Module M6.3 Solving second order differential equations

Exit module

1 Opening items

1.1 Module introduction

Suppose that we are interested in the behaviour of a mass m, free to bob up and down along the y-axis on the end of a spring of *force constant* k, and also subject to a *damping* (or resistive) force the magnitude of which is proportional to the speed of the mass, as well as to an externally imposed periodic force in the y-direction given by $F_y = F_0 \sin(\Omega t)$ where F_0 is a positive constant. (F_y is commonly abbreviated to F to avoid unnecessary confusion with the subscripts.) The *equation of motion* of the mass is obtained by applying *Newton's second law*. If y is the displacement of the mass from equilibrium, then we have

$$m\frac{d^2y}{dt^2} = -ky - b\frac{dy}{dx} + F_0\sin(\Omega t) \tag{1}$$

All the forces acting on the mass have been added together on the right-hand side of Equation 1,

$$m\frac{d^2y}{dt^2} = -ky - b\frac{dy}{dx} + F_0\sin(\Omega t) \qquad \text{(Eqn 1)}$$

and are as follows:

$-ky$	(the restoring force exerted by the spring);
$-b\dfrac{dy}{dt}$, where b is a positive constant	(the damping force);
$F_0\sin(\Omega t)$	(the periodic applied force).

Equation 1 is an example of a *second-order linear differential equation*. It belongs to a category of *linear differential equations* known as *linear equations with constant coefficients*, where the *dependent variable* and its derivatives are multiplied by constants (not by functions of the *independent variable*).

The most general second-order equation of this sort can be written as

$$\boxed{a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = f(t) \qquad (2)}$$

where a, b, c are constants and $a \neq 0$; it is equations of this type that will be discussed in this module.

The simplest of this type of equation is one for which b and c are zero, so that the equation becomes

$$a\frac{d^2y}{dt^2} = f(t) \qquad (3)$$

This can be solved by *direct integration*, as will be explained in Subsection 2.1. Another relatively simple case arises when we have zero on the right-hand side of Equation 2, instead of a function of t, so that

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = 0\text{, where } a, b, c \text{ are constants and } a \neq 0 \qquad (4)$$

Equations of this type (known as *linear homogenous equations*) have innumerable applications in physics; they arise, for example, in the discussion of simple harmonic motion and the analysis of a.c. circuits. ☞ Much of this module (Subsections 2.2 to 2.5) is therefore devoted to finding solutions to various forms of Equation 4.

Subsection 2.6 tells you how to solve some equations of the type

$$a\frac{d^2y}{dt^2}+b\frac{dy}{dt}+cy = f(t) \qquad \text{(Eqn 2)}$$

for some different forms of $f(t)$.

The method explained there will then be used in Subsection 2.7 to find the general solution of equations of the type

$$m\frac{d^2y}{dt^2} = -ky - b\frac{dy}{dx} + F_0\sin(\Omega t) \qquad \text{(Eqn 1)}$$

Study comment Having read the introduction you may feel that you are already familiar with the material covered by this module and that you do not need to study it. If so, try the *Fast track questions* given in Subsection 1.2. If not, proceed directly to *Ready to study?* in Subsection 1.3.

1.2 Fast track questions

Study comment Can you answer the following *Fast track questions*?. If you answer the questions successfully you need only glance through the module before looking at the *Module summary* (Subsection 3.1) and the *Achievements* listed in Subsection 3.2. If you are sure that you can meet each of these achievements, try the *Exit test* in Subsection 3.3. If you have difficulty with only one or two of the questions you should follow the guidance given in the answers and read the relevant parts of the module. However, *if you have difficulty with more than two of the Exit questions you are strongly advised to study the whole module*.

Question F1

The angular displacement θ of the bob of a simple pendulum satisfies the equation of motion

$$a\frac{d^2\theta}{dt^2} + b\frac{d\theta}{dt} + c\theta = 0$$

(a) Find the general solution of this equation if $a = 0.1\,\text{kg}$, $b = 0.2\,\text{kg}\,\text{s}^{-1}$ and $c = 1.0\,\text{kg}\,\text{s}^{-2}$.

(b) Find the particular solution if the bob is hit when in its rest position $\theta = 0$ such that it is given an angular speed

$$\frac{d\theta}{dt} = 0.3\,\text{rad}\,\text{s}^{-1} \quad \text{at} \quad t = 0$$

(c) Sketch this solution as a function of t.

Question F2

Find the general solution to the equation

$$\frac{d^2x}{dt^2} + 5\frac{dx}{dt} + 6x = 2t^2$$ ☞

Study comment Having seen the *Fast track questions* you may feel that it would be wiser to follow the normal route through the module and to proceed directly to *Ready to study?* in Subsection 1.3.

Alternatively, you may still be sufficiently comfortable with the material covered by the module to proceed directly to the *Closing items*.

1.3 Ready to study?

Study comment In order to study this module you will need to be familiar with the following terms: *degree* (of a polynomial), *exponential function*, *general solution*, *initial condition*, *linear differential equation* and *particular solution*. You will also need to be familiar with various *trigonometric identities* (although we will repeat them here); to be able to solve *first-order differential equations* by *direct integration* (which requires a fair knowledge of *integration* methods); to know how to check a proposed solution to a differential equation by substitution (which requires good *differentiation* skills); and to be able to use *initial conditions* to obtain a *particular solution* from a *general solution*. To appreciate the physical significance of some of the equations appearing in this module, you should know how to use *Newton's second law* to write down a differential equation describing the motion of an object if you are given information about the *forces* acting on it. Some familiarity with *simple harmonic motion* (SHM) would be particularly useful. If you are uncertain about any of these terms, you can review them now by reference to the *Glossary*, which will also indicate where in *FLAP* they are developed. The following *Ready to study questions* will allow you to establish whether you need to review some of these topics before embarking on this module.

Question R1

Find the general solution to the differential equation

$$\frac{dy}{dx} = \frac{x}{1 + x^2}$$

Question R2

If $y = 4e^{x/2} \cos 2x$, calculate dy/dx and d^2y/dx^2.

Question R3

Show by substitution that $y = \frac{1}{4}(2x - 3) + Ae^{-x}$, where A is an arbitrary constant, is a solution to

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = x$$

Is it a general solution?

Question R4

In answering this question, you should make use only of trigonometric identities, you should *not* use your calculator. ☞

(a) If $\cos\theta = 1/3$, what is the value of $\sin\theta$, given $0 < \theta < \pi/2$?

(b) If $\cos\theta = 1/3$ and $\sin\phi = -1/2$, what is the value of $\cos(\theta + \phi)$, given $0 < \theta < \pi/2$, $\pi < \phi < 3\pi/2$?

Question R5

In each of the two following expressions for y, use the given initial conditions to calculate values for the arbitrary constants A and B.

(a) $y = (At + B)e^{-2t}$; where $y = 1$ and $dy/dt = 3$ at $t = 0$.

