_e x}{\log_e 10}$$

☞

We may now easily differentiate $\log_{10} x$ because $1/\log_e 10$ is a constant, so

$$\frac{d}{dx}(\log_{10} x) = \frac{d}{dx}\left(\frac{\log_e x}{\log_e 10}\right)$$

$$= \left(\frac{1}{\log_e 10}\right)\frac{d}{dx}(\log_e x) = \left(\frac{1}{\log_e 10}\right) \times \frac{1}{x} = \frac{1}{x \log_e 10}$$

A similar argument applies for any positive base a giving the general result:

$$\frac{d}{dx}(\log_a x) = \frac{1}{x \log_e a} \tag{16}$$

☞

❖ If $a = \mathrm{e}$, check that the above formula gives our previous result for the derivative of $\log_e x$.

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✧ Differentiate $\log_\pi x$.

Question T26

Differentiate each of the following functions:

(a) $f(x) = \log_2 x$

(b) $f(x) = \log_{10} x^2$

(c) $f(x) = \dfrac{\log_2 x}{\log_{10} x}$ [*Hint*: $\log_a b = (\log_a c)(\log_c b)$]

(d) $f(x) = a^x \log_a x$ where a is a positive constant. ❏

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4.4 Resisted motion under gravity: an example

Imagine a parachutist falling to Earth with velocity, $v_x(t)$ downwards, at time t. For convenience we will choose the positive x-axis to be the downward vertical, with the origin at the point where the parachute opens. The parachutist is subject to two forces, gravity acting downwards and the resistance of the air on his parachute acting upwards. Suppose that the resistive force is proportional to $[v_x(t)]^2$ with a constant of proportionality k (which has to be determined experimentally). It follows from *Newton's second law* that the equation governing the motion is

$$m\frac{d}{dt}[v_x(t)] = mg - k[v_x(t)]^2 \tag{17}$$

where m is the mass of the parachutist and g is the magnitude of the acceleration due to gravity. The forces on the right-hand side have opposite sign because they act in opposite directions.

Now, suppose you want to check that the following expression for $v_x(t)$ satisfies Equation 17

$$v_x(t) = \sqrt{\frac{mg}{k}}\left[\frac{1 + B\exp\left(-2t\sqrt{gk/m}\,\right)}{1 - B\exp\left(-2t\sqrt{gk/m}\,\right)}\right] \tag{18}$$

where B is a constant (determined by the speed of the parachutist at time $t = 0$). It appears that we must first differentiate $v_x(t)$, which is a rather complicated multiple of the quotient of two functions.

We could certainly proceed directly, and just differentiate the function as it stands, but a little thought should tell you that the algebra is going to get very nasty; so it is worth trying to simplify the expression. To simplify the notation let $A = \sqrt{mg/k}$ and $\alpha = 2\sqrt{gk/m}$, and let us write v_x rather than $v_x(t)$. So we have a simplified version of Equation 18

$$v_x(t) = \sqrt{\frac{mg}{k}}\left[\frac{1 + B\exp\left(-2t\sqrt{gk/m}\right)}{1 - B\exp\left(-2t\sqrt{gk/m}\right)}\right] \qquad \text{(Eqn 18)}$$

$$v_x = A\left[\frac{1 + B\exp(-\alpha t)}{1 - B\exp(-\alpha t)}\right] \qquad (19)$$

Now, using the quotient rule

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]=\frac{g(x)\dfrac{df}{dx}-f(x)\dfrac{dg}{dx}}{[g(x)]^2} \qquad \text{(Eqn 12)}$$

to differentiate this equation

$$v_x = A\left[\frac{1+B\exp(-\alpha t)}{1-B\exp(-\alpha t)}\right] \qquad \text{(Eqn 19)}$$

with respect to t, we find

$$\frac{dv_x}{dt}=A\left\{\frac{\left[\dfrac{d}{dt}(1+B\mathrm{e}^{-\alpha t})\right](1-B\mathrm{e}^{-\alpha t})-\left[\dfrac{d}{dt}(1-B\mathrm{e}^{-\alpha t})\right](1+B\mathrm{e}^{-\alpha t})}{(1-B\mathrm{e}^{-\alpha t})^2}\right\}$$

$$\frac{dv_x}{dt}=A\left[\frac{(-B\alpha\mathrm{e}^{-\alpha t})(1-B\mathrm{e}^{-\alpha t})-(B\alpha\mathrm{e}^{-\alpha t})(1+B\mathrm{e}^{-\alpha t})}{(1-B\mathrm{e}^{-\alpha t})^2}\right]$$

$$\frac{dv_x}{dt} = A\left[\frac{(-B\alpha e^{-\alpha t} + B^2\alpha e^{-2\alpha t}) - (B\alpha e^{-\alpha t} + B^2\alpha e^{-2\alpha t})}{(1 - Be^{-\alpha t})^2}\right]$$

$$\frac{dv_x}{dt} = A\left[\frac{-2B\alpha e^{-\alpha t}}{(1 - Be^{-\alpha t})^2}\right]$$

