

## Module M6.2 Solving first-order differential equations

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# 1 Opening items

## 1.1 Module introduction

*First-order differential equations* such as

$$\frac{dN}{dt} = -\lambda N \quad \text{and} \quad \frac{dI}{dt} = -\frac{RC}{L} e^{-Rt/L}$$

arise in many physical contexts. The defining characteristic that is common to all such equations is that they contain no derivatives higher than the first; apart from this basic requirement they are free to take any form.

This module introduces methods that can be used to solve four different types of first-order differential equation, namely:

- 1  $\frac{dy}{dx} = f(x)$ , which can be solved by *direct integration*;
- 2  $\frac{dy}{dx} = h(y)$ , which can be solved by *inversion* followed by direct integration;
- 3  $\frac{dy}{dx} = f(x)h(y)$ , which can be solved by *separation of variables*;
- 4  $a \frac{dy}{dx} + by = f(x)$  (where  $a$  and  $b$  are constants), which can be solved by means of an *integrating factor*.

The discussion of these techniques occupies Subsections 2.1 to 2.4 of the module. The technique of *changing variables*, which can sometimes be usefully applied to equations that are not of any of the forms listed above is briefly considered in Subsection 2.5. Throughout the module physical examples are used to illustrate the various types of equation, but it is the mathematical aspects of the solution that are the main focus of interest.

**Study comment** Having read the introduction you may feel that you are already familiar with the material covered by this module and that you do not need to study it. If so, try the [Fast track questions](#) given in Subsection 1.2. If not, proceed directly to [Ready to study?](#) in Subsection 1.3.

## 1.2 Fast track questions

**Study comment** Can you answer the following *Fast track questions*?. If you answer the questions successfully you need only glance through the module before looking at the *Module summary* (Subsection 3.1) and the *Achievements* listed in Subsection 3.2. If you are sure that you can meet each of these achievements, try the *Exit test* in Subsection 3.3. If you have difficulty with only one or two of the questions you should follow the guidance given in the answers and read the relevant parts of the module. However, *if you have difficulty with more than two of the Exit questions you are strongly advised to study the whole module.*

## Question F1

Find the general solution of the following differential equations:

(a)  $\frac{dy}{dx} = \frac{x}{\sqrt{4-x^2}}$  (where  $-2 < x < 2$ )

(b)  $2x^{1/2}y \frac{dy}{dx} = 1$  (where  $x > 0$ )

(c)  $\frac{dy}{dx} = 1 - y^2$  (where  $-1 < y < 1$ )



## Question F2

Explain what is meant by an integrating factor. Use the method of integrating factors to find the general solution of the differential equation

$$2 \frac{dy}{dx} = x - 4y$$

and find the particular solution if  $y = 1/2$  at  $x = 0$ .



**Study comment** Having seen the *Fast track questions* you may feel that it would be wiser to follow the normal route through the module and to proceed directly to [Ready to study?](#) in Subsection 1.3.

Alternatively, you may still be sufficiently comfortable with the material covered by the module to proceed directly to the [Closing items](#).

## 1.3 Ready to study?

**Study comment** In order to study this module you will need to understand the following terms: constant of integration, dependent variable, derivative, general solution, independent variable, initial condition, inverse derivative (i.e. indefinite integral), linear differential equation and particular solution. You will need to have a good knowledge of differentiation (including the product and chain rules, and implicit differentiation), and of integration methods (including integration by substitution, integration by parts and the use of partial fractions); you should be familiar with the idea of checking a solution to a differential equation by substitution. If you are uncertain about any of these terms, you can review them by reference to the *Glossary*, which will also indicate where in *FLAP* they are developed. You should note that throughout this module  $\sqrt{x}$  represents the *positive* square root of  $x$ . The following *Ready to study questions* will allow you to establish whether you need to review some of these topics before embarking on this module.

## Question R1

Use the [\*product\*](#) and [\*chain rules\*](#) to express the [\*derivative\*](#)

$$\frac{d}{dx}(x^3 \log_e y)$$

in terms of  $x$ ,  $y$  and  $dy/dx$ . (This kind of differentiation is known as [\*implicit differentiation\*](#).)



## Question R2

Determine the [\*inverse derivatives\*](#) (i.e. the [\*indefinite integrals\*](#)) of the following functions of  $x$ :

(a)  $xe^{-x}$ , (b)  $e^x \cos x$ .





### Question R3

By making suitable substitutions, evaluate the following [\*indefinite integrals\*](#):

(a)  $\int x \exp\left(\frac{x^2}{2}\right) dx$ , (b)  $\int \sin x (\cos x)^6 dx$ , (c)  $\int \sqrt{1-x} dx$ .



### Question R4

By using [\*partial fractions\*](#), evaluate  $\int \frac{1}{x(x+3)} dx$ .



### Question R5

Explain the difference between a general solution and a particular solution of a first-order differential equation.



### Question R6

Show by substitution that

$$I = \frac{E}{R} + Ce^{-Rt/L}$$

where  $C$  is an arbitrary constant, is a solution of the differential equation

$$L \frac{dI}{dt} + RI = E, \text{ where } L, R \text{ and } E \text{ are given constants.}$$

Find the value of the arbitrary constant  $C$  in terms of  $E$  and  $R$ , given  $I = 0$  at  $t = 0$ .



## 2 Methods of solution for various first-order differential equations

**Notation** When a general type of differential equation is discussed in this module the *independent variable* will be denoted by  $x$  and the *dependent variable* by  $y$ . However, many of the examples given in this module are drawn from problems in physics, where the context suggests more appropriate symbols for the independent and dependent variables. Other examples involve purely ‘abstract’ differential equations and may also use a different notation. Bear in mind that the quantity being differentiated will always be the *dependent variable*.

For each of the four methods described in Subsections 2.1 to 2.4, a particular example from physics is used as an illustration.