(b) $y = A\cos 2x + B\sin 2x$; where $y = 4$ and $dy/dx = -2$ at $x = 0$.

Question R6

An object of mass m is free to move along a line; its displacement from the origin is denoted by x. It is acted on by three forces: a restoring force, the magnitude of which is proportional to the magnitude of x; a resistive (or damping) force, the magnitude of which is proportional to the cube of the object's speed; and a constant force of magnitude F_0 acting in the negative x-direction. Use Newton's second law to write down the second-order differential equation that x must satisfy.

2 Methods of solution for various second-order differential equations

Notation Very often in the problems that arise in physics the independent variable is time, and so, when discussing a general type of differential equation, the independent variable is denoted by t and the dependent variable by y. However, in some of the examples, drawn from specific problems in physics, we will use the notation for the variables that is appropriate to the situation. Bear in mind that the quantity being differentiated will *always* be the dependent variable.

2.1 Classifying second-order differential equations

There is such a wide variety of second-order differential equations that it is useful to divide them into various categories, and in this subsection we will introduce some terminology that arises when we do so. You are probably already familiar with the distinction between *linear* and *non-linear* differential equations. The most general *linear second-order differential equation* is of the form

$$a(t)\frac{d^2y}{dt^2} + b(t)\frac{dy}{dt} + c(t)y = f(t)$$

where $a(t)$, $b(t)$, $c(t)$ and $f(t)$ are functions of t.

However, in many of the second-order linear equations that you will encounter in your study of physics the dependent variable and its derivatives appear multiplied only by constants rather than functions of t. Such an equation is known as a **linear differential equation with constant coefficients** and has the form

$$\boxed{a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = f(t) \qquad (5)}$$

☞

Equation 5

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = f(t) \qquad \text{(Eqn 5)}$$

is easiest to solve when its right-hand side is zero, i.e. when

$$\boxed{a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = 0 \qquad (6)}$$

As you see, all the terms in Equation 6 contain the dependent variable y or a derivative of y; for that reason, an equation of this form is called a **linear homogeneous differential equation**. We will discuss equations of this sort in Subsections 2.3, 2.4 and 2.5. If $f(t)$ in Equation 5 is not zero, then the equation is called a **linear inhomogeneous differential equation** (as there is now a term in the equation which does not involve the dependent variable y). Equation 6 is relatively easy to solve, whereas the ease with which we are able to solve Equation 5 is crucially dependent on the form of the function $f(t)$.

Question T1

State whether the following differential equations are linear with constant coefficients and, if so, whether they are homogeneous or inhomogeneous:

(a) $\dfrac{d^2y}{dx^2} = x - 3y$ (b) $x\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + y = 0$ (c) $\dfrac{d^2x}{dt^2} - 5\dfrac{dx}{dt} = 6x$ ❏

We will first dispose of the easiest of all second-order differential equations — those that may be solved by *direct integration*. This is the subject of the next subsection.

2.2 Equations of the form $d^2y/dt^2 = f(t)$; direct integration

You should already know how to solve a *first*-order differential equation ☞ of the form

$$\frac{dy}{dt} = f(t) \tag{7}$$

where the derivative of the dependent variable is equal to a function of the independent variable. This equation may be integrated directly, to give

$$y = \int f(t)\,dt \tag{8}$$

The indefinite integral will involve one arbitrary constant of integration, this indicates that Equation 8 gives us the *general solution* to Equation 7.

It is just as straightforward to solve a second-order differential equation of the form

$$\frac{d^2y}{dt^2} = f(t) \tag{9}$$

where the *second* derivative of the dependent variable is equal to a function of the independent variable. Here, we simply integrate directly *twice*. Recalling that d^2y/dt^2 is the derivative of dy/dt, we see that if we integrate both sides of Equation 9, we obtain

$$\frac{dy}{dt} = \int f(t)\,dt = F(t) + A \tag{10}$$

where $F(t)$ is an indefinite integral ☞ of $f(t)$ (i.e. $F(t)$ is a function such that $F'(t) = dF(t)/dt = f(t)$), and A is a constant of integration. If we integrate Equation 10 again we obtain:

$$y = \int \bigl(F(t) + A\bigr)\,dt = G(t) + At + B \tag{11}$$

where $G(t)$ is an indefinite integral of $F(t)$, and B is another constant of integration. We see that Equation 11 contains *two* arbitrary constants, A and B; this indicates that it is the general solution to Equation 9.

Equations which can be directly integrated often arise when considering objects that are moving along a line. For instance, an object moving along the x-axis, with position coordinate x and acceleration $a_x(t)$, a known function of time, satisfies the equation

$$\frac{d^2x}{dt^2} = a_x(t) \tag{12}$$

The following example illustrates this.

Example 1 A car is travelling in the x-direction along a straight road. At time $t = 0$, it starts to accelerate, and for the next 8 s, its acceleration $a_x(t)$ is given by $a_x(t) = a + bt$. If $a = 2.00\,\mathrm{m\,s^{-2}}$, $b = 0.50\,\mathrm{m\,s^{-3}}$ and the car's velocity is $5.00\,\mathrm{m\,s^{-1}}$ at $t = 0$, how far does the car travel during the 8 s period?

Solution We must first solve the differential equation

$$\frac{d^2x}{dt^2} = a + bt$$

☞

where $x = x(t)$ is the position of the car and we take the origin to be the position of the car at $t = 0$, so that $x = 0$ at $t = 0$. Integrating once gives us

$$\frac{dx}{dt} = \int (a + bt)\,dt = at + b\left(\frac{t^2}{2}\right) + A = at + \frac{b}{2}t^2 + A$$

where A is an arbitrary constant. On substituting $dx/dt = 5.00\,\mathrm{m\,s^{-1}}$ at $t = 0$ into this equation, we find $A = 5\,\mathrm{m\,s^{-1}}$.

Integrating dx/dt to find x gives us

$$x = \int\left(at + \frac{b}{2}t^2 + A\right)dt = \frac{at^2}{2} + \frac{bt^3}{6} At + B$$

where B is an arbitrary constant. If we substitute $x = 0$ at $t = 0$, we find $B = 0$.

So $\quad x = (0.50\,\mathrm{m\,s^{-3}})\dfrac{t^3}{6} + (2.00\,\mathrm{m\,s^{-2}})\dfrac{t^2}{2} + (5.00\,\mathrm{m\,s^{-1}})t$

Therefore at $t = 8\,\mathrm{s}$, we find that the car has travelled 146.6 m. ❏

Sometimes, instead of being given information directly about the acceleration of the object, you may be given the force $F_x(t)$ acting on it as a function of time. However, since $F_x(t) = ma_x(t)$, where m is the mass of the object, it is very easy to write down a differential equation of the form given in Equation 12.

$$\frac{d^2x}{dt^2} = a_x(t) \qquad \text{(Eqn 12)}$$

Such a differential equation, arising from an application of Newton's second law, is often called an **equation of motion**.

Try the following question, which illustrates another example.

