Module M4.6 Hyperbolic functions and differentiation

Exit module

1 Opening items

1.1 Module introduction

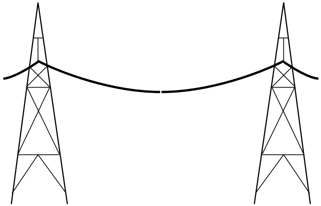

Figure 1 A heavy cable suspended between two pylons.

Figure 1 shows a situation that you have certainly seen countless times, a heavy cable suspended between two pylons. We all know that the cable sags in the middle, but is there a mathematical function which describes the shape of this curve? It turns out that there is such a function, and it is particularly relevant to this module because it is one of the so-called *hyperbolic functions*.

The hyperbolic functions: $\sinh(x)$, $\cosh(x)$, $\tanh(x)$, $\operatorname{sech}(x)$, $\operatorname{arctanh}(x)$ and so on, which have many important applications in mathematics, physics and engineering, correspond to the familiar trigonometric functions: $\sin(x)$, $\cos(x)$, $\tan(x)$, $\sec(x)$, $\arctan(x)$, etc.

In Section 2 of this module we begin by defining the basic hyperbolic functions $\sinh(x)$, $\cosh(x)$ and $\tanh(x)$, and show how the infinite series for these functions are related to those of the corresponding trigonometric functions. We also show how these two sets of functions are related through the introduction of the complex number, i (where $i^2 = -1$). Section 2 concludes with a description of how the first few terms of the infinite series for the basic hyperbolic functions can be used to provide approximations to the functions for small values of their arguments.

In Section 3 we go on to consider more advanced aspects of hyperbolic functions, including the reciprocal and inverse functions. Section 4 lists some useful identities which are analogous to those listed elsewhere in *FLAP* for the trigonometric functions. We end, in Section 5, by finding derivatives of some of the hyperbolic functions, which also provides practice in using differentiation techniques. The final example given in Section 5 leads to the function that describes the shape of a heavy cable suspended between two pylons (illustrated in Figure 1).

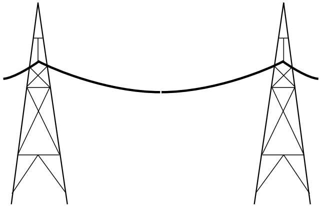

Figure 1 A heavy cable suspended between two pylons.

Study comment

Having read the introduction you may feel that you are already familiar with the material covered by this module and that you do not need to study it. If so, try the *Fast track questions* given in Subsection 1.2. If not, proceed directly to *Ready to study?* in Subsection 1.3.

1.2 Fast track questions

Study comment Can you answer the following *Fast track questions*?. If you answer the questions successfully you need only glance through the module before looking at the *Module summary* (Subsection 6.1) and the *Achievements* listed in Subsection 6.2. If you are sure that you can meet each of these achievements, try the *Exit test* in Subsection 6.3. If you have difficulty with only one or two of the questions you should follow the guidance given in the answers and read the relevant parts of the module. However, *if you have difficulty with more than two of the Exit questions you are strongly advised to study the whole module*.

Question F1

Define $\sinh(x)$, $\cosh(x)$ and $\tanh(x)$ in terms of the exponential function, $\exp(x) = e^x$.

Question F2

First define $\operatorname{sech}(x)$ and $\coth(x)$, and then use your answer to the previous question to show that

$$\operatorname{sech}(x) = \sec(ix) \quad \text{and} \quad \coth(x) = i\cot(ix)$$

Question F3

Show that

$$\operatorname{arccoth}(x) = \frac{1}{2}\log_e\left(\frac{x+1}{x-1}\right) \quad \text{for } |x| > 1$$

Use the fact that $\log_e(1-x) = -\left\{x + \dfrac{x^2}{2} + \dfrac{x^3}{3} + \dots\right\} \quad \text{for } |x| < 1$

to find a series for $\operatorname{arccoth}(1/s)$ valid for $|s| < 1$, and then use this series to estimate the value of arccoth (10).

Use the logarithmic form for $\operatorname{arccoth}(x)$ given above, and your calculator, to check your estimate for arccoth (10).

Question F4

Use the logarithmic expression for $\operatorname{arccoth}(x)$ (see Question F3) to find $\dfrac{d}{dx}\operatorname{arccoth}(x)$.

Study comment Having seen the *Fast track questions* you may feel that it would be wiser to follow the normal route through the module and to proceed directly to *Ready to study?* in Subsection 1.3.

Alternatively, you may still be sufficiently comfortable with the material covered by the module to proceed directly to the *Closing items*.

1.3 Ready to study?

Study comment

To begin the study of this module you need to be familiar with the following: the definition of a *function* and the general formula for the solution of a *quadratic equation*; also the *exponential function*, e^x, including its Taylor *series* expansion about $x = 0$ and the main features of its graph (the function e^x is often written as $\exp(x)$), plus the *trigonometric functions*, including their *graphs*, *series*, *identities*, *reciprocal* and *inverse functions*. You will also need to have some skill at *differentiation* (including the *quotient rule* and *chain rule*), and have some familiarity with *complex numbers* (including *multiplication* and *addition* of such numbers, and the use of *Euler's formula* $e^{i\theta} = \cos\theta + i\sin\theta$). It would also be helpful, though not essential, if you have met *odd* and *even* functions, and *factorials*. If you are unfamiliar with any of these terms, refer to the *Glossary*, which will indicate where they are developed in *FLAP*. The following *Ready to study questions* will help you to establish whether you need to review some of the above topics before embarking on this module. ☞

Question R1

(a) Write down the Taylor expansion of e^x about $x = 0$.

(b) Write down the solutions of the quadratic equation $x^2 - ax + 1 = 0$ where $a \geq 2$. Hence write down the solutions of the equation $e^{2w} - ae^w + 1 = 0$.

Question R2

Use your calculator to find the values of:

(a) $\arcsin(1/3)$ and $\dfrac{1}{\sin(1/3)}$, (b) $\sin[\sin(1/3)]$ and $\sin^2(1/3)$. ☞

Question R3

If $s(x)$ and $c(x)$ are defined by:

$$s(x) = \frac{e^{ix} - e^{-ix}}{2i} \quad \text{and} \quad c(x) = \frac{e^{ix} + e^{-ix}}{2}$$

show that:

(a) $s(x) \times s(x) + c(x) \times c(x) = 1$.

(b) Use Euler's formula to express $s(x)$ and $c(x)$ in more familiar form.

Question R4

Express $\dfrac{d}{dx}\left(\dfrac{u(x)}{v(x)}\right)$ and $\dfrac{d}{dx}g\big(f(x)\big)$ in terms of derivatives of u, v, g and f.

