Module M4.5 Taylor expansions and polynomial approximations

Exit module

1 Opening items

1.1 Module introduction

Many of the functions used in physics (including $\sin(x)$, $\tan(x)$ and $\log_e(x)$) are impossible to evaluate exactly for all values of their arguments. In this module we study a particular way of representing such functions by means of simpler *polynomial functions* of the form $a_0 + a_1x + a_2x^2 + a_2x^3 \dots$, where a_0, a_1, a_2, etc. are constants, the values of which depend on the function being represented. Sums of terms of this kind are generally referred to as *series*. The particular series we will be concerned with are known as *Taylor series*. We also show how, in certain circumstances, the first few terms of a Taylor series can give a useful approximation to the corresponding function.

For a physicist, the ability to find Taylor approximations is arguably the most useful skill in the whole ‘mathematical tool-kit’. It allows many complicated problems to be simplified and makes some mathematical models easier to solve and, perhaps more important, easier to understand.

Section 2 of this module begins by showing how some common functions can be approximated by *polynomials*. In each case the polynomial has the property that the values of its low-order derivatives, when evaluated at a particular point, are the same as those of the function which it approximates.

In Section 3 we show how this feature leads to a general technique for representing functions by means of series, and how taking the first few terms of such series can provide a useful approximation to the corresponding function. Subsections 3.1 and 3.2 are concerned with Taylor polynomials and series where the argument, x, of the corresponding function is close to zero. In Subsections 3.3 and 3.4 we consider polynomials and series for which x is not close to zero. Subsection 3.6 gives some useful tricks and short-cuts which can be used when finding Taylor polynomials and series. Subsection 3.5 lists some standard Taylor series, and Subsection 3.7 gives some applications of the ideas introduced in this module.

Study comment Having read the introduction you may feel that you are already familiar with the material covered by this module and that you do not need to study it. If so, try the *Fast track questions* given in Subsection 1.2. If not, proceed directly to *Ready to study?* in Subsection 1.3.

1.2 Fast track questions

Study comment Can you answer the following *Fast track questions*?. If you answer the questions successfully you need only glance through the module before looking at the *Module summary* (Subsection 4.1) and the *Achievements* listed in Subsection 4.2. If you are sure that you can meet each of these achievements, try the *Exit test* in Subsection 4.3. If you have difficulty with only one or two of the questions you should follow the guidance given in the answers and read the relevant parts of the module. However, *if you have difficulty with more than two of the Exit questions you are strongly advised to study the whole module*.

Question F1

Write down the general expression for the Taylor expansion of a function, $f(x)$, about the point $x = a$. Use this series to find the first five terms in the expansion of $\sqrt{x}$ about $x = 1$.

Question F2

Write down the Taylor expansions of $\exp(x)$ and $\sin(x)$ about $x = 0$, and hence find the first three non-zero terms in the Taylor expansion of $\exp(x^2)\sin(2x)$ about $x = 0$.

Study comment Having seen the *Fast track questions* you may feel that it would be wiser to follow the normal route through the module and to proceed directly to *Ready to study?* in Subsection 1.3.

Alternatively, you may still be sufficiently comfortable with the material covered by the module to proceed directly to the *Closing items*.

1.3 Ready to study?

Study comment To begin the study of this module you need to be familiar with the following terms: *approximation* (≈), *constant*, *factorial* (n!), *function*, *gradient* (of a *graph*), *modulus* (or *absolute value*, e.g. $|x|$), *power*, *radian*, *root*, *summation symbol* (∑), *tangent* (to a curve) and *variable*. You should also be familiar with the following topics: sketching graphs of elementary functions (such as $y = \sin(x)$); finding n^{th} order *derivatives* of simple functions; the properties (including derivatives and values at important points) of the elementary functions, such as the *trigonometric*, *exponential* and *logarithmic* functions, *simplifying*, *expanding* and *evaluating* algebraic expressions. Some familiarity with *infinite series* and the *convergence* of such series would also be useful, but this is not essential. If you are unfamiliar with any of these topics, you should consult the *Glossary*, which will indicate where in *FLAP* they are developed. The following *Ready to study questions* will help you to establish whether you need to review some of the above topics before embarking on this module.

Question R1

Find the value of the derivative of $1 + 2x + 3x^2$ at $x = 2$.

Question R2

What are the values of n! for $n = 0, 1, 2, 3, 4$?

Question R3

Simplify the following expression

$$\sum_{n=0}^{n=4} 2^n x^n - \sum_{n=0}^{n=3} x^n$$

and give your answer without using the summation symbol.

Question R4

Using primes to denote differentiation, find $f(1)$, $f'(1)$, $f''(1)$ and $f^{(3)}(1)$ for

$$f(x) = x^3 + x^4$$

Question R5

If $p(x)$ is defined by

$$p(x) = 1 + 3x + 2x^2$$

write down simplified expressions for $p(-x)$, $p(2x)$ and $p(1 + 2x)$.

Question R6

Given that $P(x) = 1 + x + x^2$ and $Q(x) = 2 - x + 3x^2$, expand and simplify $P(x)Q(x)$.

2 Polynomial approximations

2.1 Polynomials

Many different kinds of functions are used throughout mathematics and physics. Whereas some functions, such as

$$f(x) = x - \frac{x^3}{6} + \frac{x^5}{120} \tag{1}$$

$$g(x) = 1 + \frac{x}{2} - \frac{x^2}{8} \tag{2}$$

$$h(x) = 1 + x \tag{3}$$

are *polynomial functions*, others, such as

$$u(x) = \sin(x) \tag{4}$$

$$v(x) = \sqrt{1+x} \tag{5}$$

$$w(x) = \frac{1}{1-x} \tag{6}$$

are not.

A function $p(x)$ is a **polynomial function** of x (or in x) if it can be expressed in the form

$$p(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \dots + a_nx^n \tag{7}$$

in other words, as a sum of powers of x, with each power multiplied by a *coefficient* (a_0, a_1, and so on). The powers of x are non-negative integers.

While it is easy to see that expressions such as $1 + 2x - x^2 - x^3$ and $x^5 + (x - 1)^4$ are polynomials in x, it is not always so easy. For example,

$$(e^x - e^{-x})^2 - (e^x + e^{-x})^2 + (x\sin x)^2 + (x\cos x)^2$$

is actually a polynomial in x (in fact $-4 + x^2$), but it may take a few moments thought to see why this is so.

❖ Which of the following are polynomials in x?

(a) $1 + x + (x - 1)^2$, (b) $1 + e^x + e^{2x}$, (c) $\dfrac{1}{1 + x + x^2}$, (d) $1 + \dfrac{1}{x} + \dfrac{1}{x^2} + \dfrac{1}{x^3}$.

The highest power of the variable in a polynomial is known as the **degree** of the polynomial. For example, in the above expression for $p(x)$ (Equation 7),

$$p(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \dots + a_nx^n \qquad \text{(Eqn 7)}$$

the degree is n (assuming that a_n is non-zero).

Question T1

What are the degrees of the polynomial functions $f(x)$, $g(x)$ and $h(x)$, defined in Equations 1, 2 and 3?

$$f(x) = x - \frac{x^3}{6} + \frac{x^5}{120} \qquad \text{(Eqn 1)}$$

$$g(x) = 1 + \frac{x}{2} - \frac{x^2}{8} \qquad \text{(Eqn 2)}$$

$$h(x) = 1 + x \qquad \text{(Eqn 3)}$$

❏

Low-order polynomials are often given special names. Polynomials of degree zero are simply constants; polynomials of degree one are called **linear**; those of degree two are called **quadratic** and those of degree three are called **cubic**. ☞

An important property of any polynomial, $p(x)$, is that for any value of x, the value of $p(x)$ can be calculated by simply using the arithmetic operations of addition (or subtraction) and multiplication. This is not the case with many other functions (for example, $u(x)$, $v(x)$ and $w(x)$, defined in Equations 4, 5 and 6)

$$u(x) = \sin(x) \tag{Eqn 4}$$

$$v(x) = \sqrt{1+x} \tag{Eqn 5}$$

$$w(x) = \frac{1}{1-x} \tag{Eqn 6}$$

and it can be very useful to know polynomial approximations to such functions.

Question T2

Plot the functions $h(x)$ and $w(x)$ (defined in Equations 3 and 6)

$$h(x) = 1 + x \tag{Eqn 3}$$

$$w(x) = \frac{1}{1 - x} \tag{Eqn 6}$$

on the same graph for $0 \le x \le 0.9$. Compare the two curves. (*Hint*: You do not need to plot these functions 'by hand'. If you have access to a graph plotting calculator or computer program, use it!) ❏

The exercise that you have just completed shows that, for 'small' values of x, the polynomial function $h(x)$ is approximately the same as $w(x)$, so here we have an example of a *polynomial approximation* to the function $1/(1-x)$.

However, we can see from Figure 9 that the approximation becomes progressively worse as x increases towards the value 1.

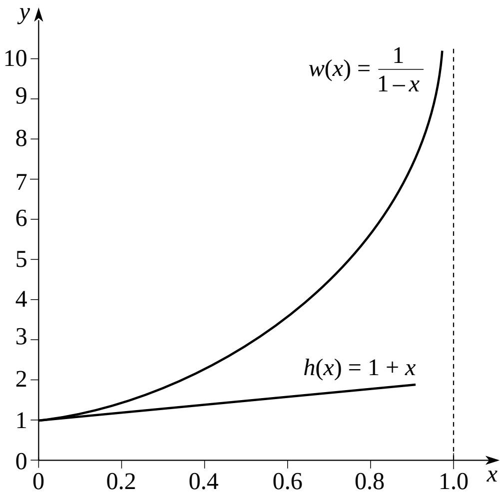

Figure 9 See Answer T2.