## 2.1 Equations of the form $dy/dx = f(x)$ ; direct integration

There is a whole class of first-order differential equations that you can already solve. In general, these equations are of the form:

$$\frac{dy}{dx} = f(x) \quad (1)$$

where the derivative of the dependent variable is equal to a given function of the independent variable. A typical example of this type of differential equation is  $dy/dx = \cos x$ . If the derivative of  $y$  is equal to  $f(x)$ , then, by definition,  $y$  is the *inverse derivative* of  $f(x)$ , or, in other words, the *indefinite integral* of  $f(x)$ . Thus the solution to Equation 1 can be written simply as an integral:

$$y = \int f(x) dx \quad (2)$$

In the particular case  $dy/dx = \cos x$  we have  $f(x) = \cos x$  and therefore

$$y = \int \cos x dx = \sin x + C \quad (\text{where } C \text{ is an arbitrary constant } \textcolor{violet}{\rightarrow}).$$

In general, when we evaluate the indefinite integral in Equation 2,

$$y = \int f(x) dx \quad (\text{Eqn 2})$$

we will introduce an arbitrary *constant of integration*; hence Equation 2 gives us the *general solution* of Equation 1.

$$\frac{dy}{dx} = f(x) \quad (\text{Eqn 1})$$

The following differential equation can be solved by this method, which is known as the method of **direct integration**:

**Example 1** Find the general solution of the differential equation

$$\frac{dy}{dx} = e^{2x}$$

*Solution* The general solution is

$$y = \int e^{2x} dx$$

that is  $y = \frac{1}{2} e^{2x} + C$  (3)  $\square$

## Question T1

Can the direct integration method be applied to the following differential equations?  
Find the general solution of the equation in those cases where the method can be applied.


(a)  $\frac{dy}{dx} = x \sin x$     (b)  $\frac{dy}{dx} = y \sin y$      $\square$




Very often the solution of such a problem in physics requires a *particular* rather than a *general* solution, which means that we need an extra piece of information in order to assign an appropriate value to the arbitrary constant. This extra piece of information is often provided in the form of an *initial condition*. For example, suppose that in Example 1 we are told that  $y = 2$  when  $x = 0$ . Substituting these values into Equation 3, we obtain

$2 = \frac{e^0}{2} + C$ , i.e.  $C = 3/2$ . Thus the *particular solution*, satisfying both the differential equation and the initial condition, is  $y = \frac{e^{2x}}{2} + \frac{3}{2}$ .


## Resisted motion under [gravity](#)

Problems involving the motion of objects often give rise to differential equations that can be solved by direct integration, and in such cases the initial condition arises in a very natural fashion. For example, we might know the [velocity](#)  $v_x(t)$  of an object as a function of time  $t$ , and want to calculate its [position](#) coordinate  $x$  as a function of time.  Since  $v_x = dx/dt$ , this immediately presents us with a first-order differential equation, the general solution of which can be found simply by integrating  $v_x$  with respect to  $t$ . If we are given a value of  $x$  at some particular time  $t$ , this will allow us to assign a specific value to the general solution's arbitrary constant, and thus solve the problem completely. The following question is an example of this sort of problem.

## Question T2

At time  $t = 0$  a stone of [mass](#)  $m$  is thrown vertically upwards with initial velocity  $u_x$   (in this case we take the vertically upwards direction to be the direction of increasing  $x$  values). The upward motion of the stone is opposed by the (downward) [force](#) of [gravity](#),  $-mg$ , and by a [resistive force](#)  $-mkv_x$  (where  $k$  is a positive constant). The height  $x$  of the stone above its point of projection at time  $t$  ( $> 0$ ) can be shown to be given by the solution of the differential equation

$$\frac{dx}{dt} = \left( u_x + \frac{g}{k} \right) e^{-kt} - \frac{g}{k}$$

- (a) Find the general solution of the differential equation.
- (b) Use the initial condition  $x = 0$  at  $t = 0$  to find an expression for the height  $x$  of the stone at time  $t$ . 





## 2.2 Equations of the form $dy/dx = h(y)$ ; inversion and direct integration

Another common sort of differential equation is one where the derivative is a function of the *dependent* variable only; such equations have the general form:

$$\frac{dy}{dx} = h(y) \quad (4)$$

The differential equation  $dy/dx = y^3$  is a typical example of this type of equation. Such equations can be converted into a more convenient form using a relationship called the [inversion rule](#):

$$\frac{dx}{dy} = \frac{1}{dy/dx} \quad (5)$$

**A mathematical aside** Since you may not be familiar with Equation 5,

$$\frac{dx}{dy} = \frac{1}{dy/dx} \quad (\text{Eqn 5})$$

we will give a quick proof of it. Suppose that  $y$  is some function of  $x$  so

$$y = f(x) \quad (6)$$

Differentiating both sides with respect to  $y$ , we obtain

$$1 = df/dy \quad (7)$$

To express  $df/dy$  in an alternative way, think of  $f$  as a [function of a function](#):  $f$  is a function of  $x$ , but Equation 6 defines  $x$  as the [inverse function](#)  $f^{-1}(y)$ . So, using the [chain rule](#) we obtain

$$\frac{df}{dy} = \frac{df}{dx} \times \frac{dx}{dy}$$

but  $df/dx$  is just the same as  $dy/dx$ , so Equation 7 gives us

$$1 = \frac{dy}{dx} \times \frac{dx}{dy}, \text{ i.e. } \frac{dx}{dy} = \frac{1}{dy/dx}$$

as claimed in Equation 5.

An example should make the matter clear. If  $y = x^2$  then we may differentiate with respect to  $x$  and obtain  $dy/dx = 2x$ . On the other hand, if  $y = x^2$  then  $x = \sqrt{y} = y^{1/2}$  and therefore  $\frac{dx}{dy} = \frac{y^{-1/2}}{2} = \frac{1}{2\sqrt{y}} = \frac{1}{2x}$ , and therefore we have  $\frac{dx}{dy} = \frac{1}{dy/dx}$ . The following question reinforces this idea.  $\square$

### Question T3

Calculate  $dy/dx$  and  $dx/dy$ , and show that they satisfy the inversion rule, in each of the following cases:

- (a)  $y = x^3$  (which has  $x = y^{1/3}$  as its inverse function);
- (b)  $y = e^x$  (which has  $x = \log_e y$  as its inverse function).  $\square$



Returning to Equation 4

$$\frac{dy}{dx} = h(y) \quad (\text{Eqn 4})$$

this can be rewritten, using the inversion rule, as

$$\frac{dx}{dy} = \frac{1}{h(y)} \quad (8)$$

which is now in the right form to be solved by direct integration; so we have

$$x = \int \frac{1}{h(y)} dy$$

Provided that we can evaluate this indefinite integral, we can obtain  $x$  as a function of  $y$ . Ideally we should like to represent  $y$  *explicitly* as a function of  $x$  but this may not always be possible, and we may have to be satisfied with an equation that determines  $y$  as an *implicit function* of  $x$ .

◆ Find the general solution of the differential equation

$$\frac{dy}{dx} = \frac{1}{5y^4 + 1}$$



## The radioactive decay equation

As a further example of this method, let us solve Equation 9, the equation of [radioactive decay](#)

$$\frac{dN}{dt} = -\lambda N \quad (9)$$

where  $N$  is the number of radioactive nuclei present at time  $t$  and  $\lambda$  is a constant characteristic of the decaying substance and is known as the [decay constant](#).