2 Basic hyperbolic functions

2.1 Defining sinh, cosh and tanh

It follows from Euler's formula (see Question R3) that the trigonometric functions sine and cosine can be expressed in terms of the exponential function as

$$\sin(x) = \frac{e^{ix} - e^{-ix}}{2i} \tag{1}$$

and $$\cos(x) = \frac{e^{ix} + e^{-ix}}{2} \tag{2}$$

Since analogous combinations of e^x and e^{-x} occur frequently in mathematics and physics, it is useful to define the **hyperbolic functions**, **sinh** (x) and **cosh** (x), ☞ by

$$\sinh(x) = \frac{e^{x} - e^{-x}}{2} \tag{3}$$

$$\cosh(x) = \frac{e^{x} + e^{-x}}{2} \tag{4}$$

These functions are given their particular names because they have much in common with the corresponding trigonometric functions, but we leave a discussion of this until later.

❖ Write down (without using your calculator) the values of $\sinh(0)$ and $\cosh(0)$.

Question T1

Use the basic definitions of $\cosh(x)$ and $\sinh(x)$ in terms of exponentials to show that

(a) $\sinh(x)$ is an odd function while $\cosh(x)$ is an even function ☞

(b) $e^x = \cosh(x) + \sinh(x)$

(c) $\cosh^2(x) - \sinh^2(x) = 1$ ☞ ❏

It is also convenient to define a third hyperbolic function, **tanh** (x), by

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} \qquad (5)$$

✧ What is the value of $\tanh(0)$?

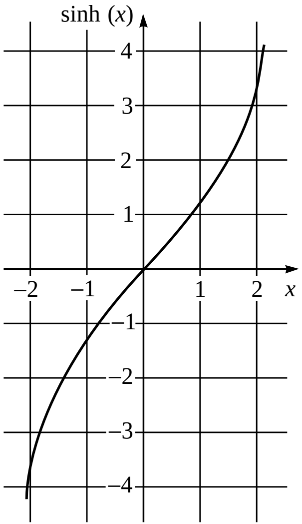

Figure 2 Graph of sinh (x).

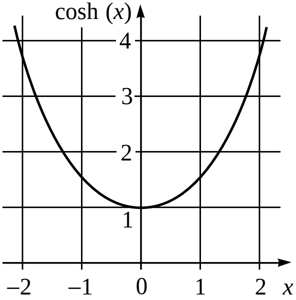

Figure 3 Graph of cosh (x).

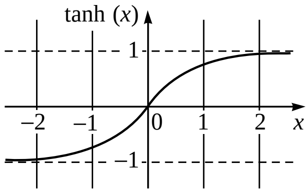

Figure 4 Graph of tanh (x).

The graphs of these three basic hyperbolic functions are given in Figures 2 to 4. Notice in Figure 3 that $\cosh(x) \geq 1$. In Figures 2 and 3 we see that $\sinh(x)$ and $\cosh(x)$ are very close in value when x is large and positive, but $\sinh(x) < \cosh(x)$ for other values of x. From Figure 4 we see that $\tanh(x)$ is approximately 1 when x is large and positive, and approximately −1 when x is large and negative, and that $-1 < \tanh(x) < 1$.

Why are they called *hyperbolic* functions?

To answer this question, we define variables x and y by

$$x = a\cosh(s) \tag{6}$$

$$y = b\sinh(s) \tag{7}$$

Note The equations for x and y in terms of s are known as *parametric equations* since s is a parameter which specifies a point on a curve (in this case, a hyperbola).

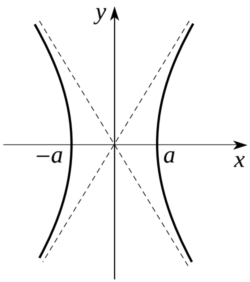

Figure 5 The hyperbola corresponding to $x^2/a^2 - y^2/b^2 = 1$.

Then use the result from Question T1(c)

$$\cosh^2(x) - \sinh^2(x) = 1$$

to obtain

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \cosh^2(s) - \sinh^2(s) = 1 \tag{8}$$

which is the equation for a hyperbola, as shown in Figure 5. Thus Equations 6 and 7 are the parametric equations of a hyperbola, and this is the reason why sinh and cosh are known as hyperbolic functions.

As we shall see, there are many other functions (such as tanh) which can be defined in terms of sinh and cosh and these are all collectively known as *hyperbolic functions*.

Notice that, unlike the corresponding trigonometric functions, the sinh, cosh and tanh functions are not periodic. Moreover, $\sinh(x)$ and $\cosh(x)$ may take very large values (unlike $\sin(x)$ and $\cos(x)$).

✥ Use the exponential function on your calculator to find $\sinh(1)$, $\cosh(1)$ and $\tanh(1)$. From these results calculate $\cosh^2(1) - \sinh^2(1)$.

Note A more accurate result for this calculation (and also for the calculation of $\tanh(1)$) could be obtained by retaining as many decimal places as possible in the intermediate steps. In fact, from Question T1(c) we know that the result for $\cosh^2(1) - \sinh^2(1)$ should be exactly 1.

Question T2

The speed of waves in shallow water is given by the following equation

$$v^2 = A\lambda \tanh\left(\frac{6.3d}{\lambda}\right)$$

where λ is the wavelength and d is the depth of the water. Find the speed, v when $d = 6\,\text{m}$, $\lambda = 18.9\,\text{m}$ and $A = 1.8\,\text{m}\,\text{s}^{-2}$. ❏

2.2 Series for sinh, cosh and tanh

So far we have defined the three basic hyperbolic functions in terms of the exponential function, e^x, which in turn can be defined in terms of an infinite series:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \tag{9}$$

Replacing x with $-x$ this equation gives us

$$e^{-x} = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \dots \tag{10}$$

and hence Equation 3

$$\sinh(x) = \frac{e^x - e^{-x}}{2} \tag{Eqn 3}$$

gives

$$\boxed{\sinh(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} = \frac{x}{1!} + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \frac{x^9}{9!} + \dots \qquad (11)}$$

Comparing Equation 11

$$\sinh(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} = \frac{x}{1!} + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \frac{x^9}{9!} + \dots \qquad \text{(Eqn 11)}$$

with the series for $\sin(x)$

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \dots \qquad (12)$$

we see that both series contain only odd powers (they are both *odd* functions) and the only difference is that the signs do not alternate for the sinh series. Likewise, from Equation 4,

$$\cosh(x) = \frac{e^x + e^{-x}}{2} \qquad \text{(Eqn 4)}$$

we find that the series for $\cosh(x)$ consists of the *even* terms in the series for e^x; that is

$$\boxed{\cosh(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \frac{x^8}{8!} + \dots \qquad (13)}$$

and comparing this

$$\cosh(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \frac{x^8}{8!} + \dots \qquad \text{(Eqn 13)}$$

with the series for $\cos(x)$

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \dots \qquad (14)$$

we again see that they both contain even powers (both are *even* functions) and the only difference is that the signs do not alternate for the cosh function.

There is a similar relationship between $\tan(x)$ and $\tanh(x)$ since it can be shown that

$$\tanh(x) = x - \frac{x^3}{3} + \frac{2x^5}{15} - \frac{17x^7}{315} + \dots \qquad (15)$$

☞

$$\tan(x) = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \dots \qquad (16)$$

We can now appreciate the motivation for giving the hyperbolic functions their names; in each case the ‘h’ is added to give the ‘hyperbolic’ version of the related trigonometric function.