In Figure 1 we have plotted the graphs of $y = \sin(x)$ and $y = x$ on the same axes. As you can see, the graphs of the two functions are very similar for small values of $|x|$, ☞ but as $|x|$ increases, the discrepancy gets progressively worse, so that for $|x|$ above about 0.7 it is very noticeable, and above $\pi/2$ (i.e. approximately 1.57) the two graphs show no similarity at all.

So there may be circumstances where we would be justified in approximating $\sin(x)$ by x, but such an approximation is only likely to be useful for small values of $|x|$.

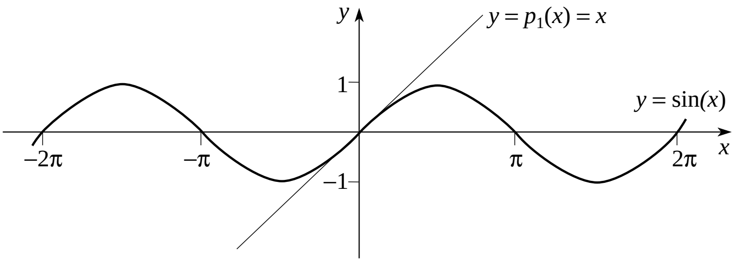

Figure 1 Graphs of the functions $y = x$ and $y = \sin(x)$. Note that throughout this module the argument of any trigonometric function is either a dimensionless variable or an angle in radians rather than degrees.

For example, an analysis of the motion of a simple pendulum gives the (apparently intractable) equation

$$\frac{d^2}{dt^2}[\theta(t)] = -\omega^2 \sin[\theta(t)] \tag{8}$$

where $\theta(t)$ is the angle shown in Figure 2 and ω is a constant (called the angular frequency). For small deviations of the pendulum from the vertical, $\theta(t)$ is small, and we are justified in making the approximation $\sin(\theta) \approx \theta$, giving

$$\frac{d^2}{dt^2}[\theta(t)] \approx -\omega^2 \theta(t)$$

and the equation $\dfrac{d^2}{dt^2}[\theta(t)] = -\omega^2 \theta(t)$ is straightforward to solve.

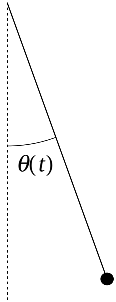

Figure 2 A simple pendulum.

Question T3

Show that $\theta(t) = \sin(\omega t)$ is a solution of

$$\frac{d^2}{dt^2}[\theta(t)] = -\omega^2 \theta(t) \qquad (9)$$

but not of

$$\frac{d^2}{dt^2}[\theta(t)] = -\omega^2 \sin[\theta(t)] \qquad \text{(Eqn 8)}$$ ❏

2.2 Increasingly accurate approximations for sin (x)

If we put $\qquad p_1(x) = x \qquad\qquad (10)$

then our previous discussion amounts to saying that $p_1(x)$ is an acceptable approximation for $\sin(x)$ provided that $|x|$ is small.

However, for many applications, particularly those involving 'large' values of x, approximating $\sin(x)$ by $p_1(x)$ is not at all satisfactory.

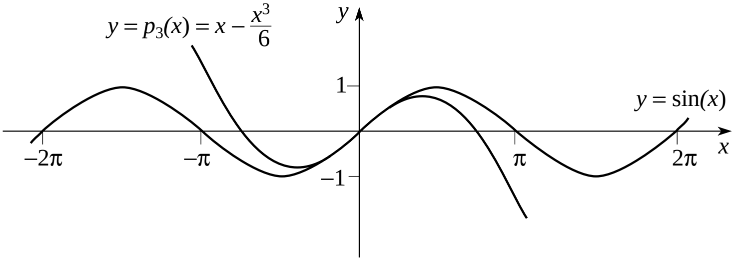

Figure 3 Graphs of the functions $y = x - (x^3/6)$ and $y = \sin(x)$.

Fortunately there are better approximations. For example, in Figure 3 we plot the polynomial

$$p_3(x) = x - \frac{x^3}{6} \qquad (11)$$

together with the $\sin(x)$ function.

If you compare Figure 3 with Figure 1, you can see that this polynomial is a much better approximation to the sine function than the linear polynomial and is particularly good for $|x|$ less than about 1.5.

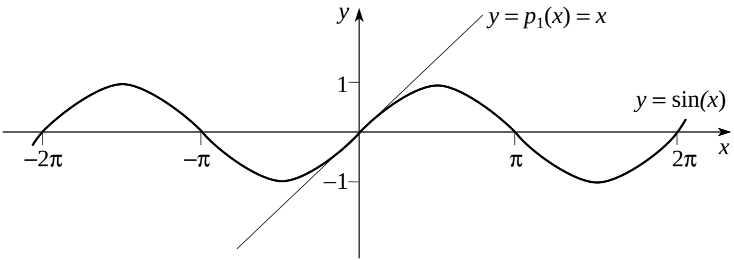

Figure 1 Graphs of the functions $y = x$ and $y = \sin(x)$. Note that throughout this module the argument of any trigonometric function is either a dimensionless variable or an angle in radians rather than degrees.

✦ How is this new approximation $p_3(x)$ related to the previous approximation $p_1(x)$?

You may be wondering how we were able to choose a polynomial $p_3(x)$ that works so well.

The answer lies in a comparison of the derivatives of $\sin(x)$ and $p_3(x)$ at $x = 0$, as follows, where for convenience we have written $f(x) = \sin(x)$, and we use primes ($'$) to indicate differentiation.

For convenience we also introduce the notation $[\quad]_{x=0}$ to indicate that the function inside the brackets should be evaluated at the value outside.

The function $f(x) = \sin(x)$, at $x = 0$ ☞	The approximating polynomial $p_3(x) = x - (x^3/6)$ at $x = 0$
$f(0) = \sin(0) = 0$	$p_3(0) = \left[x - \dfrac{x^3}{6}\right]_{x=0} = 0$
$f'(0) = \cos(0) = 1$	$p_3'(0) = \left[1 - \dfrac{x^2}{2}\right]_{x=0} = 1$
$f''(0) = -\sin(0) = 0$	$p_3''(0) = [-x]_{x=0} = 0$
$f^{(3)}(0) = -\cos(0) = -1$	$p_3^{(3)}(0) = -1$

As you can see, the values of the functions $\sin(x)$ and $p_3(x)$, and their first three derivatives, are precisely the same when evaluated at $x = 0$.

By using higher-degree polynomials it is possible to obtain increasingly accurate approximations to $\sin(x)$. For example,

$$p_5(x) = x - \frac{x^3}{6} + \frac{x^5}{120} \tag{12}$$

is a polynomial approximation of degree five to $\sin(x)$, which gives very accurate results near $x = 0$.

Shortly we will show you how we found $p_5(x)$, but first, to see how good this new approximation really is, try the following question.

Question T4

Find $p_5(0.2)$ as accurately as you can, and compare your answer to that obtained for $\sin(0.2)$ on a calculator. ❏

The polynomial $p_5(x)$ was chosen so that its value, and the values of its first five derivatives, at $x = 0$, are the same as for $\sin(x)$. ☞ We can see that this is the case by performing a calculation similar to that which we performed for $p_3(x)$.

The function $f(x) = \sin(x)$ at $x = 0$	The approximating polynomial (Equation 12) at $x = 0$
$f(0) = \sin(0) = 0$	$p_5(0) = \left[x - \dfrac{x^3}{6} + \dfrac{x^5}{120}\right]_{x=0} = 0$
$f'(0) = \cos(0) = 1$	$p_5'(0) = \left[1 - \dfrac{x^2}{2} + \dfrac{x^4}{24}\right]_{x=0} = 1$
$f''(0) = -\sin(0) = 0$	$p_5''(0) = \left[-x + \dfrac{x^3}{6}\right]_{x=0} = 0$
$f^{(3)}(0) = -\cos(0) = -1$	$p_5^{(3)}(0) = \left[-1 + \dfrac{x^2}{2}\right]_{x=0} = -1$
$f^{(4)}(0) = \sin(0) = 0$	$p_5^{(4)}(0) = [x]_{x=0} = 0$
$f^{(5)}(0) = \cos(0) = 1$	$p_5^{(5)}(0) = 1$

❖ How is this new approximation, $p_5(x)$, for $\sin(x)$ related to the previous approximation $p_3(x)$?

The previous discussion suggests that we may be able to construct a sequence of increasingly accurate approximations to $\sin(x)$

$$p_1(x) = x, \quad p_3(x) = x - \frac{x^3}{6}, \quad p_5(x) = x - \frac{x^3}{6} + \frac{x^5}{120}, \text{ and so on}$$

which are obtained by adding higher and higher powers of x.

So far we have verified that these polynomials and their derivatives behave very like $\sin(x)$ at $x = 0$, but we have not shown you how to construct them. We will remedy this deficiency shortly, but first let us try a similar idea on a function other than $\sin(x)$.

2.3 Increasingly accurate approximations for exp (x)

Since $\exp(0) = e^0 = 1$, it is clear that $\exp(x)$ ☞ is approximately equal to 1 when x is close to 0 but perhaps we can find a better approximation.

We could try a polynomial which is linear in x, that is

$$q_1(x) = a_0 + a_1 x \tag{13}$$

and look for numbers, a_0 and a_1, such that the low-order derivatives of $\exp(x)$ and $q_1(x)$ are identical.

