Module M5.3 Techniques of integration

Exit module

1 Opening items

1.1 Module introduction

You should already know that obtaining the indefinite integral of a function $f(x)$ amounts to finding a function $F(x)$ whose derivative $\dfrac{dF}{dx}$ is equal to $f(x)$, so that, for example, we know that $\int \cos x\, dx = \sin x + C$ precisely because $\dfrac{d}{dx}(\sin x + C) = \cos x$. Every time we manage to differentiate a specific function $F(x)$ to obtain the answer $\dfrac{dF}{dx} = f(x)$ we automatically discover the solution to a corresponding problem in integration, namely 'what is the indefinite integral $\int f(x)\, dx$?' Integrals of simple functions, i.e. those that can be easily recognized as the derivatives of other functions, can quickly be found in this way. For example, to find $\int \cos(2x)\, dx$, you might guess that the result was something like $\sin(2x) + C$. On differentiating $\sin(2x) + C$, you would find that the derivative was in fact $2\cos(2x)$. So your initial guess has to be adjusted by a factor of $\frac{1}{2}$, and the answer is $\int \cos(2x)\, dx = \dfrac{1}{2}\sin(2x) + C$.

Whereas there are simple rules that enable us to differentiate almost any function we meet, it is a sad fact of life that there are no such rules for integration. This is not to say that there aren't methods for finding integrals, there are plenty; it is just that these methods do not always work, and, what is more, there are many simple looking integrals for which no method could possibly work. For example, it is not possible to express the integral $\int e^{x^2}\,dx$ in terms of the elementary functions, sine, cosine, exp and so on.

The methods discussed in this module will enable you to find many useful integrals, but all integration methods ultimately amount to recognizing the integrand $f(x)$ as the derivative of some function. Indeed, many simple integrals, such as $\int \cos(2x)\,dx$, can be found by making an intelligent guess at the function $F(x)$, then differentiating your guess to see if it gives $f(x)$, and, if not, adjusting it appropriately (this may involve, for example, multiplication by a constant). Of course there are many functions $f(x)$ for which it would be very difficult to guess the form of the indefinite integral straight away. In this module, we discuss two standard methods of integration which can sometimes be used to manipulate an integral so as to bring it into a form where intelligent guesswork (or use of a table of standard integrals) will give you the final answer.

The first of these methods, *integration by parts*, is based on the *product rule* for differentiation, and is often useful when the function $f(x)$ to be integrated is the product of two simpler functions. Thus, for example, it enables you to integrate functions like xe^x.

The second method, *integration by substitution*, involves rewriting the integral $\int f(x)\,dx$ in terms of a new variable $y = g(x)$, and is based on the *chain rule* for differentiation. It will often enable you to turn a really unpleasant-looking integral — such as $\int x(1 - x^2)^{5/2}\,dx$ — into one which can be evaluated more easily.

Study comment Having read the introduction you may feel that you are already familiar with the material covered by this module and that you do not need to study it. If so, try the *Fast track questions* given in Subsection 1.2. If not, proceed directly to *Ready to study?* in Subsection 1.3.

1.2 Fast track questions

Study comment Can you answer the following *Fast track questions*?. If you answer the questions successfully you need only glance through the module before looking at the *Module summary* (Subsection 5.1) and the *Achievements* listed in Subsection 5.2. If you are sure that you can meet each of these achievements, try the *Exit test* in Subsection 5.3. If you have difficulty with only one or two of the questions you should follow the guidance given in the answers and read the relevant parts of the module. However, *if you have difficulty with more than two of the Exit questions you are strongly advised to study the whole module*.

Question F1

Find the indefinite integrals

(a) $\int x^2 e^{-x}\, dx$ (b) $\int e^{-x} \sin(3x)\, dx$.

Question F2

Find the indefinite integral $\int \sin x(1+\cos x)^4\,dx$.

Question F3

Evaluate the integral $\int_0^1 \frac{x}{\sqrt{1+x}}\,dx$ to three decimal places.

Study comment Having seen the *Fast track questions* you may feel that it would be wiser to follow the normal route through the module and to proceed directly to *Ready to study?* in Subsection 1.3.

Alternatively, you may still be sufficiently comfortable with the material covered by the module to proceed directly to the *Closing items*.

1.3 Ready to study?

Study comment In order to study this module, you will need to be familiar with the following terms: *chain rule*, *constant of integration*, *definite integral*, *derivative*, *function of a function*, *fundamental theorem of calculus*, *indefinite integral*, *integrand*, *inverse derivative*, *inverse trigonometric functions*, *product rule*, and *trigonometric identities*. If you are uncertain of any of these terms, you can review them now by referring to the *Glossary* which will indicate where in *FLAP* they are developed. In addition, you will need to have a good knowledge of differentiation (including the use of the *product* and *chain* rules). You should be able to recognize an *indefinite integral* as an *inverse derivative*, and you need to be familiar with the integrals of simple functions such as x^n, $\sin x$, $\cos x$, e^x. You should also be able to find integrals of functions like $3x$ and $\cos(2x)$ using intelligent guesswork, and you should be familiar with the procedure of evaluating *definite integrals*.

The following *Ready to study questions* will allow you to establish whether you need to review some of these topics before embarking on this module.

Question R1

Use the product rule to find $\dfrac{dy}{dx}$ in each of the following cases:

(a) $y = x^6 \log_e x$ (b) $y = e^x \cos(2x)$

Question R2

Use the chain rule to find $f'(x)$ when

(a) $f(x) = \cos(x^4)$ (b) $f(x) = \log_e(1 + x^2)$

Question R3

Differentiate the inverse trigonometric function $\arcsin(2x)$. Use your answer to find the indefinite integral of $\dfrac{1}{\sqrt{1 - 4x^2}}$.

Question R4

Find the indefinite integrals of

(a) $x^{-4/3}$ (b) $\sin(2x) + 3\cos x$ (c) $\exp(x/4)$

Question R5

Evaluate the definite integral $\displaystyle\int_1^2 x^2\,dx$.

Question R6

Without using a calculator, find the possible values of

(a) $\cos(x)$ if $\sin(x) = \dfrac{1}{3}$, (b) $\tan(x)$ if $\sec(x) = \sqrt{5}$.

Question R7

If $f(x) = \dfrac{(x+1)^2}{x+3}$ and $y = x + 3$, express $f(x)$ in terms of y.

2 Integration by parts

2.1 The method deduced from the differentiation of a product

In this subsection, we shall look at the product rule for differentiation, and derive from it the important integration technique known as **integration by parts**. We begin with an example. ☞

Example 1

Find the integral $\int x\cos x\,dx$.

Solution As you will see shortly, we can find this integral by looking at the derivative of the product $x\sin x$, the reason being that the integrand $x\cos x$, appears in this derivative. To be precise, on using the product rule to differentiate $x\sin x$, we find that

$$\frac{d}{dx}(x\sin x) = \sin x + x\cos x$$

An alternative way of expressing this result is to say that $x \sin x$ is the *indefinite integral* of $(\sin x + x \cos x)$, so that we can write

$$\int (\sin x + x \cos x)\, dx = x \sin x + C$$ ☞

where C is an arbitrary constant. (☞) Using the rule for the sum of two integrals, it follows that

$$\int \sin x\, dx + \int x \cos x\, dx = x \sin x + C$$

and a little rearrangement gives us

$$\int x \cos x\, dx = x \sin x - \int \sin x\, dx + C$$

Using the fact that the indefinite integral of $(-\sin x)$ is $\cos x + C$, we obtain the result

$$\int x \cos x\, dx = x \sin x + \cos x + C$$ ☞ ❑

Thus, by looking at the derivative of the product $x \sin x$, we have been able to find $\int x \cos x\, dx$, the indefinite integral of another product.

We will show you shortly how to generalize this method, but first you should try the following question.

Question T1

Find the derivative of the product xe^x, and use it to find $\int xe^x\,dx$. ❏

We have seen that looking at *derivatives* of products has enabled us to find some *integrals* that also involve products; however, you may feel that the method is rather 'hit or miss'. The question is:

What product should we differentiate in order to find the integral of a given product?

In order to answer this question, we must first investigate the derivative of two *arbitrary* functions $F(x)$ and $g(x)$. From the product rule, we have

$$\frac{d}{dx}\big(F(x)g(x)\big) = \frac{dF(x)}{dx}g(x) + F(x)\frac{dg(x)}{dx} \tag{1}$$

which means that $F(x)g(x)$ is an inverse derivative, or indefinite integral, of the function

$$\frac{dF(x)}{dx}g(x) + F(x)\frac{dg(x)}{dx}$$

on the right-hand side of Equation 1.