You are probably familiar with the many trigonometric functions that can be defined in terms of the sine and cosine functions, and, as you might expect, a large number of hyperbolic functions can be similarly defined in terms of sinh and cosh (as we will see in Section 3).

❖ Write down and simplify the Taylor expansions about $x = 0$ for

(a) $\sinh(x) + \sin(x)$, and (b) $\cosh(x) - \cos(x)$.

Question T3

Use the series for $\sinh(x)$ and $\cosh(x)$ to show that

$$\frac{d}{dx}\sinh(x) = \cosh(x) \tag{17}$$

$$\frac{d}{dx}\cosh(x) = \sinh(x) \tag{18}$$

❏

2.3 Connection with sine, cosine and tangent via complex numbers

The study of complex numbers (that is, numbers involving i where $i^2 = -1$) leads to the following important result, known as Euler's formula, which connects the exponential, sine and cosine functions

$$e^{ix} = \cos(x) + i\sin(x) \tag{19}$$

By replacing x by $-x$ we have $\qquad e^{-ix} = \cos(x) - i\sin(x)$

from which we can obtain concise expressions for $\cos(x)$ and $\sin(x)$ which resemble the definitions of $\cosh(x)$ and $\sinh(x)$ given in Subsection 2.1.

$$\cos(x) = \frac{e^{ix} + e^{-ix}}{2} \tag{20}$$

$$\sin(x) = \frac{e^{ix} - e^{-ix}}{2i} \tag{21}$$

This suggests that identities should exist which relate the hyperbolic and trigonometric functions. Such identities can be obtained if we replace x by ix in the basic definitions of $\cosh(x)$ and $\sinh(x)$

$$\cosh(ix) = \frac{e^{ix} + e^{-ix}}{2}$$

$$\sinh(ix) = \frac{e^{ix} - e^{-ix}}{2}$$

and then write these expressions in terms of $\cos(x)$ and $\sin(x)$.

$$\sinh(ix) = i\sin(x) \tag{22}$$

$$\cosh(ix) = \cos(x) \tag{23}$$

Question T4

Use the expressions for the sine and cosine functions in terms of the exponential function to relate (a) $\sin(ix)$ to $\sinh(x)$, and (b) $\cos(ix)$ to $\cosh(x)$. ❏

The connection between the hyperbolic and trigonometric functions can also be seen from their series. For example

$$\begin{aligned}\sinh(ix) &= \frac{ix}{1!} + \frac{(ix)^3}{3!} + \frac{(ix)^5}{5!} + \frac{(ix)^7}{7!} + \frac{(ix)^9}{9!} + \dots \\ &= \frac{ix}{1!} - i\frac{x^3}{3!} + i\frac{x^5}{5!} - i\frac{x^7}{7!} + i\frac{x^9}{9!} - \dots \\ &= i\left[\frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \dots\right] = i\sin(x)\end{aligned}$$

Question T5

Use the series for $\cos(x)$ and $\cosh(x)$ to show that $\cos(ix) = \cosh(x)$. ❏

2.4 Small argument approximations

Since the infinite series for $\sinh(x)$ and $\cosh(x)$ given in this module are Taylor expansions about $x = 0$, we would expect the first few terms in the series to provide reasonable approximations to the functions for values of x 'close' to zero. The following example provides an illustration.

Example 1 Use Equation 11

$$\sinh(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} = \frac{x}{1!} + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \frac{x^9}{9!} + \dots \qquad \text{(Eqn 11)}$$

to evaluate $\sinh(0.2)$ to three places of decimals. ☞

Solution From Equation 11 we have

$$\sinh(0.2) = 0.2 + \frac{0.008}{3!} + \frac{0.000\,32}{5!} + \dots = 0.2 + 0.0013 + \dots \approx 0.201$$

This is consistent with my calculator which gives a value of 0.201 336. ❏

Question T6

Use the series given in Equation 15

$$\tanh(x) = x - \frac{x^3}{3} + \frac{2x^5}{15} - \frac{17x^7}{315} + \dots \qquad \text{(Eqn 15)}$$

to find tanh (0.4) to three places of decimals. Check the result with your calculator. ☞ ❑

3 More advanced hyperbolic functions

3.1 Reciprocal hyperbolic functions

The **reciprocals** of the basic hyperbolic functions occur fairly frequently and so are given special names **cosech**, **sech** and **coth**: ☞

$$\operatorname{cosech}(x) = \frac{1}{\sinh(x)} \qquad \text{provided } x \neq 0 \tag{24}$$

$$\operatorname{sech}(x) = \frac{1}{\cosh(x)} \qquad \text{for all } x \tag{25}$$

$$\coth(x) = \frac{1}{\tanh(x)} \qquad \text{provided } x \neq 0 \tag{26}$$

Notice that cosech$\,(x)$ is the reciprocal of sinh$\,(x)$ and that sech$\,(x)$ is the reciprocal of cosh$\,(x)$. As in the case of trigonometric functions, this terminology may seem rather odd, but it is easily remembered by recalling that each reciprocal pair —(sinh, cosech), (cosh, sech), (tanh, coth)— involves the letters ‘co’ just once. In other words, there is just one ‘co’ between each pair. Also notice that the cosech and coth functions are undefined when their partner functions are zero. By contrast, sech$\,(x)$ is defined for all real x since cosh$\,(x)$ is never zero, see Figure 3.

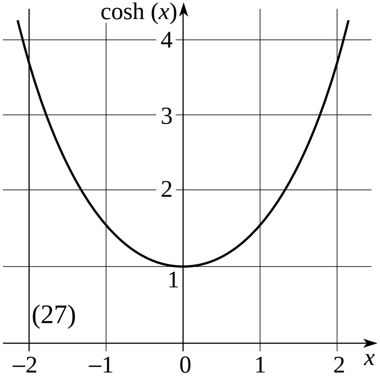

Figure 3 Graph of cosh$\,(x)$.

The reciprocal functions can be written in terms of e^x as

$$\operatorname{cosech}(x) = \frac{2}{e^x - e^{-x}} \tag{27}$$

$$\operatorname{sech}(x) = \frac{2}{e^x + e^{-x}} \tag{28}$$

$$\coth(x) = \frac{e^x + e^{-x}}{e^x - e^{-x}} \tag{29}$$

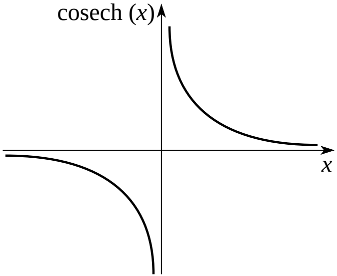

Figure 6 Graph of cosech (x).

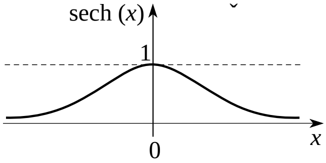

Figure 7 Graph of sech (x).