Inversion of Equation 9 gives us

$$\frac{dt}{dN} = -\frac{1}{\lambda N}$$

and direct integration gives

$$t = -\int \frac{1}{\lambda N} dN = -\frac{1}{\lambda} \int \frac{1}{N} dN = -\frac{1}{\lambda} \log_e N + C$$

so that the general solution of the differential equation is  $t + \frac{1}{\lambda} \log_e N = C$ .

However, it may be possible to write this equation in a neater form, so let us try to make the dependent variable,  $N$ , the subject of the equation.

If we rearrange the general solution we find

$$C - t = \frac{1}{\lambda} \log_e N \quad \text{i.e.} \quad \log_e N = \lambda(C - t)$$

Using the fact that  $\log_e$  and  $\exp$  are inverse functions, we can exponentiate both sides, to obtain

$$\exp(\log_e N) = \exp(\lambda C - \lambda t) \quad \text{so that} \quad N = e^{\lambda C} e^{-\lambda t}$$

This is a perfectly valid form of the general solution to Equation 9,

$$\frac{dN}{dt} = -\lambda N \quad (\text{Eqn 9})$$

but it is convenient and conventional to write it in a slightly different way. Since  $\lambda$  is a given constant and  $C$  is an arbitrary constant,  $\exp(\lambda C)$  is just another arbitrary constant, which we can call  $A$ . The general solution to Equation 9 can therefore be written as

$$N = Ae^{-\lambda t} \quad (10)$$

Having found a general solution to a differential equation, it is always a good idea to check that it is indeed a solution, by substituting it back into the original equation.

◆ Check that Equation 10 is a general solution to Equation 9.



You may find it useful to think of the technique of inversion and direct integration in four steps:

*Inversion and direct integration*

- 1    *Invert*  $\frac{dy}{dx} = h(y)$  to obtain  $\frac{dx}{dy} = \frac{1}{h(y)}$ .
- 2    *Integrate* to obtain  $x = \int \frac{1}{h(y)} dy$ .
- 3    *Manipulate* this equation to make the dependent variable  $y$  the subject.
- 4    *Check* your general solution by substitution.

There may often be one further step — the use of an initial condition to obtain a value for the arbitrary constant appearing in the general solution. You can practise these various steps by answering the following question.

### Question T4

When water flows out of a hole in a bottle of constant cross-sectional area (see Figure 1), the height  $y$  of the water above the hole varies with time  $t$  according to the differential equation

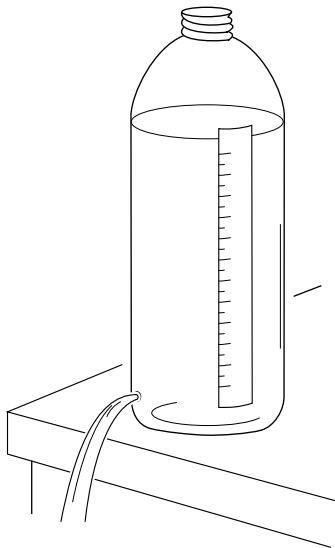
$$\frac{dy}{dt} = -k\sqrt{y}$$

where  $k$  is a positive constant. Find (a) the general solution of this differential equation, and (b) the particular solution corresponding to the initial condition  $y = y_0$  (where  $y_0 > 0$ ) at  $t = 0$ .  $\square$



We have now seen how to solve a first-order differential equation if the derivative of  $y$  is equal either to a function of  $x$  alone, or to a function of  $y$  alone.

In the next two subsections, we will consider some cases where the derivative is equal to a function of both  $x$  and  $y$ .




**Figure 1** See Question T4.



## 2.3 Equations of the form $dy/dx = f(x)h(y)$ ; separation of variables

To illustrate the mathematics discussed here we will begin this subsection by presenting you with three differential equations, drawn from completely different branches of physics, and will then show you that they are all of the same type. After explaining the general method of solving such differential equations, we will apply the method to two of these equations, and leave you to solve the third.

### Volume and pressure in a gas

The first example concerns the way in which the [pressure](#)  $P$  of a gas varies with its [volume](#)  $V$  if the gas is compressed or expanded in such a way that, though the [temperature](#) may change, no [heat](#) is allowed to enter or leave the gas.  The differential equation determining  $P$  as a function of  $V$  is

$$\frac{dP}{dV} = -\gamma \frac{P}{V} \quad (11)$$

where  $\gamma$  is a positive constant that depends on the nature of the gas.

## A space probe

For the second example, consider the case of a space probe launched vertically upwards from the Earth's surface. When the probe is at a distance  $r$  from the centre of the Earth, its outward [velocity](#)  $v_r$  satisfies the differential equation

$$\frac{dv_r}{dr} = -\frac{\mu}{v_r r^2} \quad (\text{for } r > R \text{ and } v_r > 0) \quad (12)$$

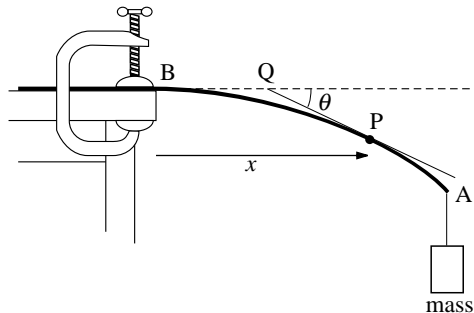
where  $R$  is the radius of the Earth and  $\mu$  is a positive constant.

## A flexible beam

Finally, Equation 13 describes the shape taken up by a thin flexible beam of length  $a$  which is caused to bend when a [mass](#) is suspended at one end, while it is clamped at the other end (see Figure 2):

$$\cos \theta \frac{d\theta}{dx} = k(a - x) \quad (13)$$

where the variables  $x$  and  $\theta$  are defined in Figure 2, and  $k$  is a positive constant.



**Figure 2** A flexible beam clamped at end B with a mass suspended at end A. The line PQ is a tangent to the beam at point P.