Question T2

An object of mass 2 kg, which is constrained to move along the x-axis, is subject to a force $F_x(t) = F_0 \cos(\Omega t)$, ☞ acting along the x-axis (in the direction of increasing x), where $F_0 = 4\,\mathrm{N}$ and $\Omega = 2\,\mathrm{s}^{-1}$. At time $t = 0$, the object's displacement is $+3\,\mathrm{m}$ and its velocity is $+1\,\mathrm{m\,s}^{-1}$. Find its displacement as a function of time. ❏

2.3 The equation for simple harmonic motion: $d^2y/dt^2 + \omega^2 y = 0$

We will now make a start at finding solutions of Equation 6, ☞

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = 0 \qquad \text{(Eqn 6)}$$

In this subsection (and the next), we will look at the simplified case where $b = 0$, so that Equation 6 takes the form

$$a\frac{d^2y}{dt^2} + cy = 0 \qquad (13)$$

In Equation 13, $a \neq 0$ (so the equation is of second order) and $c \neq 0$. (If c were zero, we could solve the equation by direct integration.) Since $a \neq 0$, we can divide both sides of the equation by a, and write $h = c/a$ to give:

$$\boxed{\frac{d^2y}{dt^2} + hy = 0} \qquad (14)$$

$$\frac{d^2y}{dt^2} + hy = 0 \qquad (14)$$

You will see shortly that the form of solution of Equation 14 depends on whether the constant h is positive or negative. We will consider these two possibilities separately, starting with the case of positive h. (The case of negative h will be discussed in Subsection 2.4.) If h is positive, we can make it clear that this is so by writing $h = \omega_0{}^2$, so that Equation 14 becomes:

$$\frac{d^2y}{dt^2} + \omega_0^2 y = 0 \quad (\textit{SHM equation}) \qquad (15)$$

Before discussing the solutions to Equation 15, we will emphasize the importance of this equation in physics. It is the equation of motion of any object undergoing **simple harmonic motion** (SHM) — motion where the force acting on the object is proportional to the object's displacement from some point and acts in the opposite direction to the displacement. For this reason, Equation 15 is often known as the **SHM equation**.

You may already know that an object undergoing SHM executes *oscillations* and we will see shortly that the *period* of the oscillations depends on the parameter ω_0 which is known as the **angular frequency** in the context of Equation 15.

$$\frac{d^2y}{dt^2} + \omega_0^2 y = 0 \qquad \text{(Eqn 15)}$$

So it is important that whenever you encounter a case of SHM, you should be able to rewrite the differential equation describing the physical situation in the form of Equation 15, and so find an expression for ω_0 in terms of the parameters appearing in the problem. The following question gives you practice in this.

Question T3

Rewrite each of the following equations in the form of Equation 15,

$$\frac{d^2y}{dt^2} + \omega_0^2 y = 0 \qquad \text{(Eqn 15)}$$

and so find an expression for ω_0:

(a) $$L\frac{d^2I}{dt^2} + \frac{I}{C} = 0$$

This equation describes the way in which the current I varies with time in a circuit containing an inductor L and a charged capacitor C, as shown in Figure 1.

(b) $$ml\frac{d^2\theta}{dt^2} = -mg\theta$$

This equation describes the motion of a simple pendulum — a mass m suspended from a fixed point by a string of length l, as shown in Figure 2. ❏

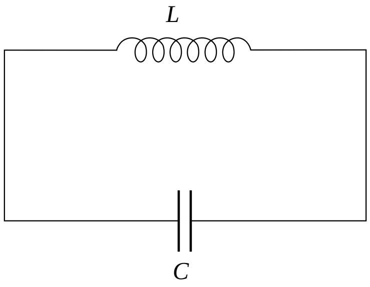

Figure 1 See Question T3(a).

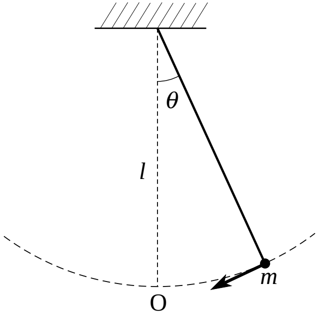

Figure 2 See Question T3(b).

It is not too difficult to guess the general solution of Equation 15.

$$\frac{d^2y}{dt^2} + \omega_0^2 y = 0 \qquad \text{(Eqn 15)}$$

Any solution of this equation must be a function such that when we differentiate it twice, we recover the function itself, multiplied by $-\omega_0{}^2$. This should remind you of a cosine or a sine function. In fact, both $\cos(\omega_0 t)$ and $\sin(\omega_0 t)$ have precisely this property, as does any function of the form $B\cos(\omega_0 t) + C\sin(\omega_0 t)$, where B and C are arbitrary constants.

❖ Show that $y = B\cos(\omega_0 t) + C\sin(\omega_0 t)$ is a solution to Equation 15.

You have now shown that

the first form of the solution to the SHM equation (Equation 15) is:

$$y(t) = B\cos(\omega_0 t) + C\sin(\omega_0 t) \qquad (16)$$

Since it contains two arbitrary constants B and C, it is the general solution.

Question T4

Write down the general solution of the following equations:

(a) $\dfrac{d^2y}{dx^2} + 25y = 0$ (b) $9\dfrac{d^2Q}{dx^2} = -4Q$ ❏

It is easy to see from Equation 16

$$y(t) = B\cos(\omega_0 t) + C\sin(\omega_0 t) \qquad \text{(Eqn 16)}$$

that $y(t)$ is a **periodic function** of t — that is, a function that 'repeats itself' each time t increases by a fixed amount, known as the **period** of the function. ☞ As you know, the value of a sine or cosine function is unchanged if the argument of the function is increased by 2π, or an integer multiple of 2π. Thus y has the same value if t increases by an amount $T = 2\pi/\omega_0$.

The quantity $T = \dfrac{2\pi}{\omega_0}$ is therefore the *period* of the function given in Equation 16.