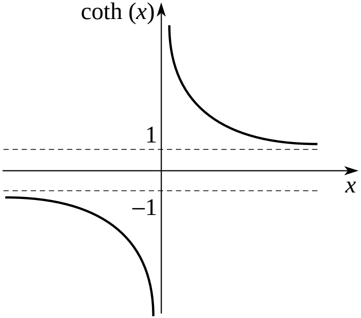

Figure 8 Graph of coth (x).

and graphs of these functions are given in Figures 6 to 8, respectively.

Question T7

Show that $\coth^2(x) - \operatorname{cosech}^2(x) = 1$. ❏

3.2 Inverse hyperbolic functions (and logarithmic forms)

A common problem is to find what argument gives rise to a particular value of some hyperbolic function. For example, if $\sinh(x) = 3.6$, what is the value of x? We can get an approximate value of x by looking at Figure 2.

✧ Use Figure 2 to find the (approximate) value of x that corresponds to $\sinh(x) = 3.6$?

To answer such questions more precisely we need to know the **inverse hyperbolic functions** which 'undo' the effect of the hyperbolic functions. These new functions are known as **arcsinh**, **arctanh** and **arccosh**, and the first two functions are defined by

$$\operatorname{arcsinh}[\sinh(x)] = x \tag{30}$$

$$\operatorname{arctanh}[\tanh(x)] = x \tag{31}$$

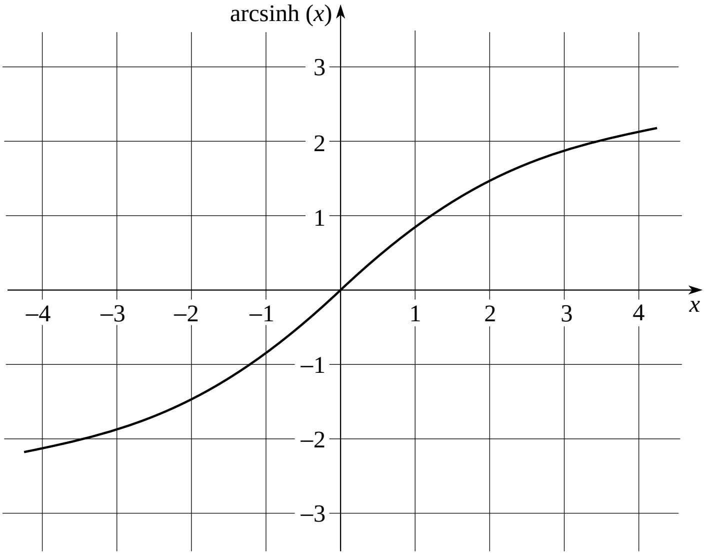

Figure 9 Graph of arcsinh (x).

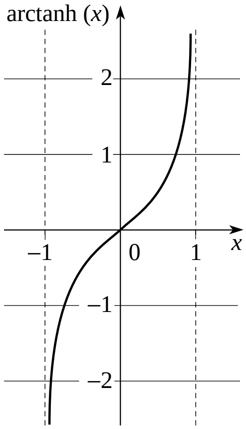

Figure 10 Graph of arctanh (x).

Graphs of these functions are given in Figures 9 and 10, respectively. Notice that the domain of arctanh (x) is $-1 < x < 1$.

✦ From Figure 3, estimate values of x that correspond to $\cosh(x) = 1$, 2 and 3?

From Figure 3, we see that for any given value of $\cosh(x) > 1$ there are *two* corresponding values of x; in other words, the equation $\cosh(x) = a > 1$ has two solutions. Consequently, in order for arccosh to be a function, we must impose a condition which picks out just one value (i.e. $x \ge 0$).

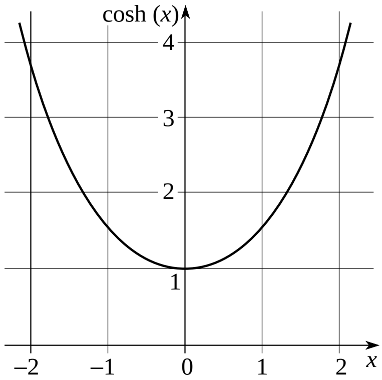

Figure 3 Graph of $\cosh(x)$.

Figure 11 shows the graph of the inverse function (the continuous line), and as you can see we choose the positive value for arccosh (x), which gives

$$\cosh[\operatorname{arccosh}(x)] = x \quad \text{for } x \geq 1 \tag{32a}$$

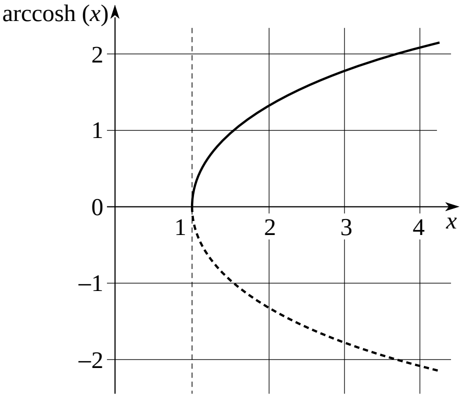

Figure 11 Graph of arccosh (x). Note that only the solid line is involved in the definition.

✦ (a) Simplify $\sqrt{x^2}$ and $\left(\sqrt{x}\right)^2$. Do you need to place any restrictions on the values of x in either case?

(b) Simplify $\operatorname{arccosh}[\cosh(x)]$ and $\cosh[\operatorname{arccosh}(x)]$. Do you need to place any restrictions on the values of x in either case?

If we agree to restrict ourselves to positive values of x, so that we can be sure that $|x| = x$, there is no problem with the inverse of cosh (just as there is no problem with the inverse of 'squaring' if we restrict ourselves to positive numbers) for then we have

$$\operatorname{arccosh}[\cosh(x)] = x \quad \text{for } x \geq 0 \tag{32b}$$

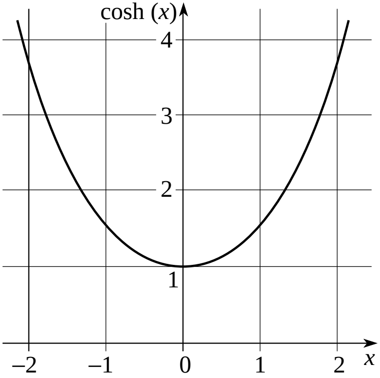

Figure 3 Graph of cosh (x).

✧ From Figure 3, what are the approximate values of arccosh (2) and arccosh (3)?

✧ Find all solutions of the equation $\cosh(x) = 2$.