Remembering that $\dfrac{d}{dx}[\exp(x)] = \exp(x)$, and writing $g(x) = \exp(x)$ for convenience, we can construct the following table:

The function $g(x) = \exp(x)$, at $x = 0$	The approximating polynomial (Equation 13) at $x = 0$
$g(0) = \exp(0) = 1$	$q_1(0) = \left[a_0 + a_1x\right]_{x=0} = a_0$
$g'(0) = \exp(0) = 1$	$q_1'(0) = \left[a_1\right]_{x=0} = a_1$

So if we choose $a_0 = 1$ and $a_1 = 1$, the approximating polynomial becomes

$$q_1(x) = 1 + x$$

and its value, and the value of its first derivative are, respectively, identical to the values of $\exp(x)$ and its first derivative at $x = 0$.

❖ Evaluate $q_1(0.1)$ and $\exp(0.1)$. What is the gradient of $y = \exp(x)$ at $x = 0$? What is the equation of the tangent to the graph of $y = \exp(x)$ at $x = 0$?

Now let us try to find a better approximation by considering a quadratic polynomial

$$q_2(x) = a_0 + a_1x + a_2x^2$$

and look for numbers, a_0, a_1, a_2, such that the low-order derivatives of $\exp(x)$ and $q_2(x)$ are identical. Constructing a table of values as before, we have:

The function $g(x) = \exp(x)$, at $x = 0$	The approximating polynomial $q_2(x) = a_0 + a_1x + a_2x^2$ at $x = 0$
$g(0) = \exp(0) = 1$	$q_2(0) = \left[a_0 + a_1x + a_2x^2\right]_{x=0} = a_0$
$g'(0) = \exp(0) = 1$	$q_2'(0) = \left[a_1 + 2a_2x\right]_{x=0} = a_1$
$g''(0) = \exp(0) = 1$	$q_2''(0) = \left[2a_2\right]_{x=0} = 2a_2$

So if we match the functions and their derivatives at $x = 0$ by choosing $a_0 = 1$, $a_1 = 1$ and $2a_2 = 1$ the approximating polynomial becomes

$$q_2(x) = 1 + x + \frac{x^2}{2} \tag{14}$$

✧

$$q_2(x) = 1 + x + \frac{x^2}{2} \qquad \text{(Eqn 14)}$$

Compare $q_2(0.1)$ with $q_1(0.1)$ and $\exp(0.1)$.

Once again we have obtained a better approximation by adding a term of higher degree, $x^2/2$ in this case, while the constant term and the coefficient of x remain unchanged. As you might imagine, we can continue this process by considering the cubic polynomial

$$q_3(x) = a_0 + a_1x + a_2x^2 + a_3x^3 \qquad (15)$$

and arranging that the low-order derivatives of $\exp(x)$ and $q_3(x)$ are identical.

✧ Calculate a_3 and hence find $q_3(x)$.

✧ Compare $q_3(0.1)$ with $q_1(0.1)$, $q_2(0.1)$ and $\exp(0.1)$.

Question T5

Use your calculator to obtain $\exp(x)$ and $q_3(x)$ (to three decimal places) for $x = 0$, 0.1, 0.5, 1.0, 2, 10. What conclusions can you draw from comparing the two sets of results? ❏

Some conclusions

So it appears that the polynomials

$$1,\quad 1+x,\quad 1+x+\frac{x}{2},\quad 1+x+\frac{x^2}{2}+\frac{x^3}{6},\text{ and so on}$$

provide a sequence of increasingly accurate approximations to $\exp(x)$ for small values of x, and in particular for $\exp(0.1)$. In each case a better approximation is obtained by adding a term with a higher power of x, and choosing the coefficient so that the low-order derivatives of the polynomial and $\exp(x)$ are identical at $x = 0$.

Sketching $\exp(x)$ and these polynomials for a range of values of x, as in Figure 4, suggests that the polynomials also provide increasingly accurate approximations for other values of x. Our aim in the next section is to generalize this technique for finding polynomial approximations.

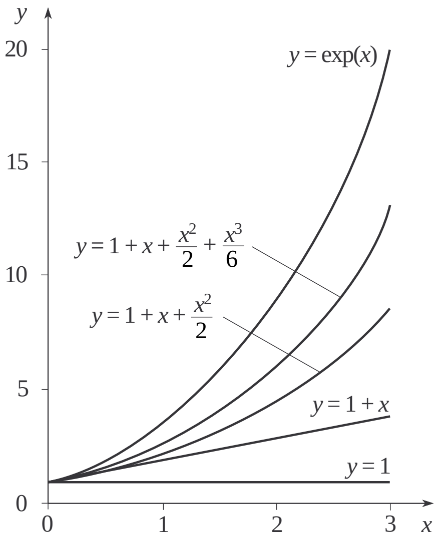

Figure 4 Polynomial approximations to $\exp(x)$.

3 Finding polynomial approximations by Taylor expansions ☞

3.1 Taylor polynomials (near $x = 0$)

So far, we have considered polynomial approximations to the sine and exponential functions, but in this subsection we intend to apply a similar method to approximate the general function $f(x)$ near to $x = 0$ by a polynomial

$$p_n(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \dots + a_nx^n \qquad (17)$$

where $n \ge 0$ indicates the *degree* of the approximating polynomial.

The purpose of this subsection is to show you how to choose the *coefficients* (i.e. the numbers a_0, a_1, a_2 and so on); but first try the following preliminary exercises.

Preliminary exercises

✦ Given that $p(x) = 1 + 3x + 5x^2 + 7x^3 + 9x^4$ write down expressions for $p'(x)$, $p''(x)$ and $p''(0)$.

✦ Elsewhere in this subsection $p_n(x)$ refers to the general polynomial defined by Equation 17,

$$p_n(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \dots + a_nx^n \qquad \text{(Eqn 17)}$$

but suppose for the purposes of this question that $p_n(x)$ is the particular polynomial defined by

$$p_n(x) = 1 + \frac{x}{1.1} + \frac{x^2}{1.2} + \frac{x^3}{1.3} + \dots + \frac{x^n}{(1 + n \times 0.1)}$$

(a) what are the coefficients a_0, a_3, a_{10} and a_n?

(b) Write down expressions for $p_n'(x)$ and $p_n''(x)$.

(c) What is the value of $p_n''(0)$?

✦ Given that $p_n(x)$ is a general polynomial defined by Equation 17, write down expressions for $p_n'(x)$ and $p_n''(x)$. What is $p_n''(0)$?

Look carefully at the following pattern, and make sure that you understand what happens to the terms as you continue to differentiate:

$$
\begin{array}{llllll}
p_n(x) = a_0 + & a_1x + & a_2x^2 + & a_3x^3 & + \dots + & a_nx^n \\
\text{differentiate} & \Downarrow & \Downarrow & \Downarrow & & \Downarrow \\
p_n'(x) = & a_1 + & 2a_2x + & 3a_3x^2 & + \dots + & na_nx^{n-1} \\
\text{differentiate} & & \Downarrow & \Downarrow & & \Downarrow \\
p_n''(x) = & & 2a_2 + & 3\times 2a_3x & + \dots + & a_nn(n-1)x^{n-2}
\end{array}
$$

The general approximation

When approximating a general function $f(x)$ by the polynomial in Equation 17,

$$p_n(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \dots + a_nx^n \qquad \text{(Eqn 17)}$$

we look for values of the coefficients, a_0, a_1, … a_n, for which the derivatives of $f(x)$ (up to the n^{th}) are the same as those of $p_n(x)$ (as we did in the previous section).

$$p_n(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \dots + a_nx^n \qquad \text{(Eqn 17)}$$

Proceeding one derivative at a time, we first calculate $p_n^{(n)}(0)$ as follows ☞

$$\begin{aligned}
p_n(0) &= [a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 \dots + a_nx^n]_{x=0} &&= a_0\\
p_n'(0) &= [a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \dots + na_nx^{n-1}]_{x=0} &&= a_1\\
p_n''(0) &= [2a_2 + 3\times 2a_3\,x + 4\times 3a_4x^2 + \dots + a_nn(n-1)x^{n-2}]_{x=0} &&= 2a_2\\
p_n^{(3)}(0) &= [(3\times 2)a_3 + (4\times 3\times 2)a_4\,x + \dots + a_nn(n-1)(n-2)x^{n-3}]_{x=0} &&= 3\times 2a_3\\
p_n^{(4)}(0) &= [(4\times 3\times 2)a_4 + \dots + a_nn(n-1)(n-2)(n-3)x^{n-4}]_{x=0} &&= 4\times 3\times 2a_4\\
&\vdots && \vdots\\
p_n^{(n)}(0) &= [a_nn(n-1)(n-2)(n-3)\dots 2]_{x=0} &&= n!\,a_n
\end{aligned}$$

☞

So in general, for the m^{th} derivative (where $0 \le m \le n$) we have

$$p_n^{(m)}(0) = m(m-1)(m-2)(m-3)\dots 2\times 1a_m = m!\,a_m$$

We must not forget the purpose of all this algebra, which is to ensure that the m^{th} derivatives of $f(x)$ and $p_n(x)$ (where $0 \le m \le n$) should be identical at $x = 0$. Thus we require

$$f^{(m)}(0) = m!\, a_m$$

So, the coefficients in the polynomial

$$p_n(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \dots + a_nx^n$$

are given by

$$a_m = \frac{f^{(m)}(0)}{m!} \tag{18}$$

or, if you prefer to write the formulae out in full they are given in the margin. ☞

We can now write down a general result for the n^{th} degree polynomial which approximates a function, $f(x)$ say near the value $x = 0$.