The solution to Equation 15 in a different form

In order to see some other features of the dependence of y on t, it is helpful to write the general solution to Equation 15

$$\frac{d^2y}{dt^2} + \omega_0^2 y = 0 \qquad \text{(Eqn 15)}$$

in a different form. This is done by expressing the arbitrary constants B and C that appear in Equation 16

$$y(t) = B\cos(\omega_0 t) + C\sin(\omega_0 t) \qquad \text{(Eqn 16)}$$

in terms of two other arbitrary constants, A and ϕ, as follows:

$$B = A\sin\phi \qquad \text{(17a)}$$

and $$C = A\cos\phi \qquad \text{(17b)}$$ ☞

With these expressions for B and C, Equation 16

$$y(t) = B\cos(\omega_0 t) + C\sin(\omega_0 t) \qquad \text{(Eqn 16)}$$

becomes

$$y = A\sin\phi\,\cos(\omega_0 t) + A\cos\phi\,\sin(\omega_0 t) = A[\sin\phi\,\cos(\omega_0 t) + \cos\phi\,\sin(\omega_0 t)] \tag{18}$$

We may now use the trigonometric identity ☞

$$\sin(\alpha + \beta) = \sin\alpha\,\cos\beta + \cos\alpha\,\sin\beta$$

to rewrite the right-hand side of Equation 18 and make it look a lot neater.

$$\frac{d^2y}{dt^2} + \omega_0^2 y = 0 \qquad \text{(Eqn 15)}$$

Thus:

the second form of the solution to the SHM equation (Equation 15) is:

$$y(t) = A\sin(\omega_0 t + \phi) \qquad (19)$$

Equation 19 shows us that, for any particular choice of the constants A and ϕ, the graph of the solution to Equation 15 always has the shape of a sine curve (though it may not pass through the origin). Such a curve is called **sinusoidal**; an example is shown in Figure 3 (next page). We can also deduce from Equation 19 that, since the maximum value attained by a sine function is 1, the maximum value of y is equal to the constant A; therefore A is the **amplitude** of the oscillations. The other constant ϕ is known as the **phase constant** or **initial phase** of the oscillations. Clearly the value of y at $t = 0$ depends on both A and ϕ since it is equal to $A\sin\phi$. Thus, if we know the value of ω_0, and we can discover the values of A and ϕ (so that we are dealing with a particular solution of Equation 15) we can easily use Equation 19 to construct the graph of the solution.

☞

The example shown in Figure 3 corresponds to the values $A = 2\,\mathrm{m}$, $\phi = \pi/6$ and $\omega_0 = 2\,\mathrm{s}^{-1}$, so that $y = (2\,\mathrm{m})\sin[(2\,\mathrm{s}^{-1})t + \pi/6]$.

(Notice that $y = 0$ when $t = -(\pi/12)\,\mathrm{s}$, and $y = 1\,\mathrm{m}$ when $t = 0$.)

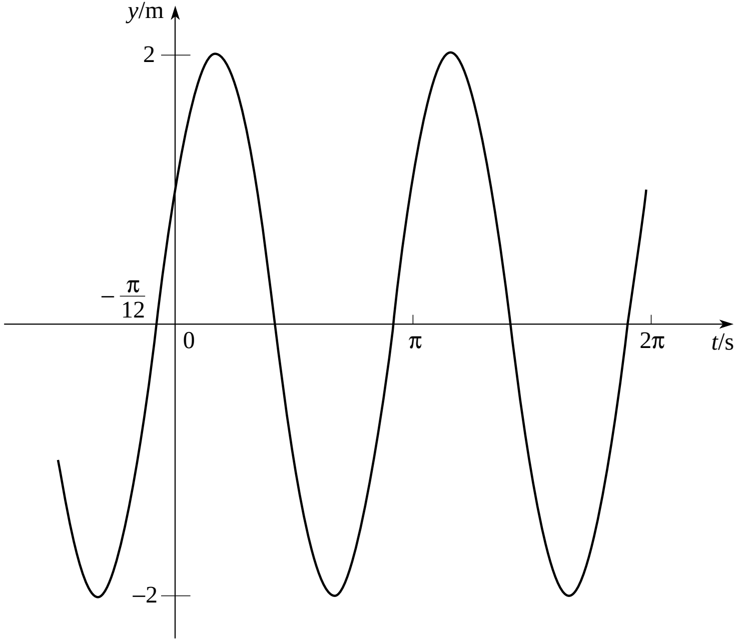

Figure 3 A sinusoidal curve with period $\pi\,\mathrm{s}^{-1}$, amplitude $2\,\mathrm{m}$ and phase constant $\pi/6$.

Converting from one form of the solution into the other

It is clearly important to be able to switch between the two forms of the solution to the SHM equation given in Equations 16 and 19.

$$y(t) = B\cos(\omega_0 t) + C\sin(\omega_0 t) \qquad \text{(Eqn 16)}$$

$$y(t) = A\sin(\omega_0 t + \phi) \qquad \text{(Eqn 19)}$$

From Equations 17a and b we can easily calculate B and C if we are given values for A and ϕ; but how do we calculate A and ϕ from known values of B and C? To see how to do this, note that if

$$B = A\sin\phi \quad \text{and } C = A\cos\phi \qquad \text{(Eqns 17a, b)}$$

then

$$B^2 + C^2 = A^2(\sin^2\phi + \cos^2\phi) = A^2$$

so that

$$A^2 = B^2 + C^2$$

By convention, A is *always* taken to be positive, so we have the following two equations for A and ϕ, where Equation 21 is obtained from Equations 17a and b

$$B = A\sin\phi \quad \text{and } C = A\cos\phi \qquad \text{(Eqns 17a, b)}$$

by dividing the expression for B by the expression for C. ☞

Equations converting from the first form of the solution of the SHM equation into the second form:

$$A = \sqrt{B^2 + C^2} \qquad \text{(positive square root)} \qquad (20)$$

$$\frac{B}{C} = \frac{A\sin\phi}{A\cos\phi} = \tan\phi \quad \text{i.e. } \phi = \arctan(B/C) \text{ but with } 0 \le \phi < 2\pi \qquad (21)$$

By convention, ϕ is *always* chosen to lie within the range $0 \le \phi < 2\pi$ in this context. You may be wondering why it cannot be chosen to lie within the range $-\pi/2 \le \phi < \pi/2$ — the standard range for the inverse tan function — the reason is that although the function $\tan\phi$ repeats itself as ϕ is increased by π, $\sin\phi$ and $\cos\phi$ do not. Thus the range $-\pi/2 \le \phi < \pi/2$ would not cover all possible values (both positive and negative) of the constants B and C.

In fact, using Equation 17,

$$B = A\sin\phi \quad \text{and } C = A\cos\phi \qquad \text{(Eqns 17a, b)}$$

and our knowledge of the properties of $\sin\phi$ and $\cos\phi$, we can lay down some rules for the range of values within which ϕ must lie, depending on whether B and C are positive or negative:

$$B \ge 0 \text{ and } C \ge 0 \quad \Rightarrow \quad 0 \le \phi \le \pi/2 \qquad (22a)$$

$$B \ge 0 \text{ and } C < 0 \quad \Rightarrow \quad \pi/2 < \phi \le \pi \qquad (22b)$$

$$B < 0 \text{ and } C < 0 \quad \Rightarrow \quad \pi < \phi < 3\pi/2 \qquad (22c)$$

$$B < 0 \text{ and } C \ge 0 \quad \Rightarrow \quad 3\pi/2 \le \phi < 2\pi \qquad (22d)$$

You can now use Equations 17, 20, 21 and 22

$$A = \sqrt{B^2 + C^2} \qquad \text{(positive square root)} \qquad \text{(Eqn 20)}$$

$$\frac{B}{C} = \frac{A\sin\phi}{A\cos\phi} = \tan\phi \quad \text{i.e. } \phi = \arctan(B/C) \text{ but with } 0 \le \phi < 2\pi \qquad \text{(Eqn 21)}$$

to answer the following questions:

Question T5

Write the particular solution $y = 6\sin(3t + \pi/3)$ in the form $y = B\cos(\omega_0 t) + C\sin(\omega_0 t)$. ❑

Question T6

Write the two following particular solutions in the form $y = A\sin(\omega_0 t + \phi)$:

(a) $y = 4\cos(2t) + 3\sin(2t)$

(b) $y = 5\sin(4t) - 12\cos(4t)$ ❑

2.4 Equations of the form $d^2y/dt^2 - \lambda^2 y = 0$

We will now return to the case of negative h in Equation 14.

$$\frac{d^2y}{dt^2} + hy = 0 \qquad \text{(Eqn 14)}$$

If h is negative, we can make it clear that this is so by writing $h = -\lambda^2$, where (by convention) $\lambda > 0$, so that Equation 14 becomes

$$\boxed{\frac{d^2y}{dt^2} - \lambda^2 y = 0 \qquad (23)}$$

You may not have yet encountered any physical situations that could be described by an equation of this sort. However, it does have important applications in quantum physics and elsewhere. So it is well worth your while to learn how to solve this equation. Moreover, the solution is very easy to find.

To solve Equation 23,

$$\frac{d^2y}{dt^2} - \lambda^2 y = 0 \qquad \text{(Eqn 23)}$$

let us proceed using the same sort of informed guesswork that we used to solve Equation 15.

$$\frac{d^2y}{dt^2} + \omega_0^2 y = 0 \qquad \text{(Eqn 15)}$$

In the case of Equation 15, we wanted to find a function such that its second derivative was equal to the function itself multiplied by a *negative* constant. Looking at Equation 23, we see that this time we want a function (or functions) the second derivative of which is equal to the function itself multiplied by a *positive* constant. *Exponential functions* have the property that *all* their derivatives are proportional to the function itself.

So let us see if an exponential function, of the form

$$y = Be^{pt} \tag{24}$$

where B is an arbitrary constant, is a solution of Equation 23.

If we differentiate Equation 24 once,

$$y = Be^{pt} \qquad \text{(Eqn 24)}$$

we find $dy/dt = pBe^{pt} = py$.

When we differentiate again, we find

$$\frac{d^2y}{dt^2} = p\frac{dy}{dt} = p^2y$$

and on substituting this result into Equation 23,

$$\frac{d^2y}{dt^2} - \lambda^2 y = 0 \qquad \text{(Eqn 23)}$$

we obtain

$$p^2y - \lambda^2 y = 0$$

which is an identity ☞ provided that $p^2 = \lambda^2$.

Thus Equation 24 is a solution to Equation 23 provided $p = +\lambda$ or $p = -\lambda$.

It follows that any function of the form

$$y = B\mathrm{e}^{\lambda t} \qquad (25a)$$

or $$y = C\mathrm{e}^{-\lambda t} \qquad (25b)$$

(We have replaced the arbitrary constant B in Equation 25a by the arbitrary constant C in Equation 25b.)

is a solution to Equation 23.

$$\frac{d^2y}{dt^2} - \lambda^2 y = 0 \qquad \text{(Eqn 23)}$$

Neither of these equations can themselves be the general solution of Equation 23, as neither contains *two* arbitrary constants. But perhaps their *sum*, which does contain two arbitrary constants, is the general solution.

✦ Show that $y = B\mathrm{e}^{\lambda t} + C\mathrm{e}^{-\lambda t}$ is a solution to Equation 23.

$$\frac{d^2y}{dt^2} - \lambda^2 y = 0 \qquad \text{(Eqn 23)}$$

So you have shown that:

The general solution of Equation 23 is

$$y(t) = Be^{\lambda t} + Ce^{-\lambda t} \qquad (26)$$

Since Equation 26 contains two arbitrary constants, it is the general solution. You may wonder what this solution looks like graphically. If either B or C is zero, the curve is exponential. Otherwise, the solution given in Equation 26 behaves like $y = B\exp(\lambda t)$ when t is large and positive (as then $\exp(-\lambda t)$ is very small), and like $y = C\exp(-\lambda t)$ when t is large and negative (as then $\exp(\lambda t)$ is very small). Figure 4 (next page) shows the four general shapes of curve you can expect, depending on the different signs of B and C.

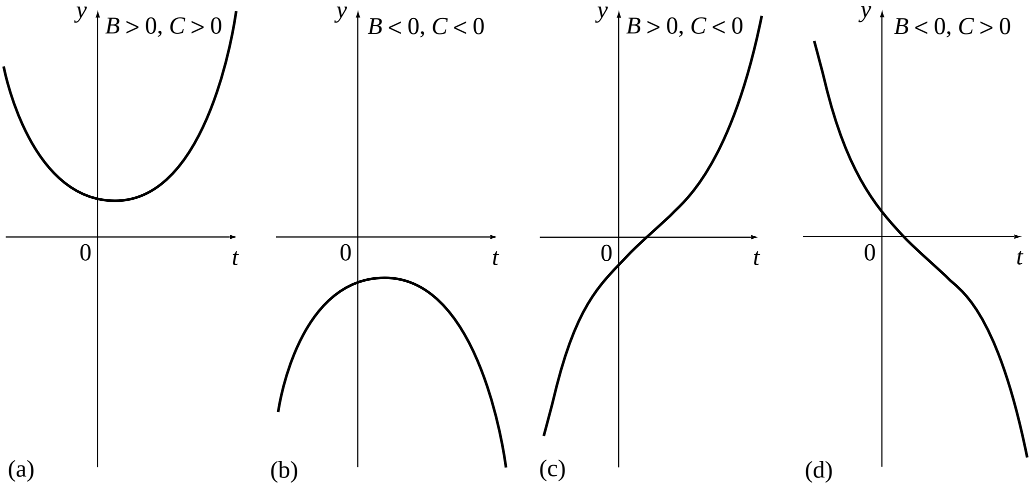

Figure 4 Solutions to Equation 23. General solution is $y(t) = Be^{\lambda t} + Ce^{-\lambda t}$ (Equation 26)

Question T7

Find the general solution of

$$\frac{d^2y}{dx^2} - 4y = 0$$

and the particular solution if $y = 6$ and $dy/dx = 0$ at $x = 0$. ❏

We obtained the general solution given in Equation 26

$$y(t) = Be^{\lambda t} + Ce^{-\lambda t} \qquad \text{(Eqn 26)}$$

by taking the *sum* of the two solutions given in Equation 25.

$$y = Be^{\lambda t} \quad \text{(25a)} \qquad \text{or} \qquad y = Ce^{-\lambda t} \quad \text{(Eqn 25b)}$$

In fact, it can be proved that, for *any* linear homogeneous equation (not necessarily one with constant coefficients) the sum of two solutions (or more than two, if the equation is of higher order than the second) is always also a solution. We will not give the proof here (although the last marginal note should provide a hint as to how the proof might go), but you should bear this useful result in mind; we will make use of it in the next subsection.