As in the case of trigonometric functions, an alternative notation is sometimes used to represent the inverse hyperbolic functions:

$$\sinh^{-1}(x) \text{ for } \operatorname{arcsinh}(x)$$

$$\cosh^{-1}(x) \text{ for } \operatorname{arccosh}(x)$$

$$\tanh^{-1}(x) \text{ for } \operatorname{arctanh}(x)$$

Notice that there is *no* connection with the *positive* index notation used to denote powers of the hyperbolic functions, for example, using $\sinh^2(x)$ to represent $[\sinh(x)]^2$. Also notice that, although this notation might make it appear otherwise, there is still a clear distinction between the inverse hyperbolic functions and the reciprocal hyperbolic functions. So

$$\sinh^{-1}[\sinh(x)] = \operatorname{arcsinh}[\sinh(x)] = x$$

but $\quad \operatorname{cosech}(x) \times \sinh(x) = \dfrac{1}{\sinh(x)} \times \sinh(x) = 1$

More inverse hyperbolic functions

There are three more inverse hyperbolic functions which we have not mentioned yet; these are **arccosech** (x), **arcsech** (x) and **arccoth** (x). As you might expect, they are defined so that

$$\operatorname{arccosech}[\operatorname{cosech}(x)] = x \qquad (33)$$

$$\operatorname{arcsech}[\operatorname{sech}(x)] = x \qquad (34)$$

$$\operatorname{arccoth}[\coth(x)] = x \qquad (35)$$

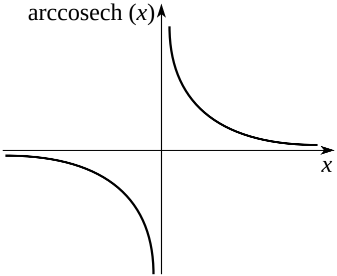

Figure 12 Graph of arccosech (x).

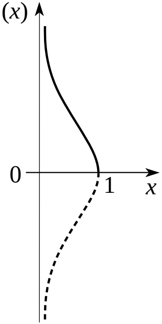

Figure 13 Graph of arcsech (x).

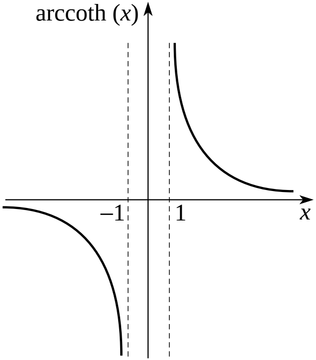

Figure 14 Graph of arccoth (x).

Graphs of these functions are given in Figures 12 to 14, respectively, and the following exercise leads to some useful identities.

Question T8

Show that (a) arccosech (x) = arcsinh $(1/x)$, (b) arcsech (x) = arccosh $(1/x)$, and (c) arccoth (x) = arctanh $(1/x)$. ❑

Connection with logarithmic function

Given the relationship between the exponential and logarithmic functions, and the fact that the hyperbolic functions are defined in terms of the exponential function, it might be expected that the inverse hyperbolic functions can be expressed in terms of the logarithmic function. This is indeed the case, and it can be shown that the three basic hyperbolic functions can be written as

$$\operatorname{arcsinh}(x) = \log_e\left(x + \sqrt{x^2 + 1}\right) \tag{36}$$

$$\operatorname{arccosh}(x) = \log_e\left(x + \sqrt{x^2 - 1}\right) \quad \text{for } x \geq 1 \tag{37}$$

$$\operatorname{arctanh}(x) = \frac{1}{2}\log_e\left(\frac{1+x}{1-x}\right) \quad \text{for } -1 < x < 1 \tag{38}$$

As an example, we now prove the second of these identities.

We start by setting $y = \operatorname{arccosh}(x)$ ☞

which implies that $x = \cosh(y) = \dfrac{e^y + e^{-y}}{2}$, and we see that $x \geq 1$. ☞

Multiplying the right- and left-hand sides of this last relationship bt $2e^y$ and rearranging, we obtain
$e^{2y} - 2xe^y + 1 = 0$,

a quadratic equation in e^y which we can solve to obtain

$$e^y = \frac{2x \pm \sqrt{4x^2 - 4}}{2} = x \pm \sqrt{x^2 - 1}$$

The question now is whether either (or both) of the signs on the right-hand side of this equation are allowable. First we note that the right-hand side must be greater than or equal to one, because $y = \operatorname{arcosh}(x) \geq 0$, so $e^y \geq 1$;

then we can eliminate the negative sign as follows. ☞

We consider the graphs of $Y_1 = \left(x + \sqrt{x^2 - 1}\right)$ and $Y_2 = \left(x - \sqrt{x^2 - 1}\right)$ for $x \geq 1$ (shown in Figure 15), and ask if it is possible for either Y_1 or Y_2 to be greater than or equal to 1 as required.

From the graphs it is clear that $Y_2 \leq 1$, which makes this choice impossible, and therefore the only acceptable solution is

$$e^y = x + \sqrt{x^2 - 1}$$

Taking the logarithm of this result gives us

$$y = \log_e\left(x + \sqrt{x^2 - 1}\right)$$

The remaining two identities can be proven in a similar way, as the following question demonstrates.

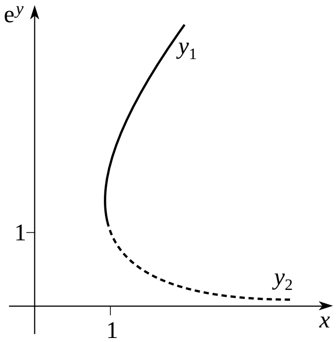

Figure 15 Graphs of $Y_1 = \left(x + \sqrt{x^2 - 1}\right)$ (full line) and $Y_2 = \left(x - \sqrt{x^2 - 1}\right)$ (dotted line) for $x \geq 1$.

Question T9

Show that

$$\operatorname{arctanh}(x) = \frac{1}{2}\log_e\left(\frac{1+x}{1-x}\right) \quad \text{for } -1 < x < 1 \quad ❏$$

Expressing the inverse hyperbolic functions in terms of the logarithmic function can be a useful way of solving certain algebraic equations. For example, suppose we want the values of x which satisfy

$$\sinh^2(x) - 3\cosh(x) + 3 = 0$$

then we can use the identity

$$\cosh^2(x) - \sinh^2(x) = 1$$

to obtain an equation involving just $\cosh(x)$

$$\cosh^2(x) - 3\cosh(x) + 2 = 0$$

This can be factorized to give

$$(\cosh(x) - 2)(\cosh(x) - 1) = 0$$

so, either $\quad x = \pm \operatorname{arccosh}(2)$ ☞

or $\quad x = \pm \operatorname{arccosh}(1)$

Using $\quad \operatorname{arccosh}(x) = \log_e\left(x + \sqrt{x^2 - 1}\right)$

gives $\quad x = \pm \operatorname{arccosh}(2) = \pm \log_e\left(2 + \sqrt{3}\right)$

or $\quad x = \pm \operatorname{arccosh}(1) = \pm \log_e(1) = 0$

(One advantage of this approach is that the logarithmic function is more likely to be available on a calculator.)

Question T10

Solve $\cosh^2(x) + \sinh(x) = 3$, in terms of the logarithmic function. ❏

4 Identities

There are many identities involving trigonometric functions, and there are an equally large number of identities involving hyperbolic functions. In fact, since the trigonometric and hyperbolic functions are related (by, for example, $\sinh(ix) = i\sin(x)$) every trigonometric identity has an analogue for hyperbolic functions. The most important **hyperbolic function identities** ☞ are listed below, and it is worth comparing them with their trigonometric partners which are given elsewhere in *FLAP*.