So we may write

$$\int\left[\frac{dF(x)}{dx}g(x)+F(x)\frac{dg(x)}{dx}\right]dx = F(x)g(x)+C$$

i.e. $$\int\frac{dF}{dx}g(x)\,dx+\int F(x)\frac{dg}{dx}\,dx = F(x)g(x)+C$$

and, rearranging this equation (and absorbing the arbitrary constant C into the integrals), it follows that

$$\int\frac{dF}{dx}g(x)\,dx = F(x)g(x)-\int F(x)\frac{dg}{dx}\,dx$$

When applying this equation, we are usually given the left-hand side, i.e. we are given $\dfrac{dF}{dx}$ and $g(x)$, and for this reason it is normal to represent the function $\dfrac{dF}{dx}$ by $f(x)$, so that the equation becomes

$$\boxed{\int f(x)g(x)\,dx = F(x)g(x)-\int F(x)\frac{dg}{dx}\,dx \quad \text{where} \quad \frac{dF}{dx} = f(x) \qquad (2)}$$

✦ Use Equation 2,

$$\int f(x)g(x)\,dx = F(x)g(x) - \int F(x)\frac{dg}{dx}\,dx \quad \text{where} \quad \frac{dF}{dx} = f(x) \qquad \text{(Eqn 2)}$$

with $f(x) = \cos x$ and $g(x) = x$, to find $\int x \cos x\,dx$.

As you can see, Equation 2 converts the problem of finding $\int f(x)g(x)\,dx$ into that of finding $\int F(x)\frac{dg}{dx}\,dx$. In order for this equation to be useful in practice, there are clearly two necessary requirements:

1 We need to be able to find $F(x)$, an indefinite integral of $f(x)$.

2 The integral on the right-hand side of Equation 2, i.e. $\int F(x)\frac{dg}{dx}\,dx$, should be easier to find than our original integral, $\int f(x)g(x)\,dx$.

The technique of using Equation 2 to find $\int f(x)g(x)\,dx$ is called the method of *integration by parts*. Some questions and examples should help to make this method clearer, and will also illustrate the importance of the two requirements just mentioned.

Example 2

Use Equation 2

$$\int f(x)g(x)\,dx = F(x)g(x) - \int F(x)\frac{dg}{dx}\,dx \quad \text{where} \quad \frac{dF}{dx} = f(x) \qquad \text{(Eqn 2)}$$

to find the indefinite integral $\int x\sin x\,dx$.

Solution We start by choosing functions $f(x)$ and $g(x)$ so that our integral $\int x\sin x\,dx$ is the integral $\int f(x)g(x)\,dx$ appearing on the left-hand side of Equation 2. There are two possibilities: we can either choose

$f(x) = \sin x$ and $g(x) = x$

or $f(x) = x$ and $g(x) = \sin x$.

Since we know how to integrate both x and $\sin x$, the requirement that $f(x)$ should be a function that is easy to integrate does not help us in our choice. Let us, therefore, decide to take $f(x) = \sin x$ and $g(x) = x$, and see what happens. With this choice of $f(x)$ and $g(x)$, we have $F(x) = -\cos x$ and $\dfrac{dg}{dx} = 1$.

$$\int f(x)\,g(x)\,dx = F(x)\,g(x) - \int F(x)\,\frac{dg}{dx}\,dx \quad \text{where} \quad \frac{dF}{dx} = f(x) \qquad \text{(Eqn 2)}$$

Substituting our expressions for $f(x)$, $g(x)$, $F(x)$ and $\dfrac{dg}{dx}$ into Equation 2, we obtain

$$\int \overset{\overset{g(x)}{\downarrow}}{x}\,\underset{\underset{f(x)}{\uparrow}}{\sin x}\,dx = \overset{\overset{g(x)}{\downarrow}}{x}(\underset{\underset{F(x)}{\uparrow}}{-\cos x}) - \int \overset{\overset{\frac{dg}{dx}}{\downarrow}}{1}(\underset{\underset{F(x)}{\uparrow}}{-\cos x})\,dx$$

so that $\displaystyle\int x \sin x\,dx = -x\cos x + \int \cos x\,dx$

and since we know that the integral of $\cos x$ is $\sin x + C$, the final result is

$$\int x \sin x\,dx = -x\cos x + \sin x + C \qquad (3)$$ ❑

Let us see what would have happened in this example if we had chosen $f(x) = x$ and $g(x) = \sin x$.

In that case, $F(x)$, the integral of x, is $\dfrac{x^2}{2}$ while $\dfrac{dg}{dx}$, the derivative of $\sin x$, is $\cos x$.

Substituting these expressions into Equation 2

$$\int f(x)\,g(x)\,dx = F(x)\,g(x) - \int F(x)\frac{dg}{dx}\,dx \quad \text{where} \quad \frac{dF}{dx} = f(x) \qquad \text{(Eqn 2)}$$

we obtain

$$\int \underset{\underset{f(x)}{\uparrow}}{x}\, \overset{\overset{g(x)}{\downarrow}}{\sin x}\, dx = \underset{\underset{F(x)}{\uparrow}}{\frac{x^2}{2}}\, \overset{\overset{g(x)}{\downarrow}}{(\sin x)} - \int \underset{\underset{F(x)}{\uparrow}}{\frac{x^2}{2}}\, \overset{\overset{\frac{dg}{dx}}{\downarrow}}{(\cos x)}\, dx \qquad (4)$$

As you can see, the integral appearing on the right-hand side of Equation 4 is no easier to find than the one we started with (in fact, it is harder!).

As we mentioned earlier, when we apply Equation 2,

$$\int f(x)\,g(x)\,dx = F(x)\,g(x) - \int F(x)\frac{dg}{dx}\,dx \quad \text{where} \quad \frac{dF}{dx} = f(x) \qquad \text{(Eqn 2)}$$

the choice of $f(x)$ and $g(x)$, in any particular situation, must be such that the integral $\displaystyle\int F(x)\frac{dg}{dx}\,dx$ is *easier* to find than the original integral, $\int f(x)\,g(x)\,dx$.

If, in using integration by parts, you find that you are left with a more complicated integral than the one you started with, this probably means that you have chosen $f(x)$ and $g(x)$ the wrong way round. Do not despair — simply change them over and start again.

✦

What would be the right choice of $f(x)$ and $g(x)$ to find the indefinite integral $\int x\mathrm{e}^{-x}\,dx$?

Example 3

Use Equation 2

$$\int f(x)g(x)\,dx = F(x)g(x) - \int F(x)\frac{dg}{dx}\,dx \quad \text{where} \quad \frac{dF}{dx} = f(x) \qquad \text{(Eqn 2)}$$

to find $\int x \log_e(x)\,dx$.

Solution Here we could either choose

$f(x) = x$, and $g(x) = \log_e x$

or $f(x) = \log_e x$ and $g(x) = x$

Remember that we need to be able to integrate $f(x)$ in order to obtain $F(x)$. It is much easier to integrate x than it is to integrate $\log_e x$; so we choose $f(x) = x$ and $g(x) = \log_e x$. Then $F(x) = \dfrac{1}{2}x^2$ and $\dfrac{dg}{dx} = \dfrac{1}{x}$. Substituting our expressions for $f(x)$, $g(x)$, $F(x)$ and $\dfrac{dg}{dx}$ into Equation 2, we obtain

$$\int x \log_e x\,dx = \frac{1}{2}x^2 \log_e x - \int\left(\frac{x^2}{2}\right)\left(\frac{1}{x}\right)dx = \frac{1}{2}x^2 \log_e x - \int \frac{1}{2}x\,dx$$

$$\int x \log_e x\,dx = \frac{1}{2}x^2 \log_e x - \int \left(\frac{x^2}{2}\right)\left(\frac{1}{x}\right)dx = \frac{1}{2}x^2 \log_e x - \int \frac{1}{2}x\,dx$$

Since the integral of $\frac{1}{2}x$ is $\frac{1}{2}\left(\frac{1}{2}x^2\right) + C = \frac{1}{4}x^2 + C$ we obtain as our final answer

$$\int x \log_e x\,dx = \frac{1}{2}x^2 \log_e x - \frac{1}{4}x^2 + C \quad ❏$$

Question T2

Use Equation 2

$$\int f(x)\,g(x)\,dx = F(x)\,g(x) - \int F(x)\frac{dg}{dx}\,dx \quad \text{where} \quad \frac{dF}{dx} = f(x) \qquad \text{(Eqn 2)}$$

to find the indefinite integral $\int x^2 \log_e x\,dx$. ❏

Definite integrals

So far, we have used integration by parts to find indefinite integrals, but the method can just as easily be used to evaluate definite integrals. Applying the *fundamental theorem of calculus* to Equation 1,

$$\frac{d}{dx}\left[F(x)g(x)\right] = \frac{dF}{dx}g(x) + F(x)\frac{dg}{dx} \qquad \text{(Eqn 1)}$$

it quickly follows that

$$\int_a^b \left[\frac{dF}{dx}g(x) + F(x)\frac{dg}{dx}\right]dx = \left[F(x)g(x)\right]_a^b$$

This can be rewritten as

$$\int_a^b \frac{dF}{dx}g(x)\,dx + \int_a^b F(x)\frac{dg}{dx}\,dx = \left[F(x)\,g(x)\right]_a^b$$

Rearranging this equation, and introducing again the notation $f(x)$ for $\dfrac{dF}{dx}$, we arrive at the following formula:

$$\int_a^b f(x)\,g(x)\,dx = \left[F(x)\,g(x)\right]_a^b - \int_a^b F(x)\frac{dg}{dx}\,dx \tag{5}$$

where $\dfrac{dF}{dx} = f(x)$

Example 4

Evaluate the integral $\int_1^2 x\mathrm{e}^{2x}\,dx$, correct to four decimal places.