The following expression is known as the **Taylor polynomial** of degree n for $f(x)$ near $x = 0$:

$$p_n(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \ldots + \frac{f^{(n)}(0)}{n!}x^n \qquad (19)$$

☞

An application of the general formula to $\exp(x)$ and $(1 - x)^{-1}$

As an example, let us find the Taylor polynomial of degree n for $\exp(x)$ near $x = 0$. In this case, $f(x) = \exp(x)$ in the general expression, Equation 19.

$$p_n(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n \qquad \text{(Eqn 19)}$$

But the n^{th} derivative of $\exp(x)$ is $\exp(x)$ and, since $\exp(0) = 1$, we have $f^{(n)}(0) = 1$ for all integers n. Consequently we find

$$\exp(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots + \frac{x^n}{n!} \qquad (20)$$

This is a generalization of the result obtained in Subsection 2.3 (Equation 16).

$$q_3(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} \qquad \text{(Eqn 16)}$$

Question T6

Show that if $f(x) = (1 - x)^{-1}$ then

$$\frac{d^n}{dx^n}f(x) = \frac{n!}{(1-x)^{n+1}}$$

and use this result to find the Taylor polynomial of degree n for $f(x)$ near $x = 0$. ❏

3.2 Taylor series (expanding about zero)

Our intention is to replace a function such as $\exp(x)$ by a simpler function, a polynomial, while maintaining an acceptable degree of accuracy; but look again at the results in Question T5, in which you were asked to compare the cubic Taylor polynomial for $\exp(x)$ with the values for the exponential function obtained on your calculator.

For $x = 2$, the polynomial gave the value $q_3(2) = 6.333$ whereas the calculator gave the value $\exp(2) = 7.389$.

Thus replacing $\exp(2)$ by $q_3(2)$ would give rise to an unacceptable discrepancy for many physical applications.

Table 2 See Answer T5.

x	$\exp(x)$	$q_3(x)$
0	1.0	1.0
0.1	1.105	1.105
0.5	1.649	1.646
1.0	2.718	2.667
2.0	7.389	6.333
10.0	22 026.466	227.667

We might imagine that we could obtain a more accurate result by using a Taylor polynomial with more terms, and this is indeed the case, as Table 1 shows.

Table 1

The degree n of the polynomial $q_n(x)$ that approximates $\exp(x)$	Result of evaluating the approximating polynomial $q_n(x)$ at $x = 2$
0	1
1	3
2	5
3	6.333
4	7
5	7.267
6	7.356

For any function, $f(x)$, we could let the number of terms in the corresponding Taylor polynomial become larger and larger in order to get an increasingly accurate result. We could even go one stage further and consider the limit as the number of terms tends to infinity, in which case we find

the **Taylor series** or **Taylor expansion** for $f(x)$ near $x = 0$

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots \qquad (21)$$

❖ In what way does the Taylor series given in Equation 21 differ from the Taylor polynomial of degree n given in Equation 19?

The series expansion of Equation 21 can be written more compactly using the summation symbol, as follows

$$f(x) = \sum_{n=0}^{\infty} f^{(n)}(0)\frac{x^n}{n!} \qquad (22)$$

Study comment The expression *Taylor expansion* is often used in place of *Taylor series*. The terms Taylor expansion of a function *about* or *at* $x = 0$ rather than *near to* $x = 0$ are all in common usage.

As an example of a Taylor series, we can consider the expansion of $\exp(x)$ about $x = 0$. This is a particularly simple function to expand since each of its derivatives is equal to the exponential function and $\exp(0) = 1$. We therefore have

$$f^{(0)}(0) = \left[\frac{d^n}{dx^n}\exp(x)\right]_{x=0} = [\exp(x)]_{x=0} = 1 \quad \text{for all values of } n$$

and putting this in the general formula for a Taylor series (Equation 21 or 22)

$$f(x) = \sum_{n=0}^{\infty} f^{(n)}(0)\frac{x^n}{n!} \qquad \text{(Eqn 22)}$$

gives

$$\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad (23)$$

Study comment For any specified value of x, this series gives increasingly accurate results for $\exp(x)$ as the number of terms in the series is increased. In such a case we say that the series *converges* to $\exp(x)$ for all values of x. Further information on the convergence of infinite series can be obtained through the *Glossary*, but in this module we are more concerned with the methods of obtaining the desired series than with their convergence.

Question T7

Show that the Taylor expansion

$$f(x) = \sum_{n=0}^{\infty} f^{(n)}(0)\frac{x^n}{n!} \qquad \text{(Eqn 22)}$$

of $(1 + x)^r$ near $x = 0$ is given by

$$(1+x)^r = 1 + rx + \frac{r(r-1)}{2!}x^2 + \frac{r(r-1)(r-2)}{3!}x^3 + \dots \qquad (24)$$

where r is any real number. ❏

The approximate error

If we ignore the higher terms in the Taylor series, so that we approximate the given function by a Taylor polynomial, then we commonly use an approximately equal symbol rather than equality, so that, for example, we may write

$$\exp(x) \approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$$

If we use a Taylor polynomial of degree n to approximate a given function then we are clearly ignoring the rest of the infinite series. But how accurate is such a polynomial approximation? Can we estimate the size of the error?

An accurate estimate of the error involved in such approximations is beyond the scope of *FLAP*, but a useful rule of thumb is that if terms involving x^{n+1} and higher powers of x are ignored, then the error is of the same order of magnitude as the first non-zero term that has been ignored. ☞

For example, if we calculate the third-order Taylor polynomial for $\exp(0.5)$, namely

$$[\exp(x)]_{x=0.5} \approx \left[1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}\right]_{x=0.5}$$

$$= 1 + (0.5) + \frac{(0.5)^2}{2!} + \frac{(0.5)^3}{3!} \approx 1.6458$$

then we would expect the approximate error to be given by

$$\left[\frac{x^4}{4!}\right]_{x=0.5} = 0.0026$$

which is about 0.16%. The value for $\exp(0.5)$ given by my calculator is approximately 1.6487, corresponding to an error in the Taylor polynomial approximation of about 0.003 which is about 0.17%. Of course, the reason for the inaccuracy in the estimation of the error is that we are neglecting all the other terms in the series, but such approximations are usually acceptable, and particularly so if we are interested in small values of x.

Question T8

Use the third-order Taylor polynomial for $\sin(x)$ near $x = 0$ to obtain an approximation to $\sin(\pi/4)$. Derive an estimate for the percentage error in your result. ❏

3.3 Taylor polynomials (near $x = a$)

If you enter $\sin(\pi)$ on your calculator you should find that you get zero, or something very close. (Try it! ☞)
Now look at the Taylor polynomial of degree 5 for $\sin(x)$ near $x = 0$

$$\sin(x) \approx x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

If we substitute $x = \pi$, we obtain

$$\sin(\pi) \approx \pi - \frac{\pi^3}{3!} + \frac{\pi^5}{5!} \approx 0.524$$

Although the infinite Taylor *series* for $\sin(\pi)$ sums to zero, we can see that the above *polynomial* does not provide anything like a reasonable approximation. The value $x = \pi$ is simply too far from the value $x = 0$ for the Taylor polynomial of degree 5 to provide an adequate approximation to the full Taylor expansion of $\sin(x)$ about $x = 0$. So, how can we find a simple polynomial representation of $\sin(x)$ that will work at $x = \pi$? The resolution of this difficulty is to find a Taylor expansion which is valid in the vicinity of a point other than zero, we can then use as many terms as we want from that expansion to provide the required polynomial approximation near $x = \pi$.

More generally, suppose that we want to approximate a function, $f(x)$, near to $x = a$ by a polynomial of order n, where $n \geq 0$. As in Subsection 3.1, we want to look for values of the coefficients for which the derivatives of $f(x)$ are the same as those of the polynomial.

Previously we calculated the derivatives at $x = 0$, but now we calculate the derivatives at $x = a$.

For this reason we consider the polynomial

$$p_n(x) = a_0 + a_1(x - a) + a_2(x - a)^2 + a_3(x - a)^3 + \dots + a_n(x - a)^n \qquad (25)$$

Such a series is sometimes known as a **power series**, or a series of powers of $(x - a)$. In order to pick out the m^{th} term in this expression we differentiate m times with respect to x and then set $x = a$. To see how this works, let us look at a specific example. Suppose that we wish to approximate $\sin(x)$ near $x = \pi/4$.

An approximation for sin (*x*) near $x = \pi/4$

We will attempt to find an approximating polynomial of degree 3 so that

$$\sin(x) \approx p_3(x) = a_0 + a_1(x - \pi/4) + a_2(x - \pi/4)^2 + a_3(x - \pi/4)^3 \tag{26}$$

First we put $x = \pi/4$ in Equation 26 so that $\boxed{p_3(\pi/4) = a_0}$

Then we differentiate Equation 26 to obtain

$$p_3'(x) = a_1 + 2a_2(x - \pi/4) + 3a_3(x - \pi/4)^2 \tag{27}$$

and put $x = \pi/4$ in Equation 27 to give $\boxed{p_3'(\pi/4) = a_1}$

Now we differentiate Equation 27 to obtain

$$p_3''(x) = 2a_2 + 6a_3(x - \pi/4) \tag{28}$$

and put $x = \pi/4$ in Equation 28 to give $\boxed{p_3''(\pi/4) = 2a_2}$

Finally we differentiate Equation 28 to obtain

$$p_3^{(3)}(x) = 6a_3 \tag{29}$$

and put $x = \pi/4$ in Equation 29 to give $\boxed{p_3^{(3)}(\pi/4) = 6a_3}$

All that remains is to ensure that the value of the polynomial $p_3(x)$ and its first three derivatives coincide with those of $\sin(x)$ at $x = \pi/4$. The relevant values of $\sin(x)$ and its derivatives are:

$$\left[\sin(x)\right]_{x=\pi/4} = 1\big/\sqrt{2}$$

$$\left[\frac{d}{dx}\sin(x)\right]_{x=\pi/4} = \left[\cos(x)\right]_{x=\pi/4} = 1\big/\sqrt{2}$$

$$\left[\frac{d^2}{dx^2}\sin(x)\right]_{x=\pi/4} = \left[-\sin(x)\right]_{x=\pi/4} = -1\big/\sqrt{2}$$

$$\left[\frac{d^3}{dx^3}\sin(x)\right]_{x=\pi/4} = \left[-\cos(x)\right]_{x=\pi/4} = -1\big/\sqrt{2}$$

and the values on the right must be made to coincide with the boxed values above. This gives

$$a_0 = 1\big/\sqrt{2}, \quad a_1 = 1\big/\sqrt{2}, \quad 2a_2 = -1\big/\sqrt{2} \quad \text{and} \quad 6a_3 = -1\big/\sqrt{2}$$

from which we see that $a_2 = -\dfrac{1}{2\sqrt{2}}$ and $a_3 = -\dfrac{1}{6\sqrt{2}}$

and the required approximating polynomial is

$$p_3(x) = \frac{1}{\sqrt{2}}\left[1 + (x - \pi/4) - \frac{(x - \pi/4)^2}{2} - \frac{(x - \pi/4)^3}{6}\right] \tag{30}$$

✦ The value 0.8 is quite close to $\pi/4 \approx 0.7854$. Use the polynomial $p_3(x)$ defined in Equation 30 to find an approximation for $\sin(0.8)$, and compare this value with the value for $\sin(0.8)$ obtained on a calculator.