2.5 The equation for damped harmonic motion: $a(d^2y/dt^2) + b(dy/dt) + cy = 0$

In Subsection 2.3, we mentioned that an equation of the form

$$a\frac{d^2y}{dt^2} + cy = 0 \quad \text{where } c/a = h > 0$$

has many different applications in physics, describing, as it does, the behaviour of an object undergoing simple harmonic motion. Its solutions are sinusoidal oscillations of constant amplitude. However, you know from experience that the vibrations of an oscillating object always die away with time (pendulum clocks run down; masses bobbing up and down on springs come to rest, and so on). This is due to the effect of *resistive* or **damping** forces, which oppose the motion of the oscillating object. How can we modify the SHM equation (Equation 15).

$$\frac{d^2y}{dt^2} + \omega_0^2 y = 0 \qquad \text{(Eqn 15)}$$

to take account of these forces? Let us return to the example mentioned in Subsection 1.1, a mass m oscillating up and down on a spring of force constant k.

If the restoring force of the spring, $-ky$, is the only force acting on the mass, then, according to Newton's second law, its displacement $y(t)$ must satisfy the equation

$$m\frac{d^2y}{dt^2} = -ky$$

To incorporate the effects of a damping force, we must add a term on the right-hand side which always acts in a direction *opposite* to the direction of the velocity of the mass (the bob). A simple way of doing this is to assume that the damping force can be written in the form $-b(dy/dt)$, where b is a positive constant, so that the equation of motion of the mass becomes

$$m\frac{d^2y}{dt^2} = -ky - b\frac{dy}{dt}$$

i.e. $$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = 0 \tag{27}$$

An equation of the form given in Equation 27

$$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = 0 \qquad \text{(Eqn 27)}$$

also arises in the theory of a.c. circuits.

The current I in the circuit shown in Figure 5, containing an inductance L, a resistor R and a capacitor C, obeys the differential equation

$$L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{I}{C} = 0 \qquad (28)$$

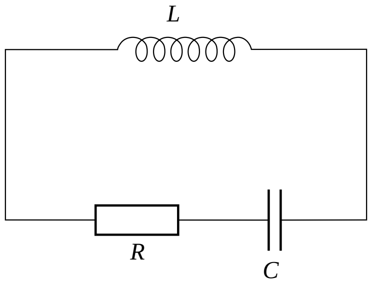

Figure 5 A circuit containing an inductance L, a resistor R and a capacitor C, connected in series.

To predict the behaviour of the mass on the spring, or the current in the circuit, you need to be able to solve differential equations of the type

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = 0 \quad \text{where } a > 0,\ b > 0 \text{ and } c > 0 \tag{29}$$

We need the ratio b/a to be positive if the coefficient of dy/dt is to represent a *resistive* force, opposing the velocity of the object; and we need the ratio c/a to be positive so that if b is zero, we recover the SHM equation (Equation 15).

$$\frac{d^2y}{dt^2} + \omega_0^2 y = 0 \qquad \text{(Eqn 15)}$$

This condition is most simply achieved by restricting all three constants to positive values. (In fact, the solutions we obtain to Equation 29 will apply equally well if any of a, b, or c is negative.)

Let us now try to find solutions to Equation 29.

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = 0 \quad \text{where } a > 0,\ b > 0 \text{ and } c > 0 \tag{Eqn 29}$$

We know that if $b = 0$, then the solution is a sinusoidal function. However, we do not *want* a purely sinusoidal solution if $b > 0$ (we want y to tend to zero as t becomes large and positive, due to the damping); moreover, if you try a solution of the form $y = \sin(pt)$ or $\cos(pt)$ in Equation 29, you will quickly find that it does not work (unless $b = 0$, of course). But perhaps an exponential function would work, just as it did for Equation 23?

$$\frac{d^2y}{dt^2} - \lambda^2 y = 0 \tag{Eqn 23}$$

We have nothing to lose by trying it, so let us substitute

$$y = Be^{pt} \tag{30}$$

into Equation 29.

The first derivative of y is equal to py, and the second derivative is equal to p^2y; so we find, on substitution,

$$ap^2y + bpy + cy = 0$$

which is an identity provided that p satisfies the equation

$$ap^2 + bp + c = 0 \qquad (\textit{auxilliary equation}) \tag{31}$$

This quadratic equation in p is known as the **auxiliary equation** of Equation 29.

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = 0 \quad \text{where } a > 0,\ b > 0 \text{ and } c > 0 \qquad \text{(Eqn 29)}$$

As it is a quadratic, the roots of Equation 31 are given by ☞

$$p_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad p_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \tag{32}$$

This formula will give us two real values for p_1 and p_2 provided that $b^2 > 4ac$. (We will deal with the cases where $b^2 < 4ac$ or $b^2 = 4ac$ shortly; but for the moment, let us assume that the two roots are real.)

We have shown, therefore, that the trial solution, Equation 30,

$$y = Be^{pt} \qquad \text{(Eqn 30)}$$

will satisfy Equation 29

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = 0 \quad \text{where } a > 0,\ b > 0 \text{ and } c > 0 \qquad \text{(Eqn 29)}$$

for any value of B provided that p is one of the two roots given in Equation 32.

$$p_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad p_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \qquad \text{(Eqn 32)}$$

Real roots of the auxiliary equation (heavy damping)

We can write the two solutions of Equation 29

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = 0 \quad \text{where } a > 0,\ b > 0 \text{ and } c > 0 \qquad \text{(Eqn 29)}$$

we have just found as

$$y = B\exp(p_1t) \quad \text{and} \quad y = C\exp(p_2t)$$

As we mentioned at the end of Subsection 2.4, we can obtain the general solution to Equation 29 by adding these two solutions together:

The general solution of Equation 29 in the case $b^2 > 4ac$ is

$$y(t) = B\exp(p_1t) + C\exp(p_2t) \qquad (33)$$

where $p_1 = \dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$ and $p_2 = \dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$

or, written out in full,

$$y = B\exp\left(\frac{-b+\sqrt{b^2-4ac}}{2a}t\right) + C\exp\left(\frac{-b-\sqrt{b^2-4ac}}{2a}t\right) \tag{34}$$

There is no need to memorize this unpleasant-looking equation—but you should remember the method that we used to derive it, and be able to apply it to a given differential equation, as in the following example.