Solution We use Equation 5,

$$\int_a^b f(x)\,g(x)\,dx = \left[F(x)\,g(x)\right]_a^b - \int_a^b F(x)\frac{dg}{dx}\,dx \qquad \text{(Eqn 5)}$$

taking $f(x) = \mathrm{e}^{2x}$ and $g(x) = x$, then $\dfrac{dg}{dx} = 1$ and

$F(x) = \dfrac{1}{2}\mathrm{e}^{2x}$. Substituting into Equation 5 we obtain

$$\int_1^2 x\mathrm{e}^{2x}\,dx = \left[\frac{1}{2}x\,\mathrm{e}^{2x}\right]_1^2 - \frac{1}{2}\int_1^2 \mathrm{e}^{2x}\,dx = (54.598150 - 3.694528) - \frac{1}{2}(27.299075 - 3.694528)$$

$$= 39.1013 \quad ❑$$

Question T3

Evaluate the integral $\displaystyle\int_0^1 x\text{e}^{3x}\,dx$, correct to four decimal places. ❏

2.2 Integrating by parts more than once

If you look back over the integrals appearing in Subsection 2.1, both in the text and in the *Text questions*, you will see that many of them are of the form $\int x\,f(x)\,dx$ where $f(x)$ is a function that is straightforward to integrate, such as an exponential, a sine or cosine. These are, perhaps, the easiest integrals to find using integration by parts, and it is not hard to see why. Referring to Equation 2,

$$\int f(x)\,g(x)\,dx = F(x)\,g(x) - \int F(x)\frac{dg}{dx}\,dx \quad \text{where} \quad \frac{dF}{dx} = f(x) \qquad \text{(Eqn 2)}$$

we see that if $g(x) = x$, then $\dfrac{dg}{dx} = 1$ and so an integral of this form becomes

$$\int x\,f(x)\,dx = x\,F(x) - \int F(x)\,dx \qquad (6)$$

Provided that we can not only integrate $f(x)$ to find $F(x)$, but also integrate $F(x)$ itself, then the right-hand side of Equation 6 is easy to find. This will certainly be the case if $f(x)$ is an exponential function, a sine or cosine — for such functions can be integrated readily, to give functions of the same form which can also be integrated.

What would happen if instead of $\int x\,f(x)\,dx$, we had to find an integral of the form $\int x^2\,f(x)\,dx$?

Example 5

Find $\int x^2 \cos x\, dx$.

Solution We will try taking $f(x) = \cos x$ and $g(x) = x^2$. (Remember that if this should prove unhelpful, we can always make the opposite choice, and see whether that works better.) With this choice, we have $F(x) = \sin x$ and $\dfrac{dg}{dx} = 2x$. Substituting these expressions into Equation 2, we find

$$\int x^2 \cos x\, dx = x^2 \sin x - \int 2x \sin x\, dx = x^2 \sin x - 2\int x \sin x\, dx \qquad (7)$$

At first sight, the integral appearing on the right-hand side (i.e. $\int x \sin x\, dx$) may not look much better than the one we started with, $\int x^2 \cos x\, dx$, but you may recognize it as an integral that you know how to do by *integration by parts*! In fact, we have already found it, in Example 2. Equation 3 tells us that

$$\int x \sin x\, dx = -x \cos x + \sin x + C \qquad \text{(Eqn 3)}$$

and substituting this into Equation 7, we find

$$\int x^2 \cos x\, dx = x^2 \sin x - 2(-x \cos x + \sin x + C) = x^2 \sin x + 2x \cos x - 2 \sin x + C$$ ❏

Any integral of the form $\int x^n f(x)\,dx$ containing the product of some positive integer power n of x and a function $f(x)$ which is an exponential, or a sine or cosine (and so can easily be integrated as many times as is necessary), can be found by repeated application of the method of integration by parts. Note that — as our notation suggests — it is important to take x^n to be $g(x)$, for it is this that ensures that every time we integrate by parts, we are left with a similar integral, but one in which the power of x has been reduced by 1.

Question T4

Find the integral $\int x^2 e^{2x}\,dx$. ❏

2.3 Integrals of various functions, including $\log_e x$ and $e^x \sin x$

You now know how to use integration by parts to find any integral of the form $\int x^n\, f(x)\, dx$, where $f(x)$ is an exponential function, a sine or cosine function. In this subsection, we shall show you some clever tricks that will enable you to use integration by parts to find some other integrals. We will concentrate here on indefinite integrals, and so use Equation 2;

$$\int f(x)\, g(x)\, dx = F(x)\, g(x) - \int F(x) \frac{dg}{dx}\, dx \quad \text{where} \quad \frac{dF}{dx} = f(x) \qquad \text{(Eqn 2)}$$

it is equally easy to use these methods to evaluate definite integrals, using Equation 5. ☞

$$\int_a^b f(x)\, g(x)\, dx = \left[F(x)\, g(x)\right]_a^b - \int_a^b F(x) \frac{dg}{dx}\, dx \qquad \text{(Eqn 5)}$$

The first trick simply involves the realization that the function $f(x)$ in Equation 2 may be taken to be 1. This means that integration by parts, which is a technique that essentially applies to integrals of *products*, can in fact be applied to an integral of *any* function; we simply regard the integrand as the product of the original function and 1. This approach may often lead to integrals worse than the one we start with, but in some cases it is extremely useful, as in the following example.

Example 6

Find the integral $\int \log_e x\,dx$.

Solution Here we choose $f(x) = 1$, $g(x) = \log_e x$. Then we have $F(x) = x$, $\dfrac{dg}{dx} = \dfrac{1}{x}$.

Substituting these expressions for $f(x)$, $g(x)$, $F(x)$ and $\dfrac{dg}{dx} = g'(x)$ into Equation 2,

$$\int f(x)\,g(x)\,dx = F(x)\,g(x) - \int F(x)\frac{dg}{dx}\,dx \quad \text{where} \quad \frac{dF}{dx} = f(x) \qquad \text{(Eqn 2)}$$

we obtain $$\int \underbrace{1}_{f(x)} \times \overbrace{\log_e x}^{g(x)}\,dx = \underbrace{x}_{F(x)} \times \overbrace{\log_e x}^{g(x)} - \int \underbrace{x}_{F(x)}\,\overbrace{(1/x)}^{g'(x)}\,dx = x\log_e x - \int 1\,dx = x\log_e x - x + C$$

So our final answer is

$$\int \log_e x\,dx = x\log_e x - x + C \quad ❑$$

It is not always easy to know in advance whether this trick will work, but it is worth bearing it in mind as part of your repertoire of integration methods. Try applying it to the integral in the next question.

Question T5

Given that $\displaystyle\int \frac{x}{1+x^2}\,dx = \frac{1}{2}\log_{\text{e}}\left(1+x^2\right)+C$

find the integral $\int \arctan x\,dx$. ❏

The second clever trick that you will often find useful involves applying Equation 2 twice,

$$\int f(x)g(x)\,dx = F(x)g(x) - \int F(x)\frac{dg}{dx}\,dx \quad \text{where} \quad \frac{dF}{dx} = f(x) \qquad \text{(Eqn 2)}$$

this time in such a way as to obtain a simple algebraic equation for the unknown integral, which can then be solved. An example is the best way to present the method. We will let I denote the integral that we wish to evaluate, and for our example we will choose

$$I = \int e^x \sin(2x)\,dx$$

If we take

$$f(x) = \sin(2x) \text{ and } g(x) = e^x \tag{8}$$

then $F(x) = -\dfrac{1}{2}\cos(2x)$ and $\dfrac{dg}{dx} = e^x$.