The argument can be generalized to polynomials of any degree if we use the result of the following question.

Question T9

Show that
$$\frac{d^m}{dx^m}(x-a)^n = \begin{cases} n(n-1)(n-2)\ \dots\ [n-(m-1)](x-a)^{n-m} & \text{if } m < n \\ n! & \text{if } m = n \\ 0 & \text{if } m > n \end{cases}$$

where $n \geq 0$. ❑

If we differentiate the polynomial

$$p_n(x) = a_0 + a_1(x-a) + a_2(x-a)^2 + a_3(x-a)^3 + \dots + a_n(x-a)^n \qquad \text{(Eqn 25)}$$

m times (where m lies between 0 and n) and then set $x = a$, only the term involving a_m can give a non-zero result. To see this, consider a typical term $a_k(x-a)^k$. If k is too small ($k < m$) then $(x-a)^k$ is eliminated by differentiation. On the other hand, if k is too big ($k > m$) then differentiation leaves a factor of $(x-a)$ raised to some power, but this gives zero on setting $x = a$. Consequently, the general result is exactly as our previous examples suggest:

$$p_n^{(m)}(a) = \left[\frac{d^m}{dx^m}p_n(x)\right]_{x=a} = m!a_m$$

Don't forget that we are trying to estimate some general function, $f(x)$ say, near $x = a$, and the requirement that the m^{th} derivatives of $f(x)$ and $p_n(x)$ (where $0 \le m \le n$) should be identical at $x = a$ leads to

$$f^{(m)}(a) = \left[\frac{d^m}{dx^m} f(x)\right]_{x=a} = m!\,a_m$$

and therefore the coefficients in the polynomial, $p_n(x)$, are given by

$$a_m = \frac{f^{(m)}(a)}{m!}$$

We can now write down the Taylor polynomial of degree n for $f(x)$ near to $x = a$.

The **Taylor polynomial of degree n for $f(x)$ near $x = a$** is

$$p_n(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \dots$$
$$\dots + \frac{f^{(n)}(a)}{n!}(x-a)^n \tag{31}$$

which can be written more compactly using the summation symbol, as follows

$$p_n(x) = \sum_{m=0}^{n} \frac{f^{(m)}(a)(x-a)^m}{m!} \tag{32}$$

This formula makes the calculation of Taylor approximations very straightforward.

✦

$$p_n(x) = \sum_{m=0}^{n} \frac{f^{(m)}(a)(x-a)^m}{m!} \qquad \text{(Eqn 32)}$$

Find the Taylor polynomial of degree 3 for sin (x) near to $x = \pi$, then estimate the value of sin (3), and compare your estimate with the value given on a calculator.

Question T10

Use Equation 32 to find:

(a) the Taylor polynomial of degree three for $\log_e(x)$ near to $x = 1$;

(b) the Taylor polynomial of degree three for exp $(2x)$ near to $x = 0$. ❏

3.4 Taylor series about a general point

We know that the Taylor polynomial of order n for $f(x)$ near $x = a$ is

$$p_n(x) = \sum_{m=0}^{n} \frac{f^{(m)}(a)(x-a)^m}{m!} \qquad \text{(Eqn 32)}$$

In general the accuracy of the approximation improves with the degree of the approximating polynomial, and in the limit as n tends to infinity we find the Taylor series (or Taylor expansion) for $f(x)$ near to $x = a$. ☞

The **Taylor series (expansion) for *f*(*x*) near *x* = *a*** is

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \dots$$
$$+ \frac{f^{(n)}(a)}{n!}(x-a)^n + \dots$$

which can be written more compactly using the summation symbol as

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)(x-a)^n}{n!} \tag{34}$$

✦ What is the distinction between approximating a function near $x = a$ by a Taylor polynomial of degree 3 say, and finding the Taylor series for the given function near $x = a$?

As an example, suppose we require the Taylor series for sin (x) near to $x = \pi$. In this case we first need to find *all* derivatives of sin (x) evaluated at $x = \pi$.

Question T11

Extend the calculations which gave rise to Equation 33

$$\sin(x) \approx -(x-\pi) + \frac{(x-\pi)^3}{3!} \qquad \text{(Eqn 33)}$$

and show that (for $n \geq 0$)

$$\left[\frac{d^{2n}}{dx^{2n}}\sin(x)\right]_{x=\pi} = 0$$

$$\left[\frac{d^{2n+1}}{dx^{2n+1}}\sin(x)\right]_{x=\pi} = -(-1)^n \quad ❏$$

Putting the results derived in Question T11 into Equation 34,

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)(x-a)^n}{n!} \qquad \text{(Eqn 34)}$$

we find

$$\sin(x) = -\sum_{n=0}^{\infty} \frac{(-1)^n (x-\pi)^{2n+1}}{(2n+1)!} = -\sum_{n=0}^{\infty} \frac{(x-\pi)^{2n+1}}{(2n+1)!}$$

$$= -(x-\pi) + \frac{(x-\pi)^3}{3!} - \frac{(x-\pi)^5}{5!} + \dots \qquad (35)$$

Question T12

Show that the Taylor expansion of x^{-1} near to $x = 1$ is given by

$$\frac{1}{x} = \sum_{n=0}^{\infty} (1-x)^n = 1 + (1-x) + (1-x)^2 + (1-x)^3 + (1-x)^4 + (1-x)^5 + \dots$$ ❏

Question T13

Use Equation 34

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)(x-a)^n}{n!} \qquad \text{(Eqn 34)}$$

to show that

$$f(x+\varepsilon) = \sum_{n=0}^{\infty} \frac{\varepsilon^n f^{(n)}(x)}{n!} \qquad (36)$$

(This form of the Taylor expansion is used in Subsection 3.7.) ❑

3.5 Some useful Taylor series

In this subsection we list some standard Taylor series which are commonly used in physics.

$$\exp(x) = e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \quad x \in \mathbb{R} \tag{37}$$

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \quad x \in \mathbb{R} \tag{38}$$

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \quad x \in \mathbb{R} \tag{39}$$

$$\log_e(1-x) = -\sum_{n=0}^{\infty}\frac{x^n}{n} = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} + \dots \quad -1 \le x < 1 \tag{40}$$

$$(1+x)^r = 1 + \frac{rx}{1!} + \frac{r(r-1)x^2}{2!} + \frac{r(r-1)(r-2)x^3}{3!} + \dots \quad -1 < x < 1 \tag{41}$$

Notice that Equations 37 to 39 are valid for all (real) values of x (in the sense that the series is convergent), whereas Equations 40 and 41 are only valid for the indicated values of x.

The series given in Equation 41 is particularly useful in the case $r = -1$, and when x is replaced by $-x$, for then

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots \quad -1 < x < 1 \tag{42}$$

This series is easy to remember and can be used to derive many other useful series.

3.6 Simplifying the derivation of Taylor expansions

In this subsection we consider a few tricks that can be used to simplify the derivation of Taylor expansions. The first is similar to the method we used in Question T13.

Method 1 Substitution into a known series

Suppose we require the Taylor expansion of $f(x) = \exp(5x)$ near $x = 0$. We could find the $n^{\rm th}$ derivative of $\exp(5x)$ and then use Equation 34. However, it is much easier to start with the result we already have for $\exp(x)$, namely

$$\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad \text{(Eqn 37)}$$

and then to substitute $x = 5u$, so we obtain

$$\exp(5u) = \sum_{n=0}^{\infty} \frac{(5u)^n}{n!}$$

We can use any symbol in place of u and in particular we may choose to replace u by x. In which case

$$\exp(5x) = \sum_{n=0}^{\infty} \frac{(5x)^n}{n!}$$

❖ Use Equation 42

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots \quad -1 < x < 1 \qquad \text{(Eqn 42)}$$

to show that $\dfrac{1}{L+3x} \approx \dfrac{1}{L} - \dfrac{3x}{L^2}$ if x is small compared to L, and $L \neq 0$.