The **symmetry relations:** ☞

$$\sinh(-x) = -\sinh(x) \tag{39}$$

$$\cosh(-x) = \cosh(x) \tag{40}$$

$$\tanh(-x) = -\tanh(x) \tag{41}$$

The **basic identities:**

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} \tag{42}$$

$$\cosh^2(x) - \sinh^2(x) = 1 \tag{43}$$

$$\coth(x) = \frac{\operatorname{cosech}(x)}{\operatorname{sech}(x)} \quad \text{for } x \neq 0 \tag{44}$$

$$1 - \tanh^2(x) = \operatorname{sech}^2(x) \tag{45}$$

The **addition identities**: ☞

$$\sinh(x+y) = \sinh(x)\cosh(y) + \cosh(x)\sinh(y) \tag{46}$$

$$\cosh(x+y) = \cosh(x)\cosh(y) + \sinh(x)\sinh(y) \tag{47}$$

$$\tanh(x+y) = \frac{\tanh(x) + \tanh(y)}{1 + \tanh(x)\tanh(y)} \tag{48}$$

The **double-argument identities**: ☞

$$\sinh(2x) = 2\sinh(x)\cosh(x) \tag{49}$$

$$\cosh(2x) = \cosh^2(x) + \sinh^2(x) \tag{50}$$

$$\cosh(2x) = 1 + 2\sinh^2(x) \tag{51}$$

$$\cosh(2x) = 2\cosh^2(x) - 1 \tag{52}$$

$$\tanh(2x) = \frac{2\tanh(x)}{1 + \tanh^2(x)} \tag{53}$$

The **half-argument identities**: ☞

$$\cosh^2\left(\frac{x}{2}\right) = \frac{1}{2}[1 + \cosh(x)] \tag{54}$$

$$\sinh^2\left(\frac{x}{2}\right) = \frac{1}{2}[-1 + \cosh(x)] \tag{55}$$

and if $t = \tanh(x/2)$ then

$$\sinh(x) = \frac{2t}{1-t^2} \tag{56}$$

$$\cosh(x) = \frac{1+t^2}{1-t^2} \tag{57}$$

$$\tanh(x) = \frac{2t}{1+t^2} \tag{58}$$

The **sum identities**:

$$\sinh(x) + \sinh(y) = 2\sinh\left(\frac{x+y}{2}\right)\cosh\left(\frac{x-y}{2}\right) \tag{59}$$

$$\sinh(x) - \sinh(y) = 2\cosh\left(\frac{x+y}{2}\right)\sinh\left(\frac{x-y}{2}\right) \tag{60}$$

$$\cosh(x) + \cosh(y) = 2\cosh\left(\frac{x+y}{2}\right)\cosh\left(\frac{x-y}{2}\right) \tag{61}$$

$$\cosh(x) - \cosh(y) = 2\sinh\left(\frac{x+y}{2}\right)\sinh\left(\frac{x-y}{2}\right) \tag{62}$$

The **product identities:**

$$2\sinh(x)\cosh(y) = \sinh(x+y) + \sinh(x-y) \tag{63}$$

$$2\cosh(x)\cosh(y) = \cosh(x+y) + \cosh(x-y) \tag{64}$$

$$2\sinh(x)\sinh(y) = \cosh(x+y) - \cosh(x-y) \tag{65}$$

All of these identities can be derived from the definitions of $\sinh(x)$ and $\cosh(x)$ in terms of e^x as the following questions demonstrate. ☞

✧ Use the definitions of $\sinh(x)$ and $\cosh(x)$ in terms of e^x to show that

$\sinh(x+y) = \sinh(x)\cosh(y) + \cosh(x)\sinh(y)$.

Question T11

Use the definition of the hyperbolic functions in terms of the exponential function to show that

$$\sinh(2x) = 2\sinh(x)\cosh(x) \quad ❏$$

In many cases it is more convenient to derive identities from other identities rather than from the basic definitions. For example, by putting $x = y$ in

$$\sinh(x + y) = \sinh(x)\cosh(y) + \cosh(x)\sinh(y)$$

we obtain $\quad \sinh(2x) = 2\sinh(x)\cosh(x)$

This is certainly more straightforward than using the basic definitions; the other double-argument identities follow from the addition identities in a similar fashion.

Question T12

Use the addition identities

$$\sinh(x+y) = \sinh(x)\cosh(y) + \cosh(x)\sinh(y) \qquad \text{(Eqn 46)}$$

$$\cosh(x+y) = \cosh(x)\cosh(y) + \sinh(x)\sinh(y) \qquad \text{(Eqn 47)}$$

$$\tanh(x+y) = \frac{\tanh(x)+\tanh(y)}{1+\tanh(x)\tanh(y)} \qquad \text{(Eqn 48)}$$

to show that

$$\cosh(2x) = 2\cosh^2(x) - 1 \quad \text{and} \quad \tanh(2x) = \frac{2\tanh(x)}{1+\tanh^2(x)}$$

5 Differentiating hyperbolic functions

Study comment This section contains some results that are important in their own right, but its main purpose is to give you further practice in using the techniques of differentiation.

As shown in Question T3, the derivatives of the basic hyperbolic functions are given by

$$\frac{d}{dx}\sinh(x) = \cosh(x) \qquad \text{(Eqn 17)}$$

$$\frac{d}{dx}\cosh(x) = \sinh(x) \qquad \text{(Eqn 18)}$$

These important results can be used to find the derivatives of other hyperbolic functions. Here we consider some specific examples.

Example 2

Show that $\dfrac{d^2}{dx^2}\sinh(3x) = 9\sinh(3x)$

Solution $\dfrac{d}{dx}\sinh(3x) = 3\cosh(3x)$ ☞ and $\dfrac{d}{dx}3\cosh(3x) = 9\sinh(3x)$ ❏

Example 3

Show that $\dfrac{d}{dx}\tanh(x) = \operatorname{sech}^2(x)$

Solution We know that $\tanh(x) = \dfrac{\sinh(x)}{\cosh(x)}$

and using the quotient rule $\dfrac{d}{dx}\left(\dfrac{u}{v}\right) = \dfrac{1}{v}\dfrac{du}{dx} - \dfrac{u}{v^2}\dfrac{dv}{dx}$

we have $\dfrac{d}{dx}\tanh(x) = \dfrac{1}{\cosh(x)} \times \cosh(x) - \dfrac{\sinh(x)}{\cosh^2(x)} \times \sinh(x)$

so $\dfrac{\cosh^2(x) - \sinh^2(x)}{\cosh^2(x)} = \dfrac{1}{\cosh^2(x)} = \operatorname{sech}^2(x)$ ❏

Question T13

Use Equations 17 and 18

$$\frac{d}{dx}\sinh(x) = \cosh(x) \qquad \text{(Eqn 17)}$$

$$\frac{d}{dx}\cosh(x) = \sinh(x) \qquad \text{(Eqn 18)}$$

to show that

$$\frac{d}{dx}\coth(x) = -\operatorname{cosech}^2(x)$$ ❏

Example 4

Show that $\dfrac{d}{dx}\operatorname{arcsinh}(x) = \dfrac{1}{\sqrt{x^2+1}}$ ☞

Solution There are two approaches:

(a) Let $y = \operatorname{arcsinh}(x)$ so that we have to find $\dfrac{dy}{dx}$.