Example 2 Find the general solution of the differential equation

$$\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y = 0$$

Solution First write down the auxiliary equation,

$$p^2 + 5p + 6 = 0$$

This equation has two real roots:

$$p = -\tfrac{5}{2} \pm \tfrac{1}{2}\sqrt{25-24} = -2 \text{ or } -3$$ ☞

Thus the general solution is $y = Be^{-2t} + Ce^{-3t}$. ❏

Question T8

Find the general solution of the differential equation

$$6\frac{d^2y}{dt^2} + 17\frac{dy}{dt} + 12y = 0 \quad ❏$$

You can see that if a, b and c are all positive quantities, the positive square root $\sqrt{b^2 - 4ac}$ must be less than b, and therefore both roots of the auxiliary equation are *negative*. Thus the solution to Equation 29

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = 0 \quad \text{where } a > 0,\ b > 0 \text{ and } c > 0 \qquad \text{(Eqn 29)}$$

that is given in Equation 34

$$y = B\exp\left(\frac{-b + \sqrt{b^2 - 4ac}}{2a}t\right) + C\exp\left(\frac{-b - \sqrt{b^2 - 4ac}}{2a}t\right) \qquad \text{(Eqn 34)}$$

is the sum of two decreasing exponentials, and as t becomes very large and positive, y tends to zero. An example of this is shown in Figure 6.

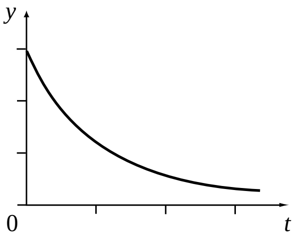

Figure 6 Heavily damped motion.

Remember that in physical situations b gives an indication of the magnitude of the damping force. If $b = 0$ the motion is *undamped* and we have oscillations corresponding to SHM (as in Equation 15),

$$\frac{d^2y}{dt^2} + \omega_0^2 y = 0 \qquad \text{(Eqn 15)}$$

but if b is so large that $b^2 > 4ac$, then there are no oscillations (as in Figure 6). In such a case, the motion is said to be *heavily damped*. You can probably guess what will happen if we choose a value for b somewhere between these two extremes. Nonetheless, we will now analyse such problems systematically.

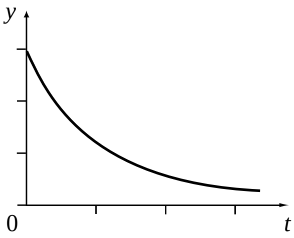

Figure 6 Heavily damped motion.

Complex roots of the auxiliary equation (light damping)

We will now solve Equation 29

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = 0 \quad \text{where } a > 0,\ b > 0 \text{ and } c > 0 \qquad \text{(Eqn 29)}$$

for the case $b^2 < 4ac$. We will give two alternative treatments of this important case.

The first treatment depends on a result that comes from the study of complex numbers, and shows that the solution in this case is merely an extension of the previous result.

The second treatment does not require as much knowledge of complex numbers, we will just give you the solution of the differential equation and ask you to verify that what we say is correct.

The first treatment requires the following result from the theory of complex numbers

$$e^{i\theta} = \cos\theta + i\sin\theta \tag{35}$$

If you are familiar with this result continue reading; if not go straight to Question T9.

If $b^2 < 4ac$ in Equation 29,

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = 0 \quad \text{where } a > 0,\ b > 0 \text{ and } c > 0 \qquad \text{(Eqn 29)}$$

then the two roots of Equation 31

$$ap^2 + bp + c = 0 \qquad \text{(auxilliary equation)} \qquad \text{(Eqn 31)}$$

are *complex* and can be written (using Equation 32)

$$p_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad p_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \qquad \text{(Eqn 32)}$$

as $\quad p_1 = -\dfrac{\gamma}{2} + i\omega \quad$ and $\quad p_2 = -\dfrac{\gamma}{2} + i\omega$

where $\gamma = b/a \quad$ and $\quad \omega = \sqrt{4ac - b^2}\,/(2a)$

(Notice that, since $b^2 < 4ac$, we can be sure that ω is a real quantity.)

Thus, the general solution to Equation 29

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = 0 \quad \text{where } a > 0,\ b > 0 \text{ and } c > 0 \qquad \text{(Eqn 29)}$$

when $b^2 < 4ac$ is similar to that given in Equation 33

$$y(t) = B\exp(p_1t) + C\exp(p_2t) \qquad \text{(Eqn 33)}$$

and we may write it as

$$y = R\exp\left[\left(-\frac{\gamma}{2}+i\omega\right)t\right] + S\exp\left[\left(-\frac{\gamma}{2}-i\omega\right)t\right] = \exp(-\gamma t/2)[R\exp(i\omega t)] + [S\exp(-i\omega t)] \qquad (36)$$

where R and S are arbitrary constants.

It looks as though this solution will give complex values to y. However, since the case $b^2 < 4ac$ frequently arises in physics problems, it must be possible to arrange Equation 36

$$y = R\exp\left[\left(-\frac{\gamma}{2} + i\omega\right)t\right] + S\exp\left[\left(-\frac{\gamma}{2} - i\omega\right)t\right] = \exp(-\gamma t/2)\left[R\exp(i\omega t)\right] + \left[S\exp(-i\omega t)\right] \qquad \text{(Eqn 36)}$$

in a form which need not involve any complex quantities. We do this by employing Equation 35,

$$e^{i\theta} = \cos\theta + i\sin\theta \qquad \text{(Eqn 35)}$$

to write

$$\exp(i\omega t) = \cos(\omega t) + i\sin(\omega t) \quad \text{and} \quad \exp(-i\omega t) = \cos(\omega t) - i\sin(\omega t)$$

if we substitute these results into Equation 36, we find

$$y = \exp(\gamma t/2)[(R + S)\cos(\omega t) + i(R - S)\sin(\omega t)]$$

We can now define new arbitrary constants $B = R + S$ and $C = i(R - S)$, and rewrite the solution as

$$y = \exp(-\gamma t/2)[B\cos(\omega t) + C\sin(\omega t)]$$

You may think that this rewriting has achieved nothing since C may be complex, so we have still not ensured that the solution $y(t)$ is a real quantity. However, at this point it is important to realize that the constants R and S in Equation 36

$$y = R\exp\left[\left(-\frac{\gamma}{2}+i\omega\right)t\right] + S\exp\left[\left(-\frac{\gamma}{2}-i\omega\right)t\right] = \exp(-\gamma t/2)\left[R\exp(i\omega t)\right] + \left[S\exp(-i\omega t)\right] \qquad \text{(Eqn 36)}$$

are quite arbitrary, *they did not even have to be real*. Equation 36 is a perfectly valid solution to Equation 29

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = 0 \quad \text{where } a > 0,\ b > 0 \text{ and } c > 0 \qquad \text{(Eqn 29)}$$

even if R and S are complex. In fact, if R and S are complex conjugates, i.e. complex numbers of the form

$$R = a + ib \quad \text{and} \quad S = a - ib \quad \text{where } a \text{ and } b \text{ are real constants}$$

then $\quad B = R + S = 2a \quad$ will be real

and $\quad C = i(R - S) = i(2ib) = -2b \quad$ will be real

Moreover, the values of a and b are unrelated, so we are free to choose arbitrary values for the constants B and C just as we were for R and S.