Equations 11 to 13 have a common property.

$$\frac{dP}{dV} = -\gamma \frac{P}{V} \quad (\text{Eqn 11})$$

$$\frac{dv_r}{dr} = -\frac{\mu}{v_r r^2} \quad (\text{for } r > R \text{ and } v_r > 0) \quad (\text{Eqn 12})$$

$$\cos \theta \frac{d\theta}{dx} = k(a - x) \quad (\text{Eqn 13})$$

They may all be written in such a way that the derivative is equal to a function of the independent variable multiplied by a function of the dependent variable. Equations 11 and 12 are already in this form, but Equation 13 requires a little rewriting to make it of this type; if we divide both sides by  $\cos \theta$ , we obtain

$$\frac{d\theta}{dx} = \frac{k(a - x)}{\cos \theta}$$

Table 1 clarifies this common structure.

$$\frac{dP}{dV} = -\gamma \frac{P}{V} \quad (\text{Eqn 11})$$

$$\frac{dv_r}{dr} = -\frac{\mu}{v_r r^2} \quad (\text{for } r > R \text{ and } v_r > 0) \quad (\text{Eqn 12})$$

$$\frac{d\theta}{dx} = \frac{k(a-x)}{\cos \theta} \quad (\text{Eqn 13})$$

**Table 1** Common features of Equations 11 to 13.

	derivative		function of dependent variable		function of independent variable
gas	$\frac{dP}{dV}$	=	$P$	×	$-\frac{\gamma}{V}$
space probe	$\frac{dv_r}{dr}$	=	$-\frac{\mu}{v_r}$	×	$\frac{1}{r^2}$
flexible beam	$\frac{d\theta}{dx}$	=	$\frac{1}{\cos \theta}$	×	$k(a-x)$

All these differential equations are, therefore, examples of the general form

$$\frac{dy}{dx} = f(x)h(y) \quad (14)$$

A differential equation of this type is said to be [separable](#), because it can be solved by a method known as [separation of variables](#), which we will now explain.

We first rewrite Equation 14 so that there is only a function of  $x$  on the right-hand side, by dividing both sides by  $h(y)$ :

$$\frac{1}{h(y)} \frac{dy}{dx} = f(x)$$

$$\text{i.e. } j(y) \frac{dy}{dx} = f(x) \quad (15)$$

where, purely for convenience, we have introduced the notation  $j(y) = 1/h(y)$ . We now integrate both sides of Equation 15 with respect to  $x$

$$\int j(y) \frac{dy}{dx} dx = \int f(x) dx \quad (16)$$

The right-hand side is no problem, but the left-hand side appears less straightforward. However, you should be familiar with the method of [integration by substitution](#), which is based on a rule of the following form

$$\int j(y) dy = \int j(y(x)) \frac{dy}{dx} dx \quad (17)$$


the implication of this is that the expression  $\left(\frac{dy}{dx}\right)dx$  inside the integral can be replaced by  $dy$ .

Using this result in Equation 16 gives

$$\int j(y) dy = \int f(x) dx$$

or, in terms of the function  $h(y)$ ,

$$\int \frac{1}{h(y)} dy = \int f(x) dx \quad (18)$$

Provided that both integrals can be evaluated, Equation 18 gives a relation between  $y$  and  $x$ , which represents the general solution of the differential equation. This relation can sometimes be manipulated to make  $y$  its subject.

The following example should make the method of separation of variables clearer.


**Example 2** Find the general solution of Equation 11,

$$\frac{dP}{dV} = -\gamma \frac{P}{V} \quad (\text{Eqn 11})$$

and express  $P$  as an explicit function of  $V$ .

*Solution* If we divide by  $P$  and integrate both sides of the equation with respect to  $V$  we get

$$\int \frac{1}{P} \frac{dP}{dV} dV = -\gamma \int \frac{1}{V} dV \quad (19)$$

The integral on the right-hand side is simply  $-\gamma \log_e V + C$ .  To evaluate the integral on the left-hand side, use

Equation 17 
$$\int j(y) dy = \int j(y(x)) \frac{dy}{dx} dx \quad (\text{Eqn 17})$$

to write Equation 19 as 
$$\int \frac{1}{P} dP = -\gamma \log_e V + C$$

which implies 
$$\log_e P = -\gamma \log_e V + C \quad (20) \quad \square$$





## Equation 20

$$\log_e P = -\gamma \log_e V + C \quad (\text{Eqn 20})$$

is the general solution, but it may be possible to write it in a neater form. To do so, first make use of the fact that a function of an arbitrary constant is itself an arbitrary constant, so it is certainly possible to define a new arbitrary constant  $C_1$  by means of the equation  $C = \log_e C_1$ . Using this new constant Equation 20 becomes

$$\log_e P = -\gamma \log_e V + \log_e C_1 = \log_e (C_1 V^{-\gamma})$$

exponentiating both sides then gives,

$$P = C_1 V^{-\gamma} \quad \text{where } C_1 \text{ is a constant}$$

◆ Check this general solution by substitution.



**A short cut** There is a short cut which many people take when using the technique of separation of variables. It involves treating  $dy/dx$  as *though it were* 'dy' divided by 'dx'. We will show you how it works on the next page.

In the text above, we went from the differential equation

$$\frac{1}{h(y)} \frac{dy}{dx} = f(x) \quad (\text{Eqn 15})$$

to the equation

$$\int \frac{1}{h(y)} dy = \int f(x) dx \quad (\text{Eqn 18})$$

by using an argument based on the rule for evaluating integrals by substitution.

However, many people would bypass this rigorous argument, and instead simply ‘multiply’ both sides by Equation 15 by ‘ $dx$ ’ to get

$$\frac{1}{h(y)} dy = f(x) dx \quad (21)$$

i.e. they would ‘separate’ the ‘ $dy$ ’ and ‘ $dx$ ’ as well. They would then integrate both sides of this equation, and so arrive at Equation 18.

There is nothing wrong with setting out your solution in this way provided you understand that it is nothing more than a notational short cut and that you appreciate the rigorous derivation (given in Equations 15–18).  $\square$

**Example 3** Find the general solution to Equation 12,

$$\frac{dv_r}{dr} = -\frac{\mu}{v_r r^2} \quad (\text{for } r > R \text{ and } v_r > 0) \quad (\text{Eqn 12})$$

*Solution* To separate variables, multiply both sides of this equation by  $v_r$  and by ' $dr$ '. This gives

$$v_r dv_r = -\mu dr/r^2$$

(the analogue of Equation 21).

$$\frac{1}{h(y)} dy = f(x) dx \quad (\text{Eqn 21})$$

Now integrate both sides, to obtain

$$\int v_r dv_r = -\mu \int \frac{1}{r^2} dr$$

Evaluating these integrals gives us the general solution

$$\frac{1}{2} v_r^2 = \mu/r + C$$

(Note that the two constants of integration have again been absorbed into one constant.)