Substituting this expression for $dv_x(t)/dt$ into the left-hand side of Equation 17

$$m\frac{d}{dt}[v_x(t)] = mg - k[v_x(t)]^2 \qquad \text{(Eqn 17)}$$

and using the fact that $A\alpha = 2g$ we obtain

$$mA\left[\frac{-2B\alpha e^{-\alpha t}}{(1 - Be^{-\alpha t})^2}\right] = \frac{-4mgBe^{-\alpha t}}{(1 - Be^{-\alpha t})^2}$$

and substituting the same expression for $dv_x(t)/dt$

$$\frac{dv_x}{dt} = A\left[\frac{-2B\alpha e^{-\alpha t}}{(1 - Be^{-\alpha t})^2}\right]$$

into the right-hand side of Equation 17

$$m\frac{d}{dt}[v_x(t)] = mg - k[v_x(t)]^2 \qquad \text{(Eqn 17)}$$

and using the fact that $kA^2 = mg$ we obtain

$$mg - mg\left(\frac{1 + Be^{-\alpha t}}{1 - Be^{-\alpha t}}\right)^2 = mg\left[\frac{(1 - Be^{-\alpha t})^2 - (1 + Be^{-\alpha t})^2}{(1 - Be^{-\alpha t})^2}\right]$$

$$= \frac{-4mgBe^{-\alpha t}}{(1 - Be^{-\alpha t})^2}$$

Thus, the two sides of Equation 17

$$m\frac{d}{dt}[v_x(t)] = mg - k[v_x(t)]^2 \qquad \text{(Eqn 17)}$$

are indeed equal if v_x has the form given in Equation 18.

$$v_x(t) = \sqrt{\frac{mg}{k}}\left[\frac{1 + B\exp\left(-2t\sqrt{gk/m}\right)}{1 - B\exp\left(-2t\sqrt{gk/m}\right)}\right] \qquad \text{(Eqn 18)}$$

This really completes the differentiation and subsequent manipulation, but having introduced an expression (Equation 18) for the downward velocity under gravity in the presence of a resistive force it is worth noting at least one of its mathematical properties.

✦

What is the behaviour of $v_x(t)$ as t becomes large? In other words, what is $\lim\limits_{t\to\infty}\left[v_x(t)\right]$?

❖ What is the terminal velocity if $m = 80\,\text{kg}$, $k = 30\,\text{kg}\,\text{m}^{-1}$ and $g = 9.81\,\text{m}\,\text{s}^{-2}$?

Question T27

Show that $y(x) = \dfrac{\log_e 2 - x}{\log_e 2 + x}$ satisfies the equation

$$2y'(x)\log_e 2 + [1 + y(x)]^2$$ ❏

5 Closing items

5.1 Module summary

1. A function, f say, is a rule that assigns a single value of the dependent variable, y say, (in the codomain) to each value of the independent variable x in the domain of the function, such that $y = f(x)$.
2. The *rate of change*, or *derivative* of a function $f(x)$ at $x = a$, can be interpreted as the *gradient* of the graph of the function at the point $x = a$. Some standard derivatives are given in Table 1a (repeated here for reference).
3. The derivative is defined more formally in terms of a *limit*, and may be represented in a variety of ways. If $y = f(x)$

$$f'(x) = \frac{dy}{dx} = \lim_{\Delta x \to 0}\left(\frac{\Delta y}{\Delta x}\right) = \lim_{h \to 0}\left[\frac{f(x+h) - f(x)}{h}\right]$$

4 The *sum*, *constant multiple*, *product* and *quotient* rules for differentiating combinations of functions are as follows:

sum rule $(f+g)' = f' + g'$

constant multiple rule $(kf)' = kf'$

product rule $(fg)' = f'g + fg'$

quotient rule $\left(\dfrac{f}{g}\right)' = \dfrac{f'g - fg'}{g^2}$

5 The derivatives of the standard functions are given in Table 1.

6 The derivatives of a^x and $\log_a x$ are given by

$$\frac{d}{dx}(a^x) = (\log_e a)a^x \qquad \text{(Eqn 15)}$$

and
$$\frac{d}{dx}(\log_a x) = \frac{1}{x\log_e a} \qquad \text{(Eqn 16)}$$

7 You should be aware that logarithms to different bases are related by

$$\log_a b = (\log_a c)(\log_c b)$$

Table 1(a) Some standard derivatives.