We have $x = \sinh(y)$, and differentiating both sides of this equation with respect to x gives us

$$1 = \frac{d}{dx}[\sinh(y)] = \frac{dy}{dx}\frac{d}{dy}[\sinh(y)] \qquad \text{(from the chain rule)}$$

and therefore $$1 = \frac{dy}{dx}\cosh(y)$$

which gives us $$\frac{dy}{dx} = \frac{1}{\cosh(y)}$$

We need the derivative in terms of x, so we use the identity of Equation 43

$$\cosh^2(x) - \sinh^2(x) = 1 \tag{Eqn 43}$$

to write

$$\cosh(y) = \sqrt{1+\sinh^2(y)} = \sqrt{1+x^2}$$

(Notice that the positive square root is taken in this expression because $\cosh(y) \geq 1$.)

We can now see that $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{1+x^2}}$ and therefore $\dfrac{d}{dx}\operatorname{arcsinh}(x) = \dfrac{1}{\sqrt{x^2+1}}$

(b) Differentiating the identity $\arcsin(x) = \log_e\left(x + \sqrt{x^2 + 1}\right)$

with respect to x, and using the fact that $\dfrac{d}{dx}\log_e(f(x)) = \dfrac{f'(x)}{f(x)}$,

we have

$$\frac{d}{dx}\operatorname{arcsinh}(x) = \frac{d}{dx}\log_e\left(x + \sqrt{x^2 + 1}\right)$$

$$= \frac{1}{x + \sqrt{x^2 + 1}}\left(1 + 2x \times \frac{1}{2} \times \frac{1}{\sqrt{x^2 + 1}}\right) = \frac{1}{\sqrt{x^2 + 1}}$$

Question T14

Show that $\dfrac{d}{dx}\operatorname{arctanh}(x) = \dfrac{1}{1 - x^2}$ by using:

(a) the known derivative of tanh (x);

(b) the identity for arctanh (x) in terms of the logarithmic function.

Example 5

Find $\dfrac{d}{dx}\operatorname{sech}(x)$

Solution

$$\frac{d}{dx}\operatorname{sech}(x) = \frac{d}{dx}[\cosh(x)]^{-1} = -[\cosh(x)]^{-2}\,\frac{d}{dx}\cosh(x) = -\frac{\sinh(x)}{\cosh^2(x)} = -\operatorname{sech}(x)\tanh(x)$$ ❏

Question T15

Find $\dfrac{d}{dx}\operatorname{cosech}(x)$ ❏

Example 6

Find $\dfrac{d}{dx}\sinh^3(2x)$

Solution

$$\frac{d}{dx}\sinh^3(2x) = 3\sinh^2(2x)\frac{d}{dx}\sinh(2x) = 6\sinh^2(2x)\cosh^2(2x) = 3\sinh(2x)\sinh(4x)$$

❏

Question T16

Find $\dfrac{d}{dx}\cosh^5(3x)$ ❏

Example 7

Find $\dfrac{d}{dx}[\sinh(x)\tanh(x)]$

Solution
$$\begin{aligned}\frac{d}{dx}[\sinh(x)\tanh(x)] &= \frac{d}{dx}\sinh(x)\tanh(x) + \sinh(x)\frac{d}{dx}\tanh(x)\\ &= \cosh(x)\tanh(x) + \sinh(x)\,\text{sech}^2(x)\\ &= \sinh(x)[1 + \text{sech}^2(x)]\end{aligned}$$ ❑

Question T17

Find $\dfrac{d}{dx}[\cosh(x)\coth(x)]$ ❑

Example 8

Find $\dfrac{d}{dx}\sinh(3x^2+1)$

Solution $$\frac{d}{dx}\sinh(3x^2+1) = \cosh(3x^2+1)\frac{d}{dx}(3x^2+1) = 6x\cosh(3x^2+1)$$ ❏

Question T18

Find $\dfrac{d}{dx}\cosh(5x^4+1)$ ❏

Example 9

Find $\dfrac{d}{dx}\left(\exp[\cosh(5x)]\right)$

Solution $$\frac{d}{dx}\left(\exp[\cosh(5x)]\right) = \exp[\cosh(5x)]\frac{d}{dx}[\cosh(5x)] = 5\exp[\cosh(5x)]\sinh(5x)$$ ❏

Question T19

Find $\dfrac{d}{dx}\left(\exp[\sinh^2(x)]\right)$ ❏

Example 10

Find $\dfrac{d}{dx}\log_{\rm e}[\cosh(x)]$

Solution

$$\frac{d}{dx}\log_{\rm e}[\cosh(x)] = \frac{1}{\cosh(x)} \times \frac{d}{dx}\cosh(x) = \frac{\sinh(x)}{\cosh(x)} = \tanh(x)$$

❏

Question T20

Find $\dfrac{d}{dx}\log_{\rm e}[\sinh(x)]$ ❏

Example 11 A heavy cable is suspended between two pylons of the same height, as shown schematically in Figure 16. ☞

It can be shown that the *y*-coordinate of the cable satisfies the (differential) equation

$$\frac{d^2y}{dx^2} = \frac{\rho g}{T_0}\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

where T_0 is the tension in the cable at $x = 0$, ρ is the density per unit length of the cable, and g is the magnitude of the acceleration due to gravity.
Show that

$$y = \frac{T_0}{\rho g}\cosh\left(\frac{\rho g x}{T_0}\right)$$

is a solution to this equation.

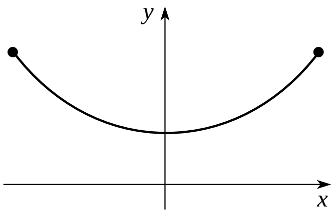

Figure 16 A heavy cable suspended between two points (such as two pylons).

Solution If $y = \dfrac{T_0}{\rho g}\cosh\left(\dfrac{\rho g x}{T_0}\right)$ then $\dfrac{dy}{dx} = \sinh\left(\dfrac{\rho g x}{T_0}\right)$

and $\dfrac{d^2y}{dx^2} = \dfrac{\rho g}{T_0}\cosh\left(\dfrac{\rho g x}{T_0}\right)$ which is the left-hand side of the differential equation.

Using the identity $\cosh^2(x) - \sinh^2(x) = 1$

we see that the right-hand side of the differential equation can be written as

$$\frac{\rho g}{T_0}\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \frac{\rho g}{T_0}\sqrt{1+\sinh^2\left(\frac{\rho g x}{T_0}\right)} = \frac{\rho g}{T_0}\cosh\left(\frac{\rho g x}{T_0}\right)$$

which verifies that $y = \dfrac{T_0}{\rho g}\cosh\left(\dfrac{\rho g x}{T_0}\right)$ is indeed a solution. ❏

6 Closing items

6.1 Module summary

1 There are *hyperbolic functions* corresponding to all the trigonometric functions, and though they are similar in some respects they are very different in others. In particular, the hyperbolic functions are not periodic, and, $\sinh(x)$ and $\cosh(x)$ are not constrained to lie between −1 and 1. Hyperbolic functions are named by appending '*h*' to the corresponding trigonometric function.