Rearranging this equation to make  $v_r$  the subject, we find

$$v_r = \pm \sqrt{2(\mu/r + C)}$$

which is the general solution of the differential equation (Equation 12).

$$\frac{dv_r}{dr} = -\frac{\mu}{v_r r^2} \quad (\text{for } r > R \text{ and } v_r > 0) \quad (\text{Eqn 12})$$

(The two possible signs for  $v_r$  appear because the space probe could be moving towards or away from the Earth. As  $v_r$  was defined as the *outward* velocity, the plus sign applies when the space probe is moving away and the minus sign when it is moving towards the Earth.)  $\square$

Before you start to apply the method of separation of variables yourself, you may find a summary of the method as a series of steps helpful:

### Separation of variables

- 1 If the differential equation is not obviously in the form

$$\frac{dy}{dx} = f(x)h(y) \quad (\text{Eqn 14})$$

try to write it in this form by appropriate multiplication, division or factorization.  
(If this cannot be done, then you must use some other method to solve the equation!)

- 2 Rewrite this in the form

$$\frac{1}{h(y)} \frac{dy}{dx} = f(x) \quad (\text{Eqn 15})$$

3 Integrate both sides with respect to  $x$  to get

$$\int \frac{1}{h(y)} dy = \int f(x) dx \quad (\text{Eqn 18})$$

4 Evaluate the integrals, not forgetting a constant of integration.

5 If possible, manipulate the resulting equation to make  $y$  the subject.

6 Check your answer is a general solution by substitution.

You can now test your understanding of the method by answering the following two questions.

### Question T5

Are the following differential equations separable?

(a)  $\frac{dy}{dx} = x^2y^3 + x^2$     (b)  $yx \frac{dy}{dx} = 3$     (c)  $\frac{dy}{dx} = 2y + x^2$     (d)  $\frac{dQ}{dt} = 2Q$      $\square$



### Question T6

Find the general solution of Equation 13,

$$\cos \theta \frac{d\theta}{dx} = k(a - x) \quad (\text{Eqn 13})$$

and the particular solution corresponding to the initial condition  $\theta = 0$  at  $x = 0$ .     $\square$



It may have occurred to you that the two types of equation discussed in Subsections 2.1 and 2.2, namely,

$$\frac{dy}{dx} = f(x) \quad (\text{Eqn 1})$$

and  $\frac{dy}{dx} = h(y) \quad (\text{Eqn 4})$

are special cases of Equation 14:

$$\frac{dy}{dx} = f(x)h(y) \quad (\text{Eqn 14})$$

In Equation 1,  $h(y) = 1$ ; and in Equation 4,  $f(x) = 1$ . Both equations could be solved by separation of variables (compare part (d) of [Question T5](#)), but this method of solution might take rather longer than direct integration or inversion.

In each of the methods that we have discussed we are, at some stage, required to evaluate integrals, and therefore the success of the methods naturally depends on the corresponding functions being integrable. Later you may meet integrals that cannot be expressed in terms of simple functions, in which case you may need to use a *numerical method*, however such methods are beyond the scope of this module.



A further problem, of which you should be aware, arises from the fact that division by zero is not defined. Thus for the differential equation

$$\frac{dy}{dx} = f(x)h(y)$$

the manipulation that gives

$$\int \frac{1}{h(y)} dy = \int f(x) dx$$

is only satisfactory if  $h(y) \neq 0$ . In [Example 3](#) the conditions  $r > R$  and  $v_r > 0$  are sufficient to ensure that our solution is valid.

## 2.4 Equations of the form $a(dy/dx) + by = f(x)$ ; integrating factors

You have now seen three types of first-order differential equations for which a general solution can always be found (provided, of course, that the integrals appearing in the solution can be evaluated). There is one other type of equation which (with the same proviso) can always be solved.

This is the [first-order linear differential equation](#), which has the general form

$$a(x) \frac{dy}{dx} + b(x)y = f(x) \quad (22)$$

where  $a(x)$ ,  $b(x)$  and  $f(x)$  are arbitrary functions of  $x$ .

Many of the linear first-order differential equations that you are likely to meet in physics are in fact rather simpler than Equation 22,

$$a(x) \frac{dy}{dx} + b(x)y = f(x) \quad (\text{Eqn 22})$$

in that  $y$  and  $dy/dx$  are multiplied simply by *constants* instead of functions of  $x$  (though a function of  $x$  may still appear on the right-hand side). In this module we restrict our discussion to differential equations of this form.

A linear equation of this sort is called a first-order [linear differential equation with constant coefficients](#). A first-order linear equation with constant coefficients is, therefore, of the form

$$a \frac{dy}{dx} + by = f(x) \quad (23)$$

where  $a$  and  $b$  are constants. In this subsection we will explain how to solve differential equations of this type.

First, we will look at two examples of linear first-order differential equations with constant coefficients that arise in physics. The first is a generalization of Equation 9.

$$\frac{dN}{dt} = -\lambda N \quad (\text{Eqn 9})$$

## Radioactive decay

If a [radioactive](#) substance A is not only decaying, but is also being formed by the decay of some other radioactive substance B, then the number  $N$  of [nuclei](#) of substance A present at time  $t$  is given by the differential equation

$$\frac{dN}{dt} + \lambda N = \mu B_0 e^{-\mu t} \quad (24) \quad \img alt="A pink hand icon pointing to the right." data-bbox="825 325 853 360"/>$$

where  $\lambda$  and  $\mu$  are the [decay constants](#) of substances A and B, respectively, and  $B_0$  is the number of B nuclei present at time  $t = 0$ .

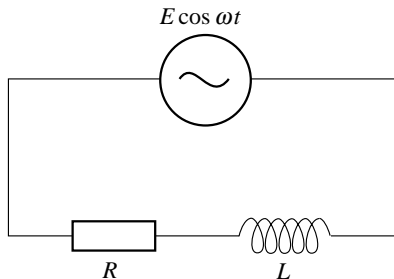
## Electric current

The second example concerns the way in which the [electric current](#)  $I$  in the circuit shown in Figure 3 varies with time.

The circuit contains an [inductor](#) of [inductance](#)  $L$ , a [resistor](#) of [resistance](#)  $R$ , and an [ideal voltage generator](#) producing an [alternating voltage](#)  $E \cos(\omega t)$ .