Equation 2

$$\int f(x)\,g(x)\,dx = F(x)\,g(x) - \int F(x)\frac{dg}{dx}\,dx \quad \text{where} \quad \frac{dF}{dx} = f(x) \tag{Eqn 2}$$

then gives us

$$I = \int e^x \sin(2x)\,dx = e^x\left[-\frac{1}{2}\cos(2x)\right] - \int\left[-\frac{1}{2}\cos(2x)\right]e^x\,dx$$

i.e. $$I = -\frac{1}{2}e^x\cos(2x) + \frac{1}{2}\int e^x\cos(2x)\,dx \tag{9}$$

At this point, it may seem that integration by parts has failed; the integral remaining on the right-hand side of Equation 9

$$I = -\frac{1}{2}\mathrm{e}^x\cos(2x) + \frac{1}{2}\int \mathrm{e}^x\cos(2x)\,dx \qquad \text{(Eqn 9)}$$

is no simpler than the one we started with. However, let us see what happens when we apply integration by parts to this integral, $\int \mathrm{e}^x\cos(2x)\,dx$.

We will take $f(x) = \cos(2x)$ and $g(x) = \mathrm{e}^x$. ☞

Then we have $F(x) = \dfrac{1}{2}\sin(2x)$ and $\dfrac{dg}{dx} = \mathrm{e}^x$, so

$$\int \mathrm{e}^x\cos(2x)\,dx = \mathrm{e}^x\left[\frac{1}{2}\sin(2x)\right] - \int\left[\frac{1}{2}\sin(2x)\right]\mathrm{e}^x\,dx = \frac{1}{2}\mathrm{e}^x\sin(2x) - \frac{1}{2}\underbrace{\int \mathrm{e}^x\sin(2x)\,dx}_{\text{This is the integral } I} \qquad (10)$$

which contains the integral I with which we started!

Substituting Equation 10

$$\int e^x \cos(2x)\,dx = e^x\left[\frac{1}{2}\sin(2x)\right] - \int\left[\frac{1}{2}\sin(2x)\right]e^x\,dx \quad = \frac{1}{2}e^x\sin(2x) - \frac{1}{2}\int e^x\sin(2x)\,dx \qquad \text{(Eqn 10)}$$

into Equation 9, $$I = -\frac{1}{2}e^x\cos(2x) + \frac{1}{2}\int e^x\cos(2x)\,dx \qquad \text{(Eqn 9)}$$

we obtain $$I = -\frac{1}{2}e^x\cos(2x) + \frac{1}{2}\left[\frac{1}{2}e^x\sin(2x) - \frac{1}{2}\int e^x\sin(2x)\,dx\right]$$

so that $$I = -\frac{1}{2}e^x\cos(2x) + \frac{1}{4}e^x\sin(2x) - \frac{1}{4}I$$ ☞

We can now collect all the terms in *I* together on the left-hand side to give

$$\frac{5}{4}I = -\frac{1}{2}e^x\cos(2x) + \frac{1}{4}e^x\sin(2x) + C$$

i.e. $$I = \frac{4}{5}\left[-\frac{1}{2}e^x\cos(2x) + \frac{1}{4}e^x\sin(2x) + C\right]$$

$$\int e^x\sin(2x)\,dx = -\frac{2}{5}e^x\cos(2x) + \frac{1}{5}e^x\sin(2x) + C$$ ☞

This method, using integration by parts twice and then solving the resulting equation to obtain the required integral, will work for any product of an exponential function with either a sine or cosine function. For such cases, it does not matter whether you choose the exponential function to be $f(x)$ or $g(x)$, as long as you make the *same* choice in both your integrations by parts. If you do not, your resulting equation will simply reduce to $0 = 0$ — true, but not very useful!

Question T6

Find the integral $\int e^{2x} \cos x \, dx$. (Start by taking $f(x) = \cos x$ and $g(x) = e^{2x}$, then let $J = \int e^{2x} \cos x \, dx$.) ❏

It is sometimes possible to obtain a simple equation for your desired integral by integrating by parts just once, as in the next example.

Example 7

Find the integral $\int \cos^2 x\, dx$.

Solution If we let $f(x) = \cos x$ and $g(x) = \cos x$ then $F(x) = \sin x$ and $\dfrac{dg}{dx} = -\sin x$. Applying Equation 2 then gives us

$$\int \cos^2 x\, dx = \sin x \cos x + \int \sin^2 x\, dx \tag{11}$$

We can now use the trigonometric identity $\cos^2 x + \sin^2 x = 1$, so that $\sin^2 x = 1 - \cos^2 x$, in Equation 11, so that it becomes

$$\int \cos^2 x\, dx = \sin x \cos x + \int \left(1 - \cos^2 x\right) dx = \sin x \cos x + \int 1\, dx - \int \cos^2 x\, dx$$

Putting $\int 1\, dx = x + C$ and collecting all $\int \cos^2 x\, dx$ terms together on the left-hand side, we find

$$2\int \cos^2 x\, dx = \sin x \cos x + x + C$$

or finally, $$\int \cos^2 x\, dx = \frac{1}{2}\sin x \cos x + \frac{1}{2}x + C \quad ❏$$

Question T7

In the integral $\int \cos^4 x\, dx$, take $f(x) = \cos x$, $g(x) = \cos^3 x$, and show that

$$\int \cos^4 x\, dx = \frac{1}{4}\sin x \cos^3 x + \frac{3}{4}\int \cos^2 x\, dx\,.$$

(*Hint*: You will need to use the identity $\cos^2 x + \sin^2 x = 1$.) ❑

3 Integration by substitution

3.1 An introduction to the method of substitution

Evaluating an integral of the form $\int f(x)\,dx$ amounts to recognizing $f(x)$ as the derivative of some function $F(x)$. However, if $F(x)$ is a rather complicated function of x, perhaps a function of a function of x, it may not be at all easy to spot that $f(x)$ is its derivative.

✥

Use the chain rule to find the derivative of $\dfrac{2}{3}(1+x)^{3/2}$.

Since the derivative of $\dfrac{2}{3}(1+x)^{3/2}$ is $(1+x)^{1/2}$, it follows that

$$\int (1+x)^{1/2}\,dx = \frac{2}{3}(1+x)^{3/2} + C \tag{12}$$

This result is probably not one that would have been obvious to you if you had not first calculated the derivative of $\dfrac{2}{3}(1+x)^{3/2}$. In this section, we explain an integration method that can often be used to bring integrals like $\int(1+x)^{1/2}\,dx$ into a much simpler form, such that you will probably be able to evaluate it by inspection, or by reference to a table of standard integrals.

We will first show you how the method works, by applying it to the integral that we have just found, i.e. $\int(1+x)^{1/2}\,dx$, and our first step is to introduce a new variable y, equal to $1 + x$. The integrand $(1 + x)^{1/2}$, is then simply equal to $y^{1/2}$. We must now decide what to do with the 'dx' inside the integral sign. Since $y = 1 + x$ it follows that $\dfrac{dy}{dx} = 1$, and if $\dfrac{dy}{dx}$ could be regarded as the *quotient* of the two quantities dy and dx, we could rearrange this equation to read

$$dy = dx \tag{13}$$

Strictly speaking, $\dfrac{dy}{dx}$ is not a quotient, and the symbols dy and dx have not been defined to have any meaning on their own. However the procedure that we outline here can certainly be justified (though a proof of its veracity lies outside the scope of *FLAP* and properly belongs in a course on mathematical analysis).

We use Equation 13

$$dy = dx \qquad \text{(Eqn 13)}$$

to transform the integral $\int (1+x)^{1/2}\ dx$ into another integral written just in terms of y, and putting $dx = dy$ and $(1 + x)^{1/2} = y^{1/2}$, we find that

$$\int (1+x)^{1/2}\, dx = \int y^{1/2}\, dy = \frac{2}{3} y^{3/2} + C$$

Finally, our answer must be written in terms of x, and so we replace y by $(1 + x)$

on the right-hand side, to obtain

$$\int (1+x)^{1/2}\, dx = \frac{2}{3}(1+x)^{3/2} + C \qquad (14)$$

Although you may be worried by the use of Equation 13 in deriving Equation 14, you can see that the result is *correct* —it is exactly the same as Equation 12.

$$\int (1+x)^{1/2}\, dx = \frac{2}{3}(1+x)^{3/2} + C \qquad \text{(Eqn 12)}$$

The method used here to find the integral $\int (1+x)^{1/2}\,dx$ is known as **integration by substitution** because we substitute the new variable y into our unknown integral to obtain another integral that we hope will be easier to evaluate. Try the method that we used to derive Equation 14

$$\int (1+x)^{1/2}\,dx = \frac{2}{3}(1+x)^{3/2} + C \qquad \text{(Eqn 14)}$$

on the next question.

Question T8

Find the integral $\int (x-2)^{4/3}\,dx$. Check your answer by differentiation. ❏

Here is a slightly harder example of integration by substitution:

Example 8

Find the integral $\int x^2 \cos\left(x^3\right) dx$ by means of the substitution $y = x^3$.