2 $\sinh(x)$ and $\cosh(x)$ are defined by

$$\sinh(x) = \frac{e^x - e^{-x}}{2} \qquad \text{(Eqn 3)}$$

$$\cosh(x) = \frac{e^x + e^{-x}}{2} \qquad \text{(Eqn 4)}$$

3 $\tanh(x)$ is defined by

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}} \qquad \text{(Eqn 5)}$$

4 A hyperbola can be defined by means of the parametric equations

$$x = a\cosh(s) \qquad \text{(Eqn 6)}$$

$$y = b\sinh(s) \qquad \text{(Eqn 7)}$$

5 The Taylor expansions of $\sinh(x)$ and $\cosh(x)$ about $x = 0$ are

$$\sinh(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} = \frac{x}{1!} + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \frac{x^9}{9!} + \dots \qquad \text{(Eqn 11)}$$

$$\cosh(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \frac{x^8}{8!} + \dots \qquad \text{(Eqn 13)}$$

6 The first few terms of the Taylor expansion of $\tanh(x)$ about $x = 0$ are

$$\tanh(x) = x - \frac{x^3}{3} + \frac{2x^5}{15} - \frac{17x^7}{315} + \dots \qquad \text{(Eqn 15)}$$

7 The derivatives of the two basic hyperbolic functions are

$$\frac{d}{dx}\sinh(x) = \cosh(x) \qquad \text{(Eqn 17)}$$

$$\frac{d}{dx}\cosh(x) = \sinh(x) \qquad \text{(Eqn 18)}$$

8 The trigonometric and hyperbolic functions are related by

$$\sinh(ix) = i\sin(x) \qquad \text{(Eqn 22)}$$

$$\cosh(ix) = \cos(x) \qquad \text{(Eqn 23)}$$

$$\sin(ix) = i\sinh(x)$$

$$\cos(ix) = \cosh(x)$$

9 *Reciprocal hyperbolic functions* are defined by

$$\operatorname{cosech}(x) = \frac{1}{\sinh(x)} \quad \text{provided } x \neq 0 \qquad \text{(Eqn 24)}$$

$$\operatorname{sech}(x) = \frac{1}{\cosh(x)} \quad \text{for all } x \qquad \text{(Eqn 25)}$$

$$\coth(x) = \frac{1}{\tanh(x)} \quad \text{provided } x \neq 0 \qquad \text{(Eqn 26)}$$

10 *Inverse hyperbolic functions* are defined by

$$\operatorname{arcsinh}(\sinh(x)) = x \qquad \text{(Eqn 30)}$$

$$\operatorname{arccosh}(\cosh(x)) = x \quad \text{for } x \geq 0 \qquad \text{(Eqn 32b)}$$

$$\operatorname{arctanh}(\tanh(x)) = x \qquad \text{(Eqn 31)}$$

11 The inverse hyperbolic functions can be written in terms of the logarithmic function

$$\operatorname{arcsinh}(x) = \log_{\mathrm{e}}\left(x + \sqrt{x^2 + 1}\right) \qquad \text{(Eqn 36)}$$

$$\operatorname{arccosh}(x) = \log_{\mathrm{e}}\left(x + \sqrt{x^2 - 1}\right) \quad \text{for } x \ge 1 \qquad \text{(Eqn 37)}$$

$$\operatorname{arctanh}(x) = \frac{1}{2}\log_{\mathrm{e}}\left(\frac{1+x}{1-x}\right) \quad \text{for } -1 < x < 1 \qquad \text{(Eqn 38)}$$

12 There are a large number of *hyperbolic function identities*, such as

$$\cosh^2(x) - \sinh^2(x) = 1 \qquad \text{(Eqn 43)}$$

The most important are listed in Section 4.

6.2 Achievements

Having completed this module, you should be able to:

A1 Define the terms that are emboldened and flagged in the margins of this module.

A2 Relate the trigonometric and hyperbolic functions by means of the complex number, i.

A3 Sketch graphs for the more common hyperbolic functions

A4 Recognize the more common identities for the hyperbolic functions and use them to solve mathematical and physical problems.

A5 Differentiate the hyperbolic functions.

A6 Derive, identify and use the Taylor series for the hyperbolic functions.

Study comment You may now wish to take the *Exit test* for this module which tests these Achievements. If you prefer to study the module further before taking this test then return to the *Module contents* to review some of the topics.

6.3 Exit test

Study comment Having completed this module, you should be able to answer the following questions, each of which tests one or more of the *Achievements*.

Question E1

(*A2*) Use the definitions of trigonometric and hyperbolic functions in terms of the exponential function to show that: (a) $\tanh(x) = -i\tan(ix)$, (b) $\operatorname{cosech}(x) = i\operatorname{cosec}(ix)$.

Question E2

(*A4*) Use the double-argument identities to show that

(a) $\cosh^2\left(\dfrac{x}{2}\right) = \dfrac{1}{2}\left[1 + \cosh(x)\right]$, (b) $\sinh^2\left(\dfrac{x}{2}\right) = \dfrac{1}{2}\left[-1 + \cosh(x)\right]$.

Question E3

($A5$) Show that $y(t) = A\sinh(\omega t) + B\cosh(\omega t)$

is a solution of $\dfrac{d^2y(t)}{dt^2} = \omega^2 t$. If $y(0) = 4$ and $y'(0) = 1$, find A and B.

Question E4

($A4$) Suppose that we are given the values of A and B. Use the identities given in Section 4 to find values of C and y (in terms of A and B) such that

$$A\sinh(x) + B\cosh(x) = C\sinh(x + y)$$

Explain any restrictions on the possible values for A and B.

Question E5

(*A3*) From the graphs given in this module, which hyperbolic functions would you expect to have a domain which excludes the point $x = 0$. Confirm your suspicions by looking at algebraic expressions for the functions that you have picked out. (You should include the reciprocal and inverse hyperbolic functions in your discussion.)

Question E6

(*A4*) Starting from the definition of $\sinh(x)$ in terms of the exponential function, show that

$$\operatorname{arcsinh}(x) = \log_e\left(x + \sqrt{x^2 + 1}\right)$$

Question E7

(*A5*) Use the known derivative of $\tanh(x)$ to express $\dfrac{d}{dx}\log_e[\tanh(x)]$ in terms of hyperbolic functions.

Question E8

($A5$) Find $\dfrac{d}{dx}\tanh^3(2x)$.

Question E9

($A4$ and $A6$) Use the series for $\sinh(x)$ and $\cosh(x)$ to find a first estimate for a solution of the equation

$$\sinh(x) = 0.01\cosh(x)$$

then use the logarithmic form of $\tanh(x)$ to find a more accurate value for the solution.

Study comment This is the final *Exit test* question. When you have completed the *Exit test* go back to Subsection 1.2 and try the *Fast track questions* if you have not already done so.

If you have completed **both** the *Fast track questions* and the *Exit test*, then you have finished the module and may leave it here.