The differential equation determining the electric current  $I$  at time  $t$  is

$$L \frac{dI}{dt} + RI = E \cos(\omega t) \quad (25)$$



**Figure 3** An electrical circuit containing a resistor, an inductor and an ideal voltage generator.

**The general approach** Look carefully at Equations 24 and 25,

$$\frac{dN}{dt} + \lambda N = \mu B_0 e^{-\mu t} \quad (\text{Eqn 24})$$

$$L \frac{dI}{dt} + RI = E \cos(\omega t) \quad (\text{Eqn 25})$$

and convince yourself that they cannot be written in any of the three ways shown in Equations 1, 4 and 14,

$$\frac{dy}{dx} = f(x) \quad (\text{Eqn 1})$$

$$\frac{dy}{dx} = h(y) \quad (\text{Eqn 4})$$

$$\frac{dy}{dx} = f(x)h(y) \quad (\text{Eqn 14})$$

but that they are of the form shown in Equation 23.

$$a \frac{dy}{dx} + by = f(x) \quad (\text{Eqn 23})$$

The general method of solving equations such as Equation 23

$$a \frac{dy}{dx} + by = f(x) \quad (\text{Eqn 23})$$

is somewhat more complicated than the other methods we have studied in this module; so, on this occasion, let us start with a particular example that may make the general method easier to understand. If you are given the information that

$$ye^x = \sin x + C \quad (26)$$

where  $C$  is a constant, you would be able to differentiate this equation with respect to  $x$  to obtain

$$e^x \frac{dy}{dx} + e^x y = \cos x \quad (27)$$

Therefore if we were asked to solve the Equation 27, we can see immediately that its general solution is the function  $y(x)$  given by Equation 26. But suppose that we alter the differential equation slightly by dividing both sides by  $e^x$  to get

$$\frac{dy}{dx} + y = e^{-x} \cos x \quad (28)$$

The general solution remains the same, it is still  $y(x)$ , but the left-hand side of Equation 28

$$\frac{dy}{dx} + y = e^{-x} \cos x \quad (\text{Eqn 28})$$

is no longer the derivative of a simple function. If you were asked to solve Equation 28 you would need to multiply both sides of the equation by  $e^x$  and then perhaps you might recognize that the left-hand side is the derivative of the product  $ye^x$ . Having done that you would then only need to integrate the right-hand side, then perform some manipulations in order to find the general solution. This is the essence of the method, and its success will depend on choosing a suitable multiplier, in this case  $e^x$ .



Now let us develop these ideas more systematically, in the context of our radioactive decay example, by trying to solve Equation 24

$$\frac{dN}{dt} + \lambda N = \mu B_0 e^{-\mu t} \quad (\text{Eqn 24})$$

Perhaps by multiplying by some suitable function  $g(t)$  we can make the left-hand side the derivative of a product. Multiplying both sides of Equation 24 by  $h(t)$  we obtain

$$h(t) \frac{dN}{dt} + [\lambda h(t)]N = \mu h(t) B_0 e^{-\mu t} \quad (29)$$

Now, the left-hand side of this equation would be the derivative of the product  $h(t)N(t)$  if the expression in square brackets was equal to  $dh/dt$ . In other words to *make* the left-hand side the derivative of a product we require

$$\frac{dh}{dt} = \lambda h(t) \quad (30)$$

But Equation 30 is just a simple first-order differential equation that can be tackled by inversion and direct integration (as in Subsection 2.2). Its solution is

$$t = \frac{1}{\lambda} \int \frac{1}{h} dh = \frac{1}{\lambda} \log_e h + C \quad (31)$$

$$t = \frac{1}{\lambda} \int \frac{1}{h} dh = \frac{1}{\lambda} \log_e h + C \quad (31)$$

We only need to find *one* function  $h(t)$  with the desired property, so we choose the simplest case, when  $C = 0$ , and Equation 31 then gives  $h(t) = e^{\lambda t}$ . Substituting this expression for  $h(t)$  in Equation 29

$$h(t) \frac{dN}{dt} + [\lambda h(t)]N = \mu h(t) B_0 e^{-\mu t} \quad (\text{Eqn 29})$$

gives us

$$e^{\lambda t} \frac{dN}{dt} + \lambda e^{\lambda t} N = \mu e^{\lambda t} B_0 e^{-\mu t}$$

where the left-hand side is now clearly seen to be the derivative of the product  $Ne^{\lambda t}$  with respect to  $t$  *because we have constructed it to be so*.

We can now see that

$$\frac{d}{dt}(Ne^{\lambda t}) = \mu e^{\lambda t} B_0 e^{-\mu t} = \mu B_0 e^{(\lambda-\mu)t}$$

Integrating both sides gives us

$$Ne^{\lambda t} = \mu B_0 \int e^{(\lambda-\mu)t} dt = \frac{\mu B_0}{\lambda - \mu} e^{(\lambda-\mu)t} + C$$

Finally, if we make  $N$  the subject of this equation by dividing both sides by  $e^{\lambda t}$  we find

$$N = \frac{\mu B_0}{\lambda - \mu} e^{-\mu t} + C e^{-\lambda t} \quad (32)$$

### Question T7

Check by substitution that Equation 32 is a solution to Equation 24.  $\square$

$$\frac{dN}{dt} + \lambda N = \mu B_0 e^{-\mu t} \quad (\text{Eqn 24})$$



It should be clear that this method would have worked whatever function of  $t$  had been present on the right-hand side of Equation 24.

$$\frac{dN}{dt} + \lambda N = \mu B_0 e^{-\mu t} \quad (\text{Eqn 24})$$

The important point of the technique is the trick of multiplying Equation 24 by  $e^{\lambda t}$  in order to bring its left-hand side into the form of Equation 27.

$$e^x \frac{dy}{dx} + e^x y = \cos x \quad (\text{Eqn 27})$$

This was what made it possible to integrate directly, and for this reason,  $e^{\lambda t}$  is called an *integrating factor*.

An **integrating factor** is a function by which each term of a differential equation is multiplied in order to make it possible to use direct integration to solve the equation.

## The general linear equation with constant coefficients

Now that we have found the integrating factor for Equation 24, we can easily find one for the general first-order linear equation with constant coefficients

$$a \frac{dy}{dx} + by = f(x) \quad (\text{Eqn 23})$$

If we divide both sides by the constant  $a$ , we obtain

$$\frac{dy}{dx} + \frac{b}{a}y = \frac{f(x)}{a}$$

The left-hand side here is just like the left-hand side of Equation 24,

$$\frac{dN}{dt} + \lambda N = \mu B_0 e^{-\mu t} \quad (\text{Eqn 24})$$

except that instead of  $\lambda$  we have the constant  $b/a$ . The following exercise shows that the integrating factor in this case must be simply  $e^{bx/a}$ , which can be written alternatively as  $\exp(bx/a)$ .