Solution We note first that if $y = x^3$, then $\dfrac{dy}{dx} = 3x^2$. Again, treating dy and dx as though they were ordinary algebraic quantities, we rewrite this equation as $dy = 3x^2\,dx$. We can rearrange this equation further if we want to, and at this point, it is convenient to notice that the 'product' $x^2\,dx$ appears in the integral we are trying to find. So we write $x^2\,dx = \dfrac{1}{3}\,dy$. We can also write $\cos(x^3) = \cos y$ so that, on making the substitution $y = x^3$,

we find that $\displaystyle\int x^2 \cos\left(x^3\right) dx = \frac{1}{3}\int \cos y\,dy = \frac{1}{3}\sin y + C$

Finally, we substitute $y = x^3$ into the right-hand side of this result, to obtain an answer for our integral in terms of x

$$\int x^2 \cos\left(x^3\right) dx = \frac{1}{3}\sin\left(x^3\right) + C$$

You can easily check that this result is correct by differentiating it. ❏

We will now list the steps involved in the method of substitution. As we do so, we shall show how we applied these to the integral in Example 8, and this will also show you how you can set your working out when you come to do problems of this sort.

Method	Example: $\int x^2 \cos\left(x^3\right) dx$
Step 1 Decide what substitution $y = g(x)$ to try.	We make the substitution $y = x^3$.
Step 2 Calculate $\dfrac{dg}{dx}$ and write down the equation $dy = \dfrac{dg}{dx}\,dx$. It may be convenient to rearrange this equation.	Here, $\dfrac{dg}{dx} = 3x^2$, so $dy = 3x^2\,dx$. We rearrange this equation into the form $x^2\,dx = \dfrac{1}{3}\,dy$
Step 3 Make the substitution, to obtain an integral in terms of y.	$\int x^2 \cos\left(x^3\right) dx = \int \underbrace{\cos\left(x^3\right)}_{\cos y} \times \underbrace{x^2\,dx}_{\frac{1}{3}dy} = \dfrac{1}{3}\int \cos y\,dy$
Step 4 Integrate with respect to y.	$= \dfrac{1}{3}\sin y + C$
Step 5 Substitute $g(x)$ for y in order to obtaina final answer that is a function of x.	$= \dfrac{1}{3}\sin\left(x^3\right) + C$

Question T9

Use integration by substitution to find the integrals

(a) $\int x(5+2x^2)^{16}\,dx$, making the substitution $y = 5 + 2x^2$;

(b) $\int \cos^4(x)\sin(x)\,dx$, making the substitution $y = \cos x$. ❏

Much the hardest part of the method of integration by substitution is to know what substitution to make — after all, we begin with an integral that we cannot find, and the point of the method is to arrive in Step 3 at an integral which is easier, and this obviously depends on our choice of $g(x)$. Skill at picking a suitable substitution will come with practice and experience, and in Subsections 3.2 and 3.3, we will present you with several examples of particular substitutions which will work for certain types of integrals.

3.2 Integrals of 'a function and its derivative'

In this subsection, we shall introduce you to a particular class of integrals that can always be simplified by a suitable substitution. We will start with an example.

Example 9

Find the integral $\int x\sqrt{1+x^2}\,dx$.

Solution *Step 1* We will try the substitution $y = g(x) = 1 + x^2$.

Step 2 $\dfrac{dg}{dx} = 2x$, so $dy = 2x\,dx$. As $x\,dx$ appears in the integral, we rearrange this equation to read

$$x\,dx = \frac{1}{2}\,dy.$$

Step 3 We also have $\sqrt{1+x^2} = y^{1/2}$. Substituting this, and $x\,dx = \dfrac{1}{2}\,dy$ into the integral, we obtain

$$\int x\sqrt{1+x^2}\,dx = \frac{1}{2}\int\sqrt{y}\,dy.$$

Step 4 Integrating with respect to y gives us $\dfrac{1}{2}\displaystyle\int \sqrt{y}\,dy = \dfrac{1}{3}y^{3/2} + C$.

Step 5 Finally, we substitute $y = 1 + x^2$ into the answer obtained in Step 4, to obtain

$$\int x\sqrt{1+x^2}\,dx = \frac{1}{3}(1+x^2)^{3/2} + C. \quad ❑$$

Question T10

Find the integral $\displaystyle\int (\cos x)\,e^{\sin x}\,dx$ by making the substitution $y = \sin x$. ❑

You may have noticed that the integral in Example 9, $\displaystyle\int x\left(1+x^2\right)^{1/2} dx$, and that of Question T10, $\displaystyle\int (\cos x)\,e^{\sin x}\,dx$, have something in common. In both cases, the integrand is not simply a function of the suggested $g(x)$ — instead it is a product of some function of $g(x)$ and the *derivative* $g'(x)\left(=\dfrac{dg}{dx}\right)$.

For example, we can write the integrand $x(1 + x^2)^{1/2}$ as $2x \times \frac{1}{2}\left(1 + x^2\right)^{1/2}$ i.e. as a product of the function of a function $\frac{1}{2}\left[g(x)\right]^{1/2}$ and the derivative $g'(x) = 2x$ so that

$$\int x(1+x^2)^{1/2}\,dx = \int \underbrace{2x}_{g'(x)}\,\underbrace{\tfrac{1}{2}(1+x^2)^{1/2}}_{\underbrace{\tfrac{1}{2}[g(x)]^{1/2}}_{\text{function of } g(x)}}\,dx$$

✥

Write $(\cos x)\,e^{\sin x}$ in the form '$g'(x) \times$ [function of $g(x)$]', where $g(x) = \sin x$.

✥

Write $e^{2x}(1 + e^{2x})^3$ in the form '$g'(x) \times$ [function of $g(x)$]', where $g(x) = e^{2x}$.

Integrals of the form $\int g'(x)\,p[g(x)]\,dx$, ☞ where the integrand is the product of some composite function $p[g(x)]$ and the derivative $g'(x)$, are sometimes called ‘integrals of a function and its derivative’. Such integrals can always be simplified by making the substitution $y = g(x)$. Let us see why.

When we make this substitution in the integral

$$\int p[g(x)]\,g'(x)\,dx,$$

the expression $p[g(x)]$ becomes simply $p(y)$; and we can also write

$$dy = g'(x)\,dx$$

Thus the integral becomes

$$\int p(y)\,dy$$

and so, if we know how to integrate the function $p(y)$, the substitution $y = g(x)$ will enable us to find the original integral $\int p[g(x)]\,g'(x)\,dx$.

Indeed, if we introduce the notation $P(y) = \int p(y)\,dy$, and remember that our final answer must be given as a function of x by writing $y = g(x)$ in $P(y)$, we can write down a neat formula for integrals of this sort:

$$\int p\big[g(x)\big]\,g'(x)\,dx = P\big[g(x)\big] \quad \text{where} \quad P(y) = \int p(y)\,dy \tag{15}$$

Example 10

Find $\int \mathrm{e}^{2x}(1 + \mathrm{e}^{2x})^3\,dx$.

Solution We make the substitution $y = g(x) = \mathrm{e}^{2x}$ so that $g'(x) = 2\mathrm{e}^{2x}$.

The integral then becomes $\int g'(x) \times \frac{1}{2}[1 + g(x)]^3\,dx$

so that in this case the function $p(y) = \frac{1}{2}(1 + y)^3$.

The integral $P(y)$ is therefore given by $P(y) = \int p(y)\,dy = \int \frac{1}{2}(1 + y)^3\,dy$ which can be simplified to $P(y) = \frac{1}{8}(1 + y)^4 + C$

and this means that $P[g(x)] = \frac{1}{8}(1 + \mathrm{e}^{2x})^4 + C$

Finally Equation 15 becomes $\int \mathrm{e}^{2x}(1 + \mathrm{e}^{2x})^3\,dx = \frac{1}{8}(1 + \mathrm{e}^{2x})^4 + C$ ❏

Whenever you encounter an integral that you think requires a substitution, you should first check whether it is of the form given on the left-hand side of Equation 15. If it is, the substitution $y = g(x)$ may well enable you to find it. There is no need to memorize Equation 15,

$$\int p\big[g(x)\big]g'(x)\,dx = P\big[g(x)\big] \quad \text{where} \quad P(y) = \int p(y)\,dy \qquad \text{(Eqn 15)}$$

though—the *method* is the important thing. Don't forget that $g'(x)$ may simply be a constant. Thus, for example, the integral $\int (3 + 2x)^{7/2}\,dx$ is of this form—we can write it as $\int \left[\tfrac{1}{2}(3 + 2x)^{7/2}\right] \times 2\,dx$, observing that the derivative of $g(x) = (3 + 2x)$ is simply 2.

Question T11

Find the integrals

(a) $\int (2 + \cos x)^7 \sin x\,dx$ (b) $\displaystyle\int \frac{x}{x^2 + 1}\,dx$ ❏ ☞

As you gain more practice with integrals of a function and its derivative, you will probably find that you can write down the answer almost immediately, by intelligent guesswork.