Method 2 Combinations of known series

Suppose that we want the expansion of ☞

$$f(x) = \frac{\exp(x) - \exp(-x)}{2}$$

near to $x = 0$, then we can use the known expansion for $\exp(x)$ together with the expansion obtained when $-x$ is substituted for x. In this way we find

$$\frac{\exp(x) - \exp(-x)}{2} = \frac{1}{2}\sum_{n=0}^{\infty}\frac{x^n}{n!} - \frac{1}{2}\sum_{n=0}^{\infty}\frac{(-x)^n}{n!} = \sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!}$$

Question T14

Use Equation 24 (or 41)

$$(1+x)^r = 1 + rx + \frac{r(r-1)}{2!}x^2 + \frac{r(r-1)(r-2)}{3!}x^3 + \dots \qquad \text{(Eqn 24)}$$

$$(1+x)^r = 1 + \frac{rx}{1!} + \frac{r(r-1)x^2}{2!} + \frac{r(r-1)(r-2)x^3}{3!} + \dots \quad -1 < x < 1 \qquad \text{(Eqn 41)}$$

to find the Taylor expansion for

$$f(x) = \frac{1}{2}[(1+x)^r + (1-x)^r]$$

about $x = 0$. ❏

Method 3 Combinations of the approximating polynomials

Sometimes we need to approximate rather unpleasant functions, which are perhaps a combination of functions whose Taylor series are known. In such a case it may be quite sufficient to find just one or two terms of the series.

Suppose, for example, that we wish to find a cubic approximation to $\exp(2x)\sin(3x)$ near to $x = 0$; we can use the first few terms of the standard Taylor series for $\sin(x)$ and $\exp(x)$ as follows

$$\exp(2x)\sin(3x) \approx \left[1 + 2x + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \dots\right]\left[3x - \frac{(3x)^3}{3!} + \dots\right]$$

If we expand the brackets and ignore the terms of degree 4 and above, we find

$$\begin{aligned}\exp(2x)\sin(3x) &\approx \left[1 + 2x + \frac{(2x)^2}{2!}\right]3x + (1)\left[-\frac{(3x)^3}{3!}\right]\\ &= 3x + 6x^2 + \frac{3}{2}x^3\end{aligned}$$

Question T15

Find the Taylor polynomial of degree 2 which approximates $\exp[\sin(x)]$ near to $x = 0$. ❏

Method 4 Differentiating a known series

In some cases it is possible to obtain one Taylor series by differentiating another series. For example, we know that

$$\frac{d}{dx}\sin(x) = \cos(x)$$

and that the Taylor series for $\sin(x)$ is

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

We also have

$$\frac{d}{dx}x^n = nx^{n-1}$$

and so putting all these results together we get the Taylor series for $\cos(x)$, we find

$$\cos(x) = \frac{d}{dx}\sin(x) = \frac{d}{dx}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots\right) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

Question T16

Use Equation 42

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots \quad -1 < x < 1 \qquad \text{(Eqn 42)}$$

to find the Taylor series for $\dfrac{1}{(1-x)^4}$ near

$x = 0$. ❏

3.7 Applications and examples

In this subsection we consider some applications of Taylor series in both mathematics and physics.

Example 1 The field strength of a magnet of length $2L$ at a point on its axis at a distance x from its centre is proportional to

$$\frac{1}{(x-L)^2}-\frac{1}{(x+L)^2}$$

Show that for $|x|<L$ this expression is approximately $\dfrac{4L}{x^3}$.

Solution Using Equation 41

$$(1+x)^r=1+\frac{rx}{1!}+\frac{r(r-1)x^2}{2!}+\frac{r(r-1)(r-2)x^3}{3!}+\dots\quad -1<x<1 \qquad \text{(Eqn 41)}$$

we can write this expression as

$$x^{-2}\left(1-\frac{L}{x}\right)^{-2}-x^{-2}\left(1+\frac{L}{x}\right)^{-2}\approx x^{-2}\left[\left(1+\frac{2L}{x}\right)-\left(1-\frac{2L}{x}\right)\right]=\frac{4L}{x^3}$$ ❏

Example 2 ☞ A particle moves in one dimension subject to a *potential*, $V(x)$, which has a minimum at $x = x_0$. Show that the motion is *simple harmonic* for small displacements from the equilibrium position.

Solution Expanding $V(x)$ as a Taylor series about the point $x = x_0$ we find (from Equation 34)

$$V(x) = V(x_0) + V'(x_0)(x - x_0) + V''(x_0)\frac{(x - x_0)^2}{2!} + \dots$$

Since $V(x)$ has a minimum at x_0, we know that $V'(x_0) = 0$ and $V''(x_0)$ must be a positive constant ☞ which we can call ω^2, hence

$$V(x) \approx V(x_0) + \omega^2\frac{(x - x_0)^2}{2}$$

so $$V(x) - V(x_0) \approx \omega^2\frac{(x - x_0)^2}{2}$$

If we write $y = x - x_0$, then we can define a new potential, $U(y)$, given by

$$U(y) = V(y + x_0) - V(x_0) \approx \frac{\omega^2}{2}y^2$$

this is the potential of simple harmonic motion about $y = 0$ (i.e. about $x = x_0$) and the corresponding component of force in the y-direction, F_y, is

$$F_y = -\frac{dU}{dy} = -\omega^2 y \quad ❏$$

Newton–Raphson method

The next example concerns a famous method of solving algebraic equations known as the **Newton–Raphson method.**

Often we wish to find an *exact* solution, $x = \alpha$ say, of an equation $f(x) = 0$, so that $f(\alpha) = 0$ but unfortunately this is not always possible, and we have to be content with an approximate solution. The Newton–Raphson technique is remarkable in that it allows us to use an approximate solution of the equation, say $x = x_1$, to construct an even better estimate of the solution, say $x = x_2$.

First we find the Taylor approximation of degree one about $x = x_1$

$$y = f(x_1) + f'(x_1)(x - x_1)$$

This is the equation of a line, and it is in fact the equation of the tangent line to the graph of $y = f(x)$ at the point $(x_1, f(x_1))$, the dashed line through the point P in Figure 5. This line meets the x-axis at the point $(x_2, 0)$, and so

$$0 = f(x_1) + f'(x_1)(x_2 - x_1)$$

which can be rearranged to give the formula

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \tag{43}$$

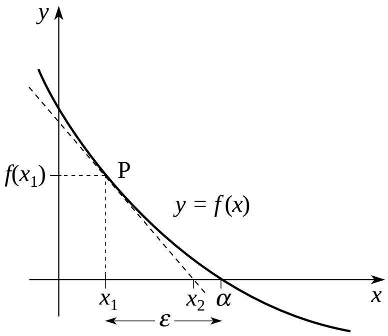

Figure 5 The graph of $y = f(x)$ showing a root at $x = \alpha$.

Although it appears from Figure 5 that x_2 is likely to be a better approximation to the root α than x_1, such an argument is unlikely to convince a mathematician and something like the following discussion is required.

Example 3 Show that if x_1 is a good approximation for a root of the equation $f(x) = 0$, then

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \qquad \text{(Eqn 43)}$$

is (generally) a better approximation to the root. Use this expression to find a better solution of the equation

$$x^2 - 2x - 5 = 0$$

starting from the approximate solution $x \approx 4$.

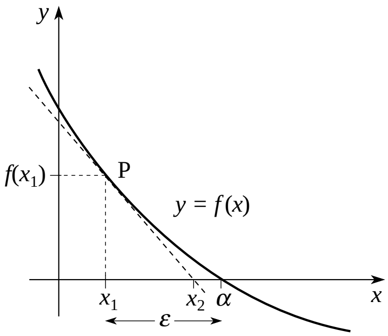

Figure 5 The graph of $y = f(x)$ showing a root at $x = \alpha$.

Solution Suppose that x_1 is a good approximate solution, then the error, ε (see Figure 5) in our approximate solution, is given by $\varepsilon = \alpha - x_1$, and ε is small.

Notice that $x_1 = \alpha - \varepsilon$, and our intention is to estimate the size of ε, so that we can (partially) correct the error in x_1.

Unusually, we regard ε as the independent variable, then consider what happens when we expand an arbitrary function of ε, say $F(\varepsilon)$, as a Taylor series (in powers of ε) about $\varepsilon = 0$. From Equation 21

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \ldots + \frac{f^{(n)}(0)}{n!}x^n + \ldots \qquad \text{(Eqn 21)}$$

we have (with ε in place of x)

$$F(\varepsilon) = F(0) + F'(0)\frac{\varepsilon}{1!} + F''(0)\frac{\varepsilon^2}{2!} + \ldots \qquad (44)$$

Now we choose $F(\varepsilon)$ to be a particular function of ε, in fact we put ☞

$$F(\varepsilon) = f(x_1 + \varepsilon)$$

so that $\quad F(0) = f(x_1) \quad$ and $\quad F'(0) = f'(x_1)$

and Equation 44 becomes

$$f(x_1 + \varepsilon) = f(x_1) + \varepsilon f'(x_1) + (\text{terms involving } \varepsilon^2 \text{ and higher powers of } \varepsilon) \qquad (45)$$

Remember that $x_1 + \varepsilon = \alpha$ is a root of the original equation, so that

$f(x_1 + \varepsilon) = 0$, so that Equation 45

$$f(x_1+\varepsilon) = f(x_1) + \varepsilon f'(x_1) + (\text{terms involving } \varepsilon^2 \text{ and higher powers of } \varepsilon) \qquad \text{(Eqn 45)}$$

implies that

$$\varepsilon f'(x_1) = -f(x_1) - (\text{terms involving } \varepsilon^2 \text{ and higher powers of } \varepsilon)$$

and this gives us an estimate for ε in terms of x_1, and the original function, as follows

$$\varepsilon = -\frac{f(x_1)}{f'(x_1)} - (\text{terms involving } \varepsilon^2 \text{ and higher powers of } \varepsilon) \tag{46}$$

It follows that the true solution

$$\alpha = x_1 + \underbrace{\varepsilon}_{\text{this is small}} = x_1 - \frac{f(x_1)}{f'(x_1)} - \underbrace{(\text{terms involving } \varepsilon^2 \text{ and higher powers of } \varepsilon)}_{\text{this is even smaller}}$$

We know that ε is small and therefore the terms involving ε^2 and higher powers of ε *must be very much smaller*, and so

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

must be even closer to the true solution than our first estimate x_1.