$f(x)$	$f'(x)$
k (constant)	0
kx^n	nkx^{n-1}
$\sin kx$	$k\cos kx$
$\cos kx$	$-k\sin kx$
$\tan kx$	$k\sec^2 kx$
$\operatorname{cosec} kx$	$-k\operatorname{cosec} kx\cot kx$
$\sec kx$	$k\sec kx\tan kx$
$\cot kx$	$-k\operatorname{cosec}^2 kx$
$\exp(kx)$	$k\exp(kx)$
$\log_e(kx)$	$1/x$

5.2 Achievements

Having completed the module, you should be able to:

A1 Define the terms that are emboldened and flagged in the margins of the module.

A2 Define the derivative of a function (at a point) as a limit and give the definition a graphical interpretation.

A3 Use the definition of the derivative as a limit to calculate the derivative of simple functions.

A4 Know the derivatives of a range of standard functions.

A5 Use addition, subtraction, multiplication and division to express a given function in terms of 'simpler' ones in appropriate cases.

A6 Differentiate a constant multiple of a function.

A7 Apply the rule for differentiating the sum (and difference) of functions.

A8 Apply the product rule of differentiation.

A9 Apply the quotient rule of differentiation.

A10 Use the properties of logarithms and exponentials to determine the derivatives of $\log_a x$ and a^x.

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Study comment You may now wish to take the *Exit test* for this module which tests these Achievements. If you prefer to study the module further before taking this test then return to the *Module contents* to review some of the topics.

5.3 Exit test

Study comment Having completed this module, you should be able to answer the following questions, each of which tests one or more of the *Achievements*.

Question E1

(*A1*, *A2* and *A4*) Write down the derivative of $f(x) = \log_e x$. As x increases from zero how does the gradient of the tangent to the graph change? What is the behaviour of $\log_e x$ as x approaches infinity?

Question E2

(*A3*) Using the definition of the derivative as a limit, differentiate $f(x) = 1/x^2$.

Question E3

(*A4*, *A5* and *A6*) Find dy/dx for each of the following functions:

(a) $y = 2x(x - 1)(x + 2)$

(b) $y = \sin(x + \pi/4)$ (*Hint*: $\sin(A + B) = \sin A \cos B + \cos A \sin B$)

(c) $y = \log_e x^2$

Question E4

($A4$ to $A9$) Find $f'(x)$ for each of the following functions:

(a) $f(x) = x\cos x + x^2 \sin x$

(b) $f(x) = \left(x - \dfrac{1}{x}\right)^2 e^x$

(c) $f(x) = \dfrac{1 + x + x^2}{1 - x^2}$

(d) $f(x) = \dfrac{\sqrt{x}\sin x}{\log_e x}$

(e) $f(x) = \dfrac{\exp(x + \log_e x)}{\sqrt{x}}$

(f) $f(x) = a^x x^a$

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Question E5

(*A4*, *A5*, *A6*, *A8* and *A9*) If n is a positive integer, the series

$$1 + x + x^2 + x^3 + \dots + x^n \quad \text{has the sum} \quad \frac{x^{n+1} - 1}{x - 1}$$

show that the sum of the series

$$1 + 2x + 3x^2 + \dots + nx^{n-1} \quad \text{is} \quad \frac{nx^{n+1} - (n+1)x^n + 1}{(x-1)^2}$$

Question E6

(*A7* and *A8*) If f and g are functions show that

$$\left(\frac{1}{f(x)g(x)}\right)' = \left(\frac{1}{f(x)}\right)'\frac{1}{g(x)} + \frac{1}{f(x)}\left(\frac{1}{g(x)}\right)'$$

Study comment This is the final *Exit test* question. When you have completed the *Exit test* go back to Subsection 1.2 and try the *Fast track questions* if you have not already done so.

If you have completed **both** the *Fast track questions* and the *Exit test*, then you have finished the module and may leave it here.