3.3 More examples of integration by substitution

In this subsection, we will show you some other examples of integration by substitution. Each example will be followed by a similar *Text question*, which should help you to become familiar with the lines of attack likely to work with different types of integral.

Example 11

Find the integral $\displaystyle\int \frac{x}{(1+2x)^2}\,dx$.

Solution *Step 1* Here we will try the substitution $y = g(x) = 1 + 2x$

Step 2 Then $\dfrac{dg}{dx} = 2$, so $dy = 2\,dx$, i.e. $dx = \frac{1}{2}\,dy$.

Step 3 The integral can now be written in the form $\displaystyle\int \left[\frac{x}{(1+2x)^2}\right]\frac{1}{2}\,dy$

The integrand, which is in square brackets, must be converted into a function of y, and in order to do this we need to invert the relation $y = 1 + 2x$ to obtain x in terms of y. ☞ This gives $x = \dfrac{1}{2}(y - 1)$. Thus on making the substitution $y = 1 + 2x$, the integral becomes

$$\int \frac{x}{(1+2x)^2}\,dx = \int \left[\frac{1}{y^2} \times \frac{(y-1)}{2}\right] \times \left(\frac{1}{2}\,dy\right) = \frac{1}{4}\int \left(\frac{1}{y} - \frac{1}{y^2}\right) dy$$

Step 4 Evaluating the integral with respect to y (on the right) gives

$$\frac{1}{4}\int \left(\frac{1}{y} - \frac{1}{y^2}\right) dy = \frac{1}{4}\left(\log_e y + \frac{1}{y}\right) + C.$$

Step 5 Substituting $y = 1 + 2x$ in this result gives the final answer

$$\int \frac{x}{(1+2x)^2}\,dx = \frac{1}{4}\left(\log_e (1+2x) + \frac{1}{1+2x}\right) + C$$

This example illustrates the point that if a particular function of x, in this case $(1 + 2x)$, features prominently in the integrand, it is worth trying that function as $g(x)$. ❏

Question T12

Find $\int x(1+x)^{5/2}\,dx$ by making the substitution $y = 1 + x$. ❏

In Example 11 and Question T12, it was necessary at one stage to find x in terms of y. Sometimes it is simpler to *start* with x as a function of y, and write the substitution in the form $x = h(y)$. In that case, we simply substitute $h(y)$ into the integrand, and replace dx by $\dfrac{dh}{dy}\,dy$. The next example is of this type.

Example 12

Find the integral $\displaystyle\int \frac{1}{\sqrt{1-x^2}}\,dx$.

Solution *Step 1* The presence of $\sqrt{1-x^2}$ in the denominator suggests we should make use of the trigonometric relationship $\cos^2 y = 1 - \sin^2 y$.

So we try the substitution $x = h(y) = \sin y$.

Step 2 Then $\dfrac{dh}{dy} = \cos y$, and so $dx = \cos y\,dy$ and the integral can be written in the form

$$\int \left[\frac{1}{\sqrt{1-x^2}}\right](\cos y\,dy)$$

where again the expression in square brackets must be converted into a function of y.

Step 3 With this substitution, $\sqrt{1-x^2} = \sqrt{1-\sin^2 y} = \cos y$ so the integral becomes ☞

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \int \frac{\cos y}{\cos y}\,dy = \int 1\,dy$$

Step 4 The integral is easy: $\int 1\,dy = y + C$.

Step 5 Finally, we express the result in terms of x, using the fact that if

$x = \sin y$, then $y = \arcsin x$, and so

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \int 1\,dy = y + C = \arcsin x + C \quad ❑$$

You may already have known the integral in Example 12 — it appears in every table of standard integrals. The method used to find it in Example 12 opens the way to finding many other integrals of the same type. Consider, for example, the integral $\displaystyle\int \frac{1}{\sqrt{9-x^2}}\,dx$. What substitution might we make here?

The experience that we gained in Example 12 suggests that we want something like $x = a\sin y$, where a is a suitable constant. We must choose a so that the identity $\cos^2 y + \sin^2 y = 1$ can be used to turn $\sqrt{9 - x^2}$ into a multiple of $\cos y$. Then this will cancel the $\cos y$ term in $dx = a\cos y\,dy$, leaving us with a very simple integral. Putting $x = a\sin y$ into $\sqrt{9 - x^2}$, it becomes $\sqrt{9 - a^2\sin^2 y}$, and if we choose $a = 3$ it will be equal to $\sqrt{9 - 9\sin^2 y} = 3\cos y$. So the substitution $x = 3\sin y$ can be used to simplify the integral.

The following integral is of the same type; so try replacing x by $a\sin y$ for a suitable choice of the constant a.

Question T13

Find the integral $\displaystyle\int \frac{1}{\sqrt{1 - 16x^2}}\,dx$. ❏

The substitution $x = \sin y$ can also be used to find integrals where the integrand is some other function of $\sqrt{1 - x^2}$ as in the next question.

Question T14

Find the integral $\int \sqrt{1-x^2}\,dx$. (*Hint*: You will find the solution to Example 7 helpful here; you will also need to use the identity $\cos^2 y + \sin^2 y = 1$, both in making the substitution and in expressing the final answer in terms of x.) ❏

Our next example is one of another class of integrals that can be found by means of a different trigonometric substitution.

Example 13

Find the integral $\displaystyle\int \frac{1}{1+x^2}\,dx$.

Solution *Step 1* Here, we will make the substitution $x = h(y) = \tan y$.

Step 2 Then $\dfrac{dx}{dy} = \dfrac{dh}{dy} = \sec^2 y$, so $dx = \sec^2 y\,dy$.

Step 3 With this substitution, $1 + x^2 = 1 + \tan^2 y$. At this point, we use the trigonometric identity $1 + \tan^2 y = \sec^2 y$. Thus the integrand is simply $\dfrac{1}{\sec^2 y}$.

Using this, and $dx = \sec^2 y\,dy$, we find that the integral becomes

$$\int \frac{1}{1+x^2}\,dx = \int \frac{\sec^2 y}{\sec^2 y}\,dy = \int 1\,dy$$

Step 4 $\displaystyle\int 1\,dy = y + C$

Step 5 If $x = \tan y$, then $y = \arctan x$, so the final answer is

$$\int \frac{1}{1+x^2}\,dx = \arctan x + C \quad ❏$$

The integral in Example 13 may also be one that you have seen before — but again, the point is that the method can be generalized to other integrals of the same form.

Consider $\displaystyle\int \frac{1}{4+x^2}\,dx$. Here, we might reason that a substitution of the form $x = a\tan y$ should work; we choose the constant a so that the identity $1 + \tan^2 y = \sec^2 y$ can be used to transform $(4 + x^2)$ into a multiple of $\sec^2 y$. Putting $x = a\tan y$ into $(4 + x^2)$, it becomes $(4 + a^2\tan^2 y)$; and if we choose $a = 2$, this becomes $4(1 + \tan^2 y) = 4\sec^2 y$. So $x = 2\tan y$ is the substitution we want here.

Question T15

Find the integral $\displaystyle\int \frac{1}{5+4x^2}\,dx$. ❏

3.4 Definite integrals; transforming the limits

So far in this section, we have dealt exclusively with *indefinite* integrals. Of course, once you have found the appropriate indefinite integral, finding a definite integral is just a matter of putting in the limits of integration. For example, let us return to the integral of Example 8,

$$\int x^2 \cos\left(x^3\right) dx = \frac{1}{3}\sin\left(x^3\right) + C$$

If, say, we want to evaluate the *definite* integral $\int_0^2 x^2 \cos\left(x^3\right) dx$, we simply put the limits in:

$$\begin{aligned}\int_0^2 x^2 \cos\left(x^3\right) dx &= \left[\frac{1}{3}\sin\left(x^3\right)\right]_0^2 \\ &= \frac{1}{3}\left(\sin\left(2^3\right) - \sin 0\right) \\ &= \frac{1}{3}\sin 8 = 0.3298 \text{ (to 4 decimal places)} \qquad (16)\end{aligned}$$

However, there is sometimes slightly less work involved if we find the definite integral directly, in terms of the new variable y. This involves *transforming the limits* of the definite integral. To see how it works, let us go through the process of substituting $y = x^3$ into the above integral $\int_0^2 x^2 \cos\left(x^3\right) dx$.

As before, we write $\cos(x^3) = \cos y$ and $x^2\,dx = \dfrac{1}{3}\,dy$. We must now note that if x ranges from 0 to 2, then $y = x^3$ ranges from 0^3 to 2^3, i.e. between 0 and 8. So the limits of integration in the integral with respect to y are 0 and 8. Thus we obtain

$$\int_0^2 x^2 \cos\left(x^3\right) dx = \frac{1}{3}\int_0^8 \cos y\,dy = \frac{1}{3}\left[\sin y\right]_0^8 = \frac{1}{3}\left(\sin 8 - \sin 0\right) = \frac{1}{3}\sin 8$$

as in Equation 16.

$$\int_0^2 x^2 \cos\left(x^3\right) dx = \frac{1}{3}\sin 8 = 0.3298 \text{ (to 4 decimal places)} \qquad \text{(Eqn 16)}$$

In Example 9, you saw that the indefinite integral $\int x\left(1+x^2\right)^{1/2}dx$ could be found by means of the substitution $y = 1 + x^2$. If we want to evaluate the *definite* integral $\int\limits_1^3 x\left(1+x^2\right)^{1/2}dx$ using this substitution, what are the limits in the integral with respect to y?