### Question T8

Show that the following is true

$$\frac{d}{dx}(ye^{bx/a}) = e^{bx/a} \frac{dy}{dx} + \frac{b}{a} ye^{bx/a}$$

and explain why this means that  $\exp(bx/a)$  is the integrating factor for Equation 23.  $\square$



### Question T9

Find the integrating factor for the following differential equation:

$$2 \frac{dy}{dx} - x = y \quad \square$$



We will now summarize the steps involved in using the method of integrating factors to solve the first-order linear differential equation with constant coefficients,

$$a \frac{dy}{dx} + by = f(x) \quad (\text{Eqn 23})$$

### *Integrating factor method*

- 1 *Divide* both sides of the equation by  $a$ , the coefficient of  $dy/dx$ , to obtain

$$\frac{dy}{dx} + \frac{b}{a}y = \frac{f(x)}{a}$$

- 2 *Multiply* both sides of the equation by  $e^{bx/a}$ , the integrating factor. This gives

$$e^{bx/a} \frac{dy}{dx} + \frac{b}{a}ye^{bx/a} = \frac{f(x)}{a}e^{bx/a}$$

which can be written as

$$\frac{d}{dx}(ye^{bx/a}) = e^{bx/a} \frac{f(x)}{a}$$

- 3 *Integrate* this equation directly, to obtain

$$ye^{bx/a} = \frac{1}{a} \int f(x) e^{bx/a} dx$$

- 4 *Evaluate* the integral, remembering to include the constant of integration.  
5 If possible, *make  $y$  the subject*, by dividing all terms — *including* the constant of integration — by  $e^{bx/a}$ .  
6 *Check* that your answer is a solution by substitution.

You can now practise these six steps by answering the following questions.

### Question T10

Use the integrating factor you found for the equation given in Question T9  $2 \frac{dy}{dx} - x = y$

to find the general solution of that equation.  $\square$



### Question T11

Find the general solution to Equation 25,

$$L \frac{dI}{dt} + RI = E \cos(\omega t) \quad (\text{Eqn 25})$$

You will need to use the following standard integral:

$$\frac{E}{L} \int e^{Rt/L} \cos(\omega t) dt = \frac{E}{R^2 + L^2 \omega^2} e^{Rt/L} [\omega L \sin(\omega t) + R \cos(\omega t)] + C \quad \square$$





## 2.5 Choosing the correct method; what to do if none of them work

So far we have discussed differential equations of the following forms:

- 1  $\frac{dy}{dx} = f(x)$ , which can be solved by direct integration;
- 2  $\frac{dy}{dx} = h(y)$ , which can be solved by inversion and then direct integration;
- 3  $\frac{dy}{dx} = f(x)h(y)$ , which can be solved by separation of variables;
- 4  $a\frac{dy}{dx} + by = f(x)$ , the linear equation with constant coefficients, which can be solved by means of an integrating factor.

When faced with a first-order differential equation, you should first see whether it can be written in any of the above four forms (making sure that you have correctly identified the independent and dependent variables); if it can, you should then use the appropriate method to find its general solution.

The following exercise is intended to test your ability to choose the appropriate method (you are *not* asked to solve the differential equations).

### Question T12

What method would you use to solve the following differential equations? (In some cases, more than one method is possible; but you should not use separation of variables if direct integration or inversion and direct integration would work.)

(a)  $y \frac{dy}{dx} = \frac{1}{x^2} + \frac{y^2}{x^2}$

(b)  $\frac{dv}{dt} = -g + \frac{\alpha}{\beta - \gamma t}$  where  $g$ ,  $\alpha$ ,  $\beta$  and  $\gamma$  are constants.

(c)  $\frac{dy}{dx} = 5(y + 3x)$

(d)  $\frac{dN}{dt} + \lambda N = R$  where  $\lambda$  and  $R$  are constants.  $\square$



## Change of variable

There are, of course, many first-order differential equations that cannot be written in any of the forms 1 to 4 above. Sometimes, however, an equation of that sort can be transformed into one that we already know how to solve by changing the (dependent) variable. The best way to explain this technique is by means of an example.

Consider the differential equation

$$\frac{dy}{dx} = \frac{y}{x} + \frac{x}{y} \quad (33)$$


This equation is neither separable nor linear. However, let us see what happens if we define a new dependent variable  $v$  in terms of  $y$  and  $x$  by

$$y = vx, \quad \text{which can be rewritten as } v = y/x \quad (34)$$

where  $v$  is to be regarded as a (so far unknown) function of  $x$ . Since  $y = vx$  we can use the product rule to obtain the following expression for the left-hand side of Equation 33

$$\frac{dy}{dx} = \frac{d}{dx}(vx) = x \frac{dv}{dx} + v$$

We can now use Equation 34 to rewrite the right-hand side of Equation 33 in the following way

$$\frac{y}{x} + \frac{x}{y} = v + \frac{1}{v}$$

Using these results Equation 33

$$\frac{dy}{dx} = \frac{y}{x} + \frac{x}{y} \quad (\text{Eqn 33})$$

becomes

$$x \frac{dv}{dx} + v = v + \frac{1}{v}$$

that is  $x \frac{dv}{dx} = \frac{1}{v}$

which *can* be separated and integrated to obtain

$$\int \frac{1}{x} dx = \int v dv$$

giving  $\log_e x + C = \frac{1}{2} v^2$

Rearranging this equation to make  $v$  the subject, we find

$$v = \pm \sqrt{2 \log_e x + C_1}$$

(where the new constant  $C_1 = 2C$ ). Finally, we express the solution in terms of our original independent variable, by putting  $v = y/x$ , to obtain the general solution

$$y = \pm x \sqrt{2 \log_e x + C_1}$$

There is no systematic way of finding a change of variable that will convert an otherwise insoluble differential equation into one that can be solved by one of the techniques you have learnt in this module. Often it is necessary to proceed simply by trial and error, though we can give you one hint: if the equation contains a particular *combination* of the variables  $y$  and  $x$ , it is worth seeing what happens if you make that combination your new dependent variable. (That was why we tried  $y = vx$  in Equation 33;

$$\frac{dy}{dx} = \frac{y}{x} + \frac{x}{y} \quad (\text{Eqn 33})$$

the right-hand side is simply a function of  $y/x$ .)