The calculation can go wrong if $f'(x_1)$ is very small, but generally this improved solution can be used as the starting point for the calculation of an even better solution. We therefore have an *iterative* technique for solving equations, and the iteration can be continued indefinitely and so provide solutions that are as accurate as we please.

In the particular case of $f(x) = x^2 - 2x - 5$ and $x_1 = 4$, we have

$$f(x_1) = 16 - 8 - 5 = 3$$

and $$f'(x_1) = [2x - 2]_{x=4} = 6$$

For the first iteration we therefore obtain (from Equation 43)

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \qquad \text{(Eqn 43)}$$

$$x_2 = 4 - \frac{3}{6} = 3.5$$

We can now repeat the process and use this value as a starting point for another approximation

$$f(x_2) = \frac{49}{4} - 7 - 5 = 0.25$$

$$f'(x_2) = [2x - 2]_{x=3.5} = 5$$

and using Equation 43 again

(with x_3 in place of x_2, and x_2 in place of x_1) gives us

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 3.5 - 0.05 = 3.45$$

Continuing in this fashion we could obtain the following approximations to the root

$$4.0, \quad 3.5, \quad 3.45, \quad 3.449\,489\,80, \quad 3.449\,489\,74, \ \ldots$$

and thereafter the first eight decimal places will not change.

If x_n is the n^{th} approximation to the root of the equation $f(x) = 0$, then the next approximation is given by

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$

This is known as the **Newton–Raphson formula**.

Since $f(x) = 0$ is a quadratic equation, the standard formula for such equations gives the exact roots $\alpha = 1 \pm \sqrt{6}$.

(The positive root gives good agreement with our result.) ❏ ☞

Question T17

If a function, $f(x)$, is defined by

$$f(x) = x\log_e(x) - 1$$

use three iterations of the Newton–Raphson technique, ☞ with the starting value of 2, to find an approximate solution to the equation $f(x) = 0$. ❏

Thermal expansion and anharmonicity

Many materials expand when they are heated, and it is possible to construct a mathematical model which explains why this happens in terms of the behaviour of their molecules. Taylor series are central to this mathematical model, and therefore crucial to an understanding of the mechanism which causes thermal expansion.

Two atoms in close proximity exert forces on each other, rather like the tension or compression in a spring. In the case of atoms the force between them is composed of two components, one of repulsion and one of attraction. The repulsive forces between two atoms, caused by the overlapping of two electron clouds, act over a short range; while the attractive force (the van der Waals force), due to the distortion of the electron cloud of one molecule because of the presence of the other, acts over a rather greater distance.

Just as a stretched spring can store energy, so too can a pair of atoms — energy is required to pull them apart, or to push them together — and this (potential) energy, V say, is a function of the distance r between the atoms. In Figure 6 we show two examples of such functions.

The mid-point of the line AB in Figure 6a represents the mean distance between two atoms at this particular energy level.

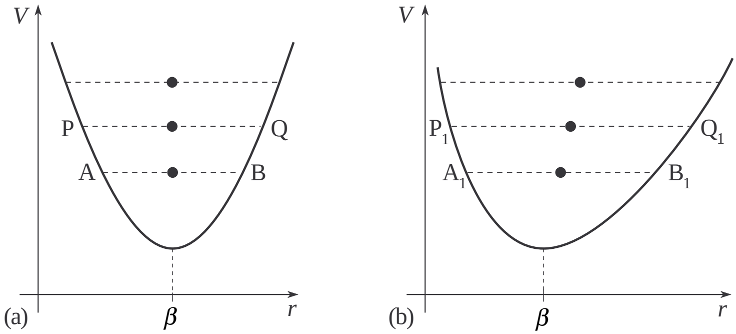

Figure 6 Potential energy V as a function of the distance r between two atoms.

Notice that as we increase the energy the mid-point (of the line PQ say) remains directly above the value $r = \beta$. Such a case represents a material in which it is just as difficult to push the atoms together as it is to pull them apart, and when they vibrate (which they will do as the temperature rises) they will do so in an harmonic fashion, so that their mean separation remains unchanged. In other words, the material does not expand as the temperature is raised. This corresponds to a graph which is symmetric about the line $r = \beta$.

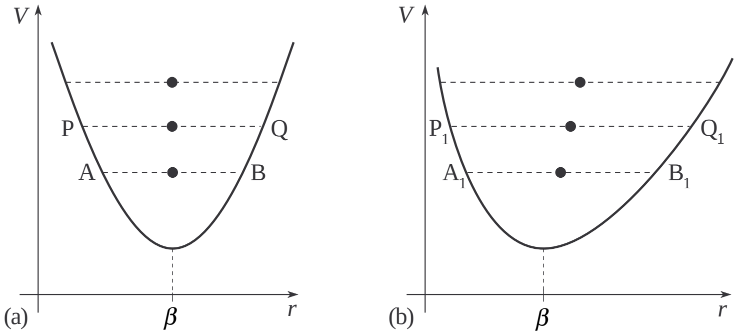

Figure 6 Potential energy V as a function of the distance r between two atoms.

On the other hand, Figure 6b represents the behaviour of a material in which two atoms can be more easily pulled apart than pushed together. In this case the mid-point of the line moves to the right as the energy increases (from the mid-point of A_1B_1 to the mid-point of P_1Q_1 say) so that the mean displacement between the atoms increases with temperature, i.e. the material expands. Such systems are said to be anharmonic, and correspond to a graph which is not symmetric about a vertical axis through $r = \beta$.

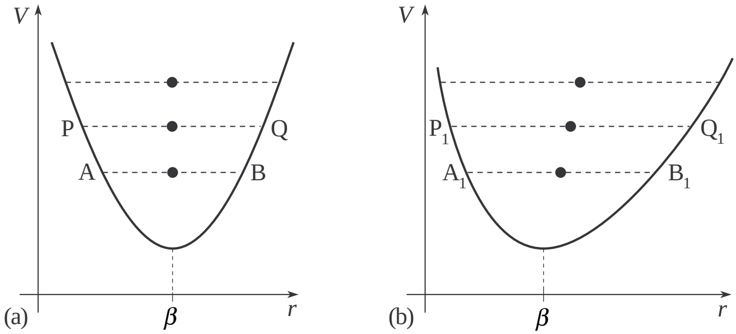

Figure 6 Potential energy V as a function of the distance r between two atoms.

The function $V(r)$ may be expanded as a Taylor series about a point β say, so that Equation 34 becomes

$$V(r) = V(\beta) + V'(\beta)(r-\beta) + V''(\beta)\frac{(r-\beta)^2}{2!} + V^{(3)}(\beta)\frac{(r-\beta)^3}{3!} + \dots \tag{47}$$

The potential energy is often modelled by the Lennard–Jones 6–12 function (shown in Figure 7). One form of this function is given below:

$$V(r) = \varepsilon\left[\left(\frac{a}{r}\right)^{12} - 2\left(\frac{a}{r}\right)^{6}\right] \tag{48}$$

☞

where ε and a are constants and r is the distance between the atoms. Such a mathematical model is appropriate for molecular solids, such as argon (and less so for metals).

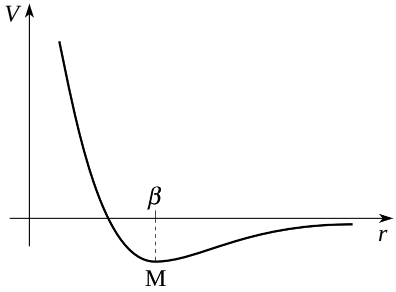

Figure 7 The Lennard–Jones 6–12 function.

We will examine the behaviour of this function close to its minimum value at $r = \beta$ and hence determine how a material modelled by Equation 48

$$V(r) = \varepsilon\left[\left(\frac{a}{r}\right)^{12} - 2\left(\frac{a}{r}\right)^{6}\right] \qquad \text{(Eqn 48)}$$

behaves as the energy increases. To do so we will need the first three derivatives of the function $V(r)$ evaluated at the point $r = \beta$ (bearing in mind that the first derivative must be zero at M because this is a minimum point on the graph).

First we rearrange Equation 48 a little to obtain

$$V(r) = 12\varepsilon\left[\frac{1}{12}\left(\frac{a}{r}\right)^{12} - \frac{1}{6}\left(\frac{a}{r}\right)^{6}\right]$$

then we use the fact that $r = \beta$ corresponds to the minimum point on the graph of $V(r)$, so that

$$V'(\beta) = -\frac{12\varepsilon}{a}\left[\left(\frac{a}{r}\right)^{13} - \left(\frac{a}{r}\right)^{7}\right]_{r=\beta} = -\frac{12\varepsilon}{a}\left[\left(\frac{a}{\beta}\right)^{13} - \left(\frac{a}{\beta}\right)^{7}\right] = 0$$

from which it follows immediately that $\beta = a$.