So, for definite integrals, when we make the substitution $y = g(x)$, we have two ways of proceeding. We can either follow Steps 1–5 of Subsection 3.1, finding the indefinite integral first, and putting in the x limits of integration at the end of the calculation.

Alternatively, when carrying out Step 3 and writing the integral in terms of y, we can at the same time transform the x limits of integration (let us call the lower limit a and the upper limit b) into the corresponding y limits of integration, $g(a)$ and $g(b)$. Often this will save a little time because there is then no need to carry out Step 5. ☞

Use the second procedure to answer the following question.

Question T16

Evaluate the integral $\int_0^4 x\sqrt{4-x}\,dx$ by means of the substitution $y = 4 - x$. ❏

An application to electricity and magnetism

We will end this subsection with an application of integration by substitution to electricity and magnetism. ☞

Example 14
Figure 1 shows a wire AB carrying a current I (denoted i on the figure). Such a current produces a magnetic field around the wire, and each small element of the wire, Δx say, makes a small contribution to this field. ☞
The object of this example is to determine the magnitude B of the resulting field at P due to the contributions from all such elements from A to B.

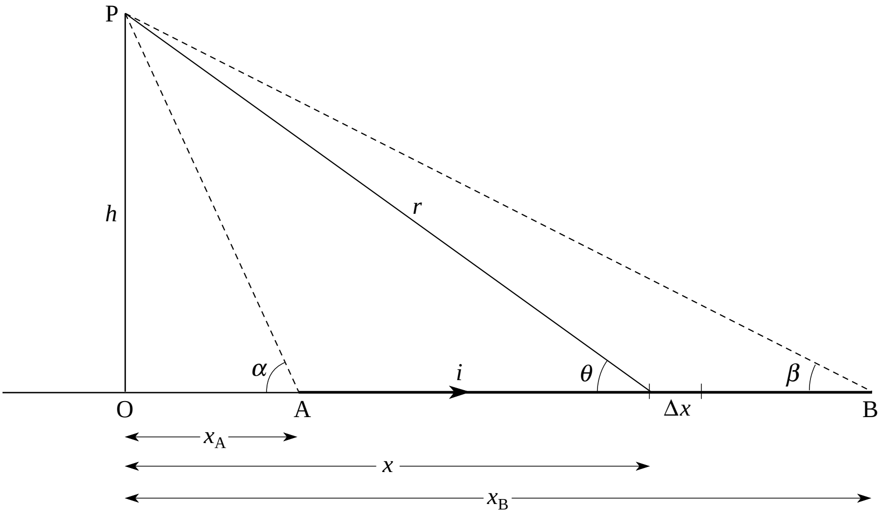

Figure 1 An electrical current through a segment of wire AB.

Solution First we need to know that the current through the element Δx produces a contribution to the field at P perpendicular to the plane of the paper, and of magnitude $\dfrac{I\sin\theta\,\Delta x}{r^2}$. Thus the total magnitude B of the field at P is $\displaystyle\int_{x_\mathrm{A}}^{x_\mathrm{B}}\frac{I\sin\theta}{r^2}\,dx$ and, since $h = r\sin\theta$ while I and h are constants, we have

$$B = \int_{x_\mathrm{A}}^{x_\mathrm{B}}\frac{I\sin\theta}{r^2}\,dx = I\int_{x_\mathrm{A}}^{x_\mathrm{B}}\frac{h}{r^3}\,dx = I\int_{x_\mathrm{A}}^{x_\mathrm{B}}\frac{h}{\left(x^2+h^2\right)^{3/2}}\,dx$$

In order to evaluate the integral on the right-hand side, we make the substitution $x = h\tan\phi$ so that $dx = h\sec^2\phi\,d\phi$, and therefore

$$I\int\frac{h}{\left(x^2+h^2\right)^{3/2}}\,dx = I\int\frac{h^2\sec^2\phi}{\left(h^2\tan^2\phi+h^2\right)^{3/2}}\,d\phi = I\int\frac{h^2\sec^2\phi}{h^3\sec^3\phi}\,d\phi = \frac{I}{h}\int\cos\phi\,d\phi = \frac{I}{h}\sin\phi + C \qquad (17)$$

In deriving this equation we have made use of the identity $\sec^2\phi = 1 + \tan^2\phi$.

Now we substitute for ϕ in terms of x in Equation 17

$$I\int \frac{h}{\left(x^2 + h^2\right)^{3/2}}\,dx = \frac{I}{h}\sin\phi + C \qquad \text{(Eqn 17)}$$

using the fact that $\tan\phi = \dfrac{x}{h}$ implies that $\sin\phi = \dfrac{x}{\sqrt{h^2 + x^2}}$, see Figure 2.

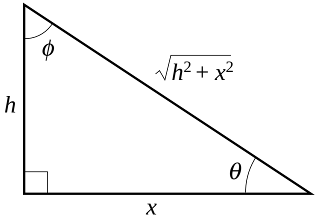

Figure 2 Geometrical meaning of the substitution $x = h\tan\phi$. Notice that $\sin\phi = \cos\theta$.

It follows that

$$I\int \frac{h}{\left(x^2 + h^2\right)^{3/2}}\,dx = \frac{I}{h}\sin\phi + C = \frac{I\,x}{h\sqrt{h^2 + x^2}} + C$$

We can now evaluate the magnitude of the field at P, and write

$$B = \int\limits_{x_{\rm A}}^{x_{\rm B}} \frac{I\sin\theta}{r^2}\,dx = \left[\frac{I\,x}{h\sqrt{h^2 + x^2}}\right]_{x_{\rm A}}^{x_{\rm B}} = \frac{I}{h}\left[\frac{x_{\rm B}}{\sqrt{h^2 + {x_{\rm B}}^2}} - \frac{x_{\rm A}}{\sqrt{h^2 + {x_{\rm A}}^2}}\right]$$

Using the angles α and β defined by Figure 1, we obtain the neat result

$$B = \frac{I}{h}\left(\cos\beta - \cos\alpha\right) \tag{18}$$

There is one circumstance that is of particular interest, namely when we wish to estimate the field at points close to a very long straight wire; in which case we can assume that the wire is of infinite length (since the contributions from distant sections of the wire are insignificant).

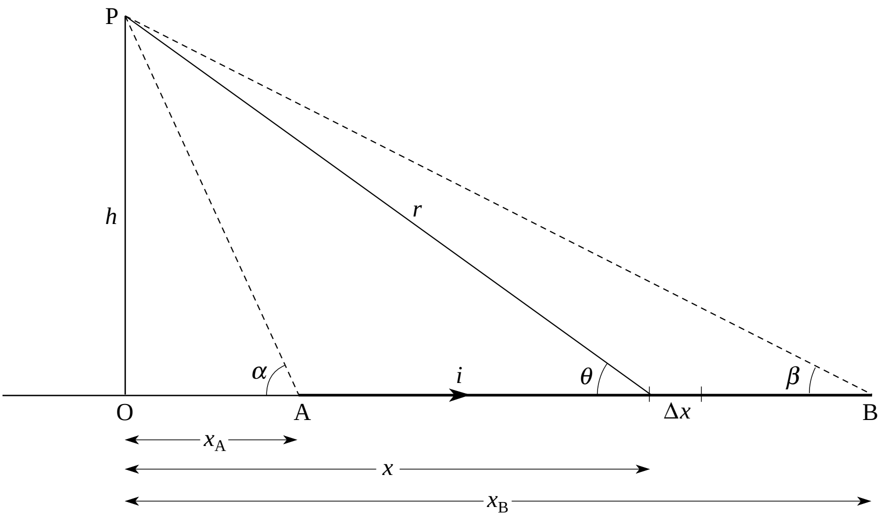

Figure 1 An electrical current through a segment of wire AB.

This corresponds to choosing $\alpha = \pi$ and $\beta = 0$,

in $$B = \frac{I}{h}\left(\cos\beta - \cos\alpha\right) \qquad \text{(Eqn 18)}$$

and we then have the useful result

$$B \approx \frac{2I}{h} \qquad (19)$$ ❏

4 Which method to try?

In the *Text questions* you have done so far in this module, either you have been told what integration method to use, or it has been clear by comparison with what has gone immediately before. It is time for you now to start deciding for yourself what the best line of attack will be and, if you decide to use substitution, to choose for yourself what substitution to try. To help you with this, we shall list below some types of integrals and comment on the best methods of finding them. (As we are concerned here with the *method* of integration, we shall consider only indefinite integrals.)