### Question T13

Show that the differential equation

$$\frac{dy}{dx} = \frac{1}{(y+x)^2}$$

can be converted into an equation that can be solved by inversion by means of the change of variable  $v = y + x$ .  
(You are not required to find the solution.)  $\square$



As you become more experienced in solving differential equations, you will find it easier to spot the change of variable that makes the equation soluble, if there is one (there may not be; for example, we know of no change of variable that makes it possible to solve the equation  $dy/dx = y^x$ ).

Of course, you can always seek help from textbooks, which often discuss changes of variable that will work for particular types of differential equations. You should regard a differential equation that cannot be solved by any of the four systematic methods listed above as an interesting challenge; it is, after all, at this point that the process of solving differential equations ceases simply to be a routine matter of applying standard technique and becomes something of an art, requiring ingenuity, patience and low animal cunning.

## 3 Closing items

### 3.1 Module summary

- 1 *First-order differential equations* of the form  $dy/dx = f(x)$  can be solved at once by *direct integration*, to give  $y = \int f(x) dx$ .
- 2 Equations of the form  $dy/dx = h(y)$  can be solved by applying the *inversion rule*,  $dy/dx = 1/(dx/dy)$ , to obtain  $dx/dy = 1/h(y)$ , which can be directly integrated.
- 3 Equations of the form  $dy/dx = f(x)h(y)$  can be solved by the technique of *separation of variables*, which leads to an equation of the form

$$\int \frac{1}{h(y)} dy = \int f(x) dx \quad (\text{Eqn 18})$$

which may be integrated and rearranged to find  $y$ .



- 4 Linear first-order differential equations with constant coefficients of the form  $a(dy/dx) + by = f(x)$  can be solved by dividing both sides by  $a$  and then multiplying both sides by an integrating factor  $\exp(bx/a)$ , to bring them into a form where direct integration becomes possible.
- 5 The technique of changing the dependent variable is almost the only option available if none of the methods listed above works. This is a trial and error procedure rather than a systematic one.

## 3.2 Achievements

Having completed this module, you should be able to:

- A1 Define the terms that are emboldened and flagged in the margins of the module.
- A2 Solve differential equations of the form  $dy/dx = f(x)$ , by direct integration.
- A3 Use the inversion rule to solve differential equations of the form  $dy/dx = h(y)$ , by inversion and direct integration.
- A4 Explain what is meant by the term separable, and solve differential equations of the form  $dy/dx = f(x)h(y)$ , by separation of variables.
- A5 Explain what is meant by an integrating factor, find the integrating factor for a linear first-order equation with constant coefficients and use it to solve the equation.
- A6 Decide which (if any) of the above methods can be used to solve a given first-order differential equation.
- A7 Use a given change of variable to transform a first-order differential equation into one that can more easily be solved.

**Study comment** You may now wish to take the [Exit test](#) for this module which tests these Achievements. If you prefer to study the module further before taking this test then return to the [Module contents](#) to review some of the topics.

### 3.3 Exit test

**Study comment** Having completed this module, you should be able to answer the following questions each of which tests one or more of the *Achievements*.

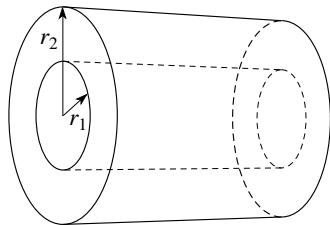
#### Question E1

(A2) The differential equation

$$H = -2\pi r \kappa \frac{dT}{dr}$$

describes the way in which [temperature](#)  $T$  varies with radius  $r$  within the walls of a hollow cylindrical pipe, of inner radius  $r_1$  and outer radius  $r_2$  (see Figure 4).  $H$  (the [heat](#) loss per unit length) and  $\kappa$  (the [thermal conductivity](#) of the pipe) are constants. Find the general solution of this equation by direct integration.

If the temperature of the inner wall is  $T_1$ , find an expression for the temperature  $T_2$  of the outer wall in terms of  $T_1$ ,  $r_1$ ,  $r_2$  and constants.



**Figure 4** See Question E1.



### Question E2

(A3) Use the technique of inversion and integration to find the general solution to the differential equation

$$\frac{dy}{dx} = \sqrt{a^2 - y^2} \quad \text{where } a \text{ is a constant}$$



### Question E3

(A4) The following equation describes how the (absolute) temperature  $T$  at which a liquid boils varies with the external pressure  $P$  (under certain assumptions, which we shall not go into here):

$$\frac{dT}{dP} = \frac{RT^2}{LP}$$

where  $L$  (the molar latent heat of vaporization of the liquid) and  $R$  (the molar gas constant) are constants. Find the general solution to this equation using the technique of separation of variables.



### Question E4

(A3 and A5) A stone of mass  $m$  is thrown vertically upwards with initial velocity  $u_x$ . The stone is subject to the (downward) force of gravity, of magnitude  $-mg$ , and a resistive force of magnitude  $-mkv_x$  (where  $k$  is a positive constant), the stone's upward velocity  $v_x$  at time  $t$  satisfies the differential equation

$$\frac{dv_x}{dt} = -g - kv_x \quad (v_x > 0)$$

Find the general solution to this equation:

- (a) by using inversion and direct integration (treating the cases when  $(g + kv_x)$  is positive and negative separately); and
- (b) by using an integrating factor.
- (c) Use the initial condition  $v_x = u_x$  at  $t = 0$  to find the particular solution.



### Question E5

(A6) Find the general solutions of the following differential equations:

- (a)  $\frac{du}{dT} = 4 \frac{u}{T}$  (where  $u$  and  $T$  are both always positive);
- (b)  $R \frac{dQ}{dt} + \frac{Q}{C_0} = Ee^{-\lambda t}$  where  $R$ ,  $C_0$ ,  $E$  and  $\lambda$  are constants.



## Question E6

(A4, A5, A6 and A7) A differential equation of the form

$$\frac{dy}{dx} + y = f(x)y^n$$

where  $n$  is a constant (not necessarily an integer), is called a *Bernoulli equation*.

- (a) For one value of  $n$ , the equation is linear with constant coefficients. What is this value?
- (b) For another value of  $n$ , the equation is separable. What is this value?
- (c) If  $n$  is not equal to either of these two values, the equation cannot be solved as it stands. Show that the substitution  $y = v^m$ , where  $m = 1/(1 - n)$ , transforms the equation into a linear equation.
- (d) Find the general solution of the equation if  $n = 2$  and  $f(x) = e^{2x}$ .



**Study comment** This is the final *Exit test* question. When you have completed the *Exit test* go back to Subsection 1.2 and try the [\*Fast track questions\*](#) if you have not already done so.

If you have completed **both** the *Fast track questions* and the *Exit test*, then you have finished the module and may leave it here.