Now we differentiate again to find

$$V''(\beta)=\frac{12\varepsilon}{a^2}\left[13\left(\frac{a}{r}\right)^{14}-7\left(\frac{a}{r}\right)^{8}\right]_{r=\beta}=\frac{72\varepsilon}{a^2}$$ ☞

and $$V^{(3)}(\beta)=-\frac{12\varepsilon}{a^3}\left[13\times14\left(\frac{a}{r}\right)^{15}-7\times8\left(\frac{a}{r}\right)^{9}\right]_{r=\beta}=\frac{-1512\varepsilon}{a^3}$$

The purpose of these calculations is to simplify Equation 47,

$$V(r)=V(\beta)+V'(\beta)(r-\beta)+V''(\beta)\frac{(r-\beta)^2}{2!}+V^{(3)}(\beta)\frac{(r-\beta)^3}{3!}+\dots \qquad \text{(Eqn 47)}$$

and we can simplify it still further if we let $x = (r - \beta)$ and $E_{\text{pot}}(x) = V(r) - V(\beta)$ to obtain ☞)

$$E_{\text{pot}}(x)\approx 36\varepsilon\left(\frac{x}{a}\right)^2-252\varepsilon\left(\frac{x}{a}\right)^3 \qquad (49)$$

and as a further simplification we let $P = 36\varepsilon$ and $Q = 252\varepsilon$, so that Equation 49 can be written as

$$E_{\text{pot}}(x)\approx P\left(\frac{x}{a}\right)^2-Q\left(\frac{x}{a}\right)^3 \qquad (50)$$

Effectively this means that we have moved the graph of Figure 7 so that the point M is at the origin, and the right-hand side of Equation 50

$$E_{\rm pot}(x) \approx P\left(\frac{x}{a}\right)^2 - Q\left(\frac{x}{a}\right)^3 \qquad \text{(Eqn 50)}$$

is the cubic approximation illustrated in Figure 8.

Now consider a pair of vibrating atoms with fixed total energy E_0. At any instant

$$E_0 = E_{\rm pot}(x) + E_{\rm kin}(x)$$

where $E_{\rm kin}(x)$ represents their instantaneous kinetic energy when their separation is $r = \beta + x$.

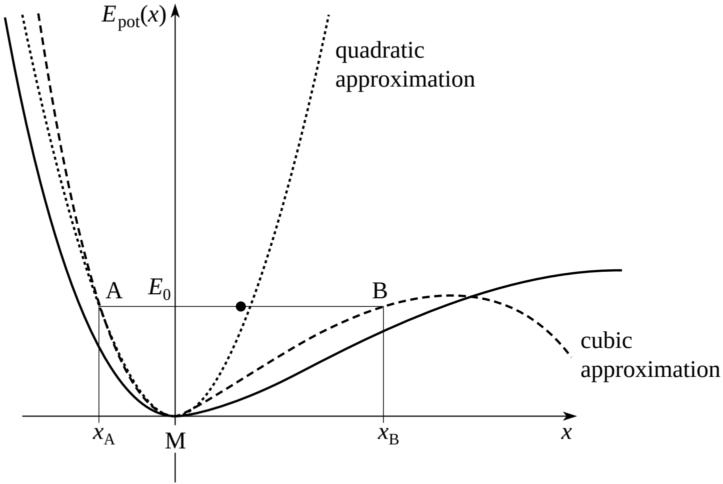

Figure 8 A cubic approximation to $E_{\rm pot}(x)$.

The quantity $E_{\text{kin}}(x)$ must be positive, so the greatest and least possible values of x for the atoms will be given by the roots (i.e. solutions) of the equation $E_0 = E_{\text{pot}}(x)$, and their approximate values will be given by $x = x_{\text{A}}$ and $x = x_{\text{B}}$ the solutions of the equation

$$E_0 = P\left(\frac{x}{a}\right)^2 - Q\left(\frac{x}{a}\right)^3 \tag{51}$$

Once we have found x_{A} and x_{B}, the amount by which the average separation at energy E_0 exceeds the minimum energy separation will be given by $\dfrac{x_{\text{A}} + x_{\text{B}}}{2}$.

We can easily obtain a first estimate of x_{A} and x_{B} by ignoring the term involving x^3 in Equation 51, so that x_{A} and x_{B} are approximately the roots of the equation $E_0 = P\left(\dfrac{x}{a}\right)^2$.

then $$x_{\text{A}} \approx -a\sqrt{\frac{E_0}{P}}, \quad x_{\text{B}} \approx +a\sqrt{\frac{E_0}{P}} \tag{52}$$

and $$\frac{x_{\text{A}} + x_{\text{B}}}{2} \approx 0 \tag{53}$$

This estimate is not sufficiently accurate for our purpose, but the estimates for $x_{\rm A}$ and $x_{\rm B}$ in Equation 52

$$x_{\rm A} \approx -a\sqrt{\frac{E_0}{P}}, \quad x_{\rm B} \approx +a\sqrt{\frac{E_0}{P}} \qquad \text{(Eqn 52)}$$

can be used to obtain an improved estimate as follows. Since $x_{\rm A}$ and $x_{\rm B}$ are roots of Equation 51

$$E_0 = P\left(\frac{x}{a}\right)^2 - Q\left(\frac{x}{a}\right)^3 \qquad \text{(Eqn 51)}$$

we have

$$Px_{\rm A}^2 - \frac{Q}{a}x_{\rm A}^3 = a^2E_0 \qquad (54)$$

and

$$Px_{\rm B}^2 - \frac{Q}{a}x_{\rm B}^3 = a^2E_0 \qquad (55)$$

Subtracting these equations, and factorizing the result, we obtain

$$P(x_{\rm A} - x_{\rm B})(x_{\rm A} + x_{\rm B}) - \frac{Q}{a}(x_{\rm A} - x_{\rm B})(x_{\rm A}^2 + x_{\rm A}x_{\rm B} + x_{\rm B}^2) = 0$$

and therefore (since $x_A \neq x_B$)

$$\frac{x_A + x_B}{2} = \frac{Q}{2aP}(x_A^2 + x_A x_B + x_B^2) \tag{56}$$

Substituting the estimates for x_A and x_B from Equation 52

$$x_A \approx -a\sqrt{\frac{E_0}{P}}, \quad x_B \approx +a\sqrt{\frac{E_0}{P}} \tag{Eqn 52}$$

into the right-hand side of Equation 56 we obtain

$$\frac{x_A + x_B}{2} \approx \frac{Q}{2aP}\left(\frac{a^2 E_0}{P} - \frac{a\sqrt{E_0}}{\sqrt{P}}\frac{a\sqrt{E_0}}{\sqrt{P}} + \frac{a^2 E_0}{P}\right) = \frac{aQE_0}{2P^2}$$

and after substituting $P = 36\varepsilon$ and $Q = 252\varepsilon$ we obtain

$$\frac{x_A + x_B}{2} \approx \frac{7aE_0}{72\varepsilon} \tag{57}$$

From Equation 57 it follows that as the energy E_0 increases, the mean distance between the atoms also increases, in other words the material expands.

Study comment In this module we have studied Taylor polynomials as approximations to various functions. Such approximations are valuable in certain circumstances, such as obtaining the equation for a simple pendulum near to its equilibrium position. However, it is important that you should realize that in other circumstances a Taylor polynomial may not be the best approximation since it becomes less accurate as we move further away from the point about which we are expanding. You should be aware that there are other polynomial approximations which may be better in some circumstances. However, such polynomials are not considered within *FLAP*.

4 Closing items

4.1 Module summary

1 The *Taylor polynomial of degree n for f(x) near x = 0* is

$$p_n(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n \qquad \text{(Eqn 19)}$$

2 The *Taylor series or expansion for f(x) near x = 0* is

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots \qquad \text{(Eqn 21)}$$

☞

3 The *Taylor polynomial of degree n for f(x) near x = a* is

$$p_n(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \dots\dots + \frac{f^{(n)}(a)}{n!}(x-a)^n \qquad \text{(Eqn 31)}$$

4 The *Taylor series*, or *expansion, for f(x) near x = a* is

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \ldots\ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \ldots$$

5 For a convergent series, the error in using a Taylor polynomial is approximately equal to the next (non-zero) term in the corresponding Taylor series.

6 If x_n is the n^{th} approximation to the root of the equation $f(x) = 0$, then the next approximation is given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

This is known as the *Newton–Raphson formula*.

7 New Taylor series can be found from known series by various methods, including:

- substitution;
- combinations of series;
- differentiation.

4.2 Achievements

Having completed this module, you should be able to:

A1 Define the terms that are emboldened and flagged in the margins of the module.

A2 Use the general form of a Taylor series to find series expansions for given functions about $x = 0$ or about $x = a$.

A3 Describe and estimate the approximation involved in replacing a Taylor expansion by the corresponding polynomial.

A4 Derive new Taylor series from known series.

Study comment You may now wish to take the *Exit test* for this module which tests these Achievements. If you prefer to study the module further before taking this test then return to the *Module contents* to review some of the topics.

4.3 Exit test

Study comment Having completed this module, you should be able to answer the following questions, each of which tests one or more of the *Achievements*.

Question E1

(*A2*) Find the Taylor polynomial of degree 4 for $\sqrt{1+x}$ near $x = 0$.

Question E2

(*A2* and *A3*) Obtain an approximate value of $\sin(50°)$ by taking the first two non-zero terms in the Taylor expansion of $\sin(x)$ about $x = 45°$. Give an estimate of the likely error in your approximation.

Question E3

(*A2* and *A3*) Find the Taylor expansion of $\log_e(x)$ near $x = 1$.

Question E4

(*A4*) Use the *known* Taylor expansion for $\exp(x)$ about $x = 0$ to obtain the expansion about $x = 1$.

Question E5

(*A4*) Use the *known* Taylor expansion for $\log_e(1 - x)$ to obtain the expansion for

$$\frac{1}{2}\log_e\left(\frac{1+x}{1-x}\right)$$

about $x = 0$.

Study comment This is the final *Exit test* question. When you have completed the *Exit test* go back to Subsection 1.2 and try the *Fast track questions* if you have not already done so.

If you have completed **both** the *Fast track questions* and the *Exit test*, then you have finished the module and may leave it here.