4.1 Some helpful advice

1 Integrals of products of certain functions.

Integration by parts will always work if the integrand is a product of any two of: an integer power of x; a sine or cosine; an exponential. There are certain other products for which it works: for example, $x\ \log_e x$. However, integration by parts is certainly not applicable to all products (try applying it, for example, to $\int x^2 \cos\left(x^3\right) dx$ and you will soon discover that it doesn't work).

2 Integrals of the form $\int p[g(x)]g'(x)\, dx$.

Make the substitution $y = g(x)$ this will transform the integral into $\int p\left(y\right) dy$

3 Integrals of the form $\displaystyle\int \frac{1}{\sqrt{A - Bx^2}}\, dx$, *or* $\int \sqrt{A - Bx^2}\, dx$, *where A and B are positive constants.*

Use a substitution of the form $x = a \sin y$, where a is chosen so that $\sqrt{A - Bx^2}$ turns into a multiple of $\cos y$ when you make the substitution. In this case, we can guarantee that this substitution will work.

4 Integrals of the form $\displaystyle\int \frac{1}{A + Bx^2}\,dx$ *where A and B are positive constants.*

Use a substitution of the form $x = a\tan y$, where a is chosen so that $A + Bx^2$ turns into a multiple of $\sec^2 y$ when you make the substitution. Here too, a substitution of this sort will certainly work.

5 Integrals in which the integrand contains some particular function of x.

For example, in the integral $\int e^{2x}(1 + e^{2x})^3\,dx$ we might try $g(x) = e^{2x}$. Of course, there may be several functions of x appearing in the integrand. However, if a particular function of x appears in the denominator, or is raised to some power, that is a good one to try as $g(x)$.

4.2 Exercises

In the following question, all the integrals except one can be found using the methods introduced in this module; we leave it to you to choose which method to use. The exception is an integral that can be done in a very simple fashion, it doesn't require integration by parts or the method of substitution.

Question T17

Find the indefinite integrals

(a) $\displaystyle\int \frac{(x+3)^2}{(x+5)^4}\,dx$ (b) $\displaystyle\int 3x\cos\left(\tfrac{1}{2}x\right)dx$ (c) $\displaystyle\int \frac{1}{\sqrt{2-x^2}}\,dx$

(d) $\displaystyle\int \frac{x}{\sqrt{2-x^2}}\,dx$ (e) $\displaystyle\int \sqrt{x}\,(1+x)\,dx$ ❏

As you carry your study of mathematics and physics further, you will probably encounter more substitutions that can be used for certain classes of integral; you may also learn how to combine various types of substitution with clever algebraic manipulations, or crafty use of trigonometric identities, and this will enlarge your integration repertoire even more. However, the most important techniques of integration are the two that have been presented in this module: integration by parts and integration by substitution. Consequently, if you have mastered these (as we hope), you have, in a sense, nothing further to learn about integration; all that remains for you to do is to gain greater familiarity with the ingenious tricks that can be used to bring an even greater number of integrals under your control.

5 Closing items

5.1 Module summary

1 A product $f(x)g(x)$ may often be integrated using the method of *integration by parts*, which is based on the formula

$$\int f(x)\,g(x)\,dx = F(x)\,g(x) - \int F(x)\frac{dg}{dx}\,dx\,,\ \text{where } \frac{dF}{dx} = f(x) \qquad \text{(Eqn 2)}$$

2 If this formula is to be useful, we must be able to integrate the function $f(x)$ to obtain $F(x)$; also, we want the integral on the right-hand side to be easier than the original integral. These considerations must be borne in mind when choosing $f(x)$ and $g(x)$ in any given case.

3 In the case of a definite integral, the formula becomes

$$\int_a^b f(x)\,g(x)\,dx = \Big[F(x)\,g(x)\Big]_a^b - \int_a^b F(x)\frac{dg}{dx}\,dx\,,\ \text{where } \frac{dF}{dx} = f(x) \quad \text{(Eqn 5)}$$

4 It is sometimes necessary to apply the technique of integrating by parts more than once. This can either be because the remaining integral to be evaluated also requires the use of integration by parts; or, in the case of integrals like $\int e^x \sin x\,dx$, by integrating by parts twice we can arrive at a simple equation involving the original integral, which can then be solved.

5 A small class of integrals can be found by taking the function $f(x)$ to be 1.

6 *Integration by substitution*. An integral of the form $\int f(x)\,dx$ may often be transformed into a simpler integral by writing $y = g(x)$ and $dy = \dfrac{dg}{dx}\,dx$, and substituting these relations into the integral to obtain an integral in terms of the new variable y which (if $g(x)$ has been intelligently chosen) should be easy to find. Once it is found, the answer may then be expressed in terms of x by substituting $y = g(x)$.

7 Integrals of a function and its derivative, of the form $\int p\big[g(x)\big]g'(x)\,dx$ may always be simplified by means of the substitution $y = g(x)$.

8 If some function of x appears raised to some power, or as a denominator, in the integrand, that may be a good choice for $g(x)$.

9 There are also certain types of substitution which work for particular classes of integrals; for example, a substitution of the form $x = a\sin y$, where a is a suitable constant, can be used to find integrals of the form $\displaystyle\int \frac{1}{\sqrt{A - Bx^2}}\,dx$ where A and B are positive constants.

10 When a substitution $y = g(x)$ is used to evaluate definite integrals, it is necessary to transform the limits of integration when writing down the integral with respect to y.

5.2 Achievements

Having completed this module, you should be able to:

A1 Define the terms that are emboldened and flagged in the margins of the module.

A2 Use the method of integration by parts to find indefinite integrals involving products of simple functions, such as $\int x \sin x \, dx$ or $\int x^3 \log_e x \, dx$.

A3 Apply integration by parts more than once to find indefinite integrals of the form $\int x^n f(x)\, dx$, where $f(x)$ is an exponential, or a sine or cosine.

A4 Apply integration by parts in such a way as to obtain a simple equation for the original unknown integral, which can then be solved, and recall that this method is particularly useful when the integrand is a product of an exponential function and a sine or cosine function.

A5 Remember that, in the formula for integration by parts (Equation 2), the function $f(x)$ may be taken to be 1, and use this trick to find certain integrals.

A6 Use integration by parts to evaluate definite integrals.

A7 Use a given substitution to transform a difficult indefinite integral into an easier one, and so find the original integral.

A8 Recognize an integral of the form $\int p[g(x)]\,g'(x)\,dx$ and use the substitution $y = g(x)$ to find it.

A9 Use the method of substitution to evaluate definite integrals, remembering to transform the limits correctly.

A10 Choose, for a given integral, a suitable substitution that will enable you to evaluate it.

Study comment You may now wish to take the *Exit test* for this module which tests these Achievements. If you prefer to study the module further before taking this test then return to the *Module contents* to review some of the topics.

5.3 Exit test

Study comment Having completed this module, you should be able to answer the following questions, each of which tests one or more of the *Achievements*.

Question E1

($A2$ and $A3$) Find the integral $\int x^2 \sin(kx)\,dx$ where k is a constant.

Question E2

($A4$) Find the integral $\int e^{3x}\cos(3x)\,dx$. (*Hint*: Start by taking

$f(x) = e^{3x}$ and $g(x) = \cos(3x)$.)

Question E3

(A6) Evaluate the integral $\displaystyle\int_0^1 x\cos(5x)\,dx$ to four decimal places.

Question E4

(A7) Use the substitution $y = x^2 + 2$ to find the integral

$$\int (x^2 + 2)^6\, x^3\, dx.$$

Question E5

(A8) Find the integrals

(a) $\displaystyle\int \frac{2}{3x+4}\,dx$ (b) $\displaystyle\int x\exp\left(x^2\right)dx$ (c) $\displaystyle\int e^{2x}\sin\left(e^{2x}\right)dx$

Question E6

(A9 and A10) Evaluate the integral $\displaystyle\int_1^2 x(2+x)^{3/2}\,dx$ to four decimal places.

Question E7

(A5 and A8) Find the integral $\displaystyle\int \arcsin x\,dx$. (*Hint*: Start by integrating by parts.)

Question E8

(A10) Find the integrals

(a) $\displaystyle\int \sec^2(3x)\tan(3x)\,dx$ (b) $\displaystyle\int \frac{1}{1+9x^2}\,dx$ (c) $\displaystyle\int \frac{x^5}{1+2x^3}\,dx$

Study comment This is the final *Exit test* question. When you have completed the *Exit test* go back to Subsection 1.2 and try the *Fast track questions* if you have not already done so.

If you have completed **both** the *Fast track questions* and the *Exit test*, then you have finished the module and may leave it here.