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = 0 \quad \text{where } a > 0,\ b > 0 \text{ and } c > 0 \qquad \text{(Eqn 29)}$$

Thus:

The general solution of Equation 29 in the case $b^2 < 4ac$ is

$$y(t) = \exp(-\gamma t/2)[(B\cos(\omega t) + C\sin(\omega t)] \tag{37}$$

with $\gamma = b/a$ (38a)

and $\omega = \sqrt{4ac - b^2}\,/(2a)$ (38b)

Question T9

Show by substitution that, for arbitrary constants B and C,

$$y(t) = \exp(-\gamma t/2)[B\cos(\omega t) + C\sin(\omega t)]$$

where

$$\gamma = b/a \quad \text{and} \quad \omega = \sqrt{4ac - b^2}/(2a)$$

is a solution to Equation 29.

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = 0 \quad \text{where } a > 0,\ b > 0 \text{ and } c > 0 \qquad \text{(Eqn 29)}$$

(Notice that ω is a real quantity if $b^2 < 4ac$.) ❏

The combination $[B\cos(\omega t) + C\sin(\omega t)]$ appears in Equation 37,

$$y(t) = \exp(-\gamma t/2)[(B\cos(\omega t) + C\sin(\omega t)] \qquad \text{(Eqn 37)}$$

and it may be convenient to write this instead in the form $A\sin(\omega t + \phi)$, as we did in Subsection 2.3, by putting

$$A = \sqrt{B^2 + C^2} \qquad \text{(Eqn 20)}$$

and $$\phi = \arctan(B/C) \quad \text{but with } 0 \le \phi < 2\pi \qquad \text{(Eqn 21)}$$ ☞

This gives:

The alternative form of the solution of Equation 29 in the case $b^2 < 4ac$

$$y(t) = A\exp(-\gamma t/2)\sin(\omega t + \phi) \qquad (39)$$

with $\gamma = b/2$ and $\omega = \sqrt{4ac - b^2}\,/(2a)$ (Eqn 38a and b)

From Equation 39 it is easy to determine the behaviour of the function $y(t)$. Since b/a is positive, γ is also positive — so $y(t)$ is given by a sinusoidal function multiplied by an exponential function that decays as t becomes large and positive.

Figure 7 shows the form of the solution in this case. You can see that the effect of multiplying the sinusoidal function by the exponential is to produce oscillations with amplitudes that *decrease* with time.

This is just the sort of behaviour we expect of a system where the resistive forces are not too large — remember that b^2 must be *less than* $4ac$ for the solution in Equation 39 to apply.

$$y(t) = A\exp(-\gamma t/2)\sin(\omega t + \phi) \qquad \text{(Eqn 39)}$$

In this case, the system is said to be *lightly damped*. Note that the values of t for which $y = 0$ are still equally spaced, with separation, Δt say, given by $\omega\Delta t = \pi$, or $\Delta t = \pi/\omega$. Successive maxima and minima are also equally spaced, by an interval in t equal to $2\pi/\omega$ (though they no longer occur exactly half-way between the points where $y = 0$).
For these reasons, we still speak of the quantity $2\pi/\omega$ as the *period* of the oscillations.

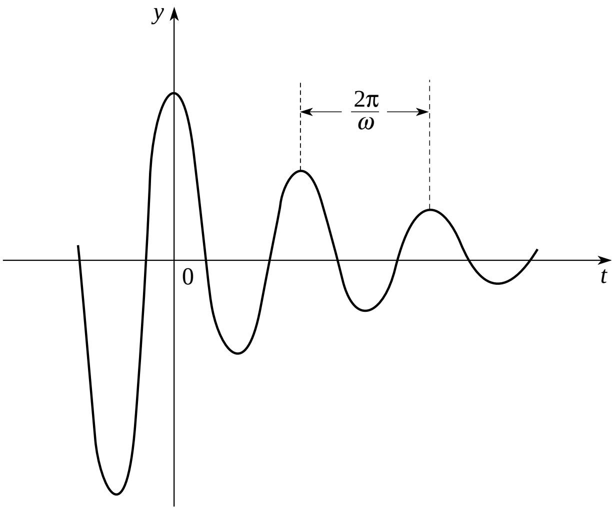

Figure 7 Lightly damped oscillations.

Equal roots of the auxiliary equation (critical damping)

Finally, we will consider the case where $b^2 = 4ac$ which (although it is somewhat artificial in that it rarely occurs in practice) is of interest because it marks the transition from light to heavy damping. In this case, the auxiliary Equation 31

$$ap^2 + bp + c = 0 \qquad \text{(auxilliary equation)} \qquad \text{(Eqn 31)}$$

has only one root; from Equation 32,

$$p_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad p_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \qquad \text{(Eqn 32)}$$

we see that $p_1 = p_2 = -b/(2a)$. We deduce that $y = B\exp[-bt/(2a)]$ is a solution of the differential equation Equation 29.

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = 0 \quad \text{where } a > 0,\ b > 0 \text{ and } c > 0 \qquad \text{(Eqn 29)}$$

However, it cannot be the *general* solution, since it only contains *one* arbitrary constant. The other solution that we must add to it is not at all obvious, so we will just tell you the answer, and leave you to check it by substitution.

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = 0 \quad \text{where } a > 0,\ b > 0 \text{ and } c > 0 \qquad \text{(Eqn 29)}$$

It turns out

The general solution of Equation 29 in the case $b^2 = 4ac$ is

$$y(t) = (B + Ct)\exp\left[-bt/(2a)\right] \qquad (40)$$

Question T10

Show that Equation 40 is a general solution of Equation 29 in the case $b^2 = 4ac$. ❏

This has been a long and quite complicated subsection, which is perhaps misleading since in practice the solution of Equation 29 is fairly straightforward, and just a matter of knowing how to deal with three cases. Here then is a summary of the steps you need to follow in order to solve the equation of damped harmonic motion:

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = 0 \qquad \text{(Eqn 29)}$$

Step 1 Evaluate the quantity $b^2 - 4ac$.

Step 2

- ○ If $b^2 > 4ac$, find the roots p_1 and p_2 of the auxiliary equation, Equation 31.

$$\mathrm{ap}^2 + \mathrm{bp} + \mathrm{c} = 0 \qquad \text{(auxilliary equation)} \qquad \text{(Eqn 31)}$$

The general solution is then given by Equation 33.

$$y(t) = B\exp(p_1t) + C\exp(p_2t) \qquad \text{(Eqn 33)}$$

○ If $b^2 < 4ac$, find the quantities γ and ω, using Equation 38.

with $\quad \gamma = b/a \qquad$ (Eqn 38a)

and $\quad \omega = \sqrt{4ac - b^2}/(2a) \qquad$ (Eqn 38b)

The general solution is then given by Equation 37

$$y(t) = \exp(-\gamma t/2)[(B\cos(\omega t) + C\sin(\omega t)] \qquad \text{(Eqn 37)}$$

(and then use Equations 20 and 21

$$A = \sqrt{B^2 + C^2} \qquad \text{(Eqn 20)}$$

and $\quad \phi = \arctan(B/C) \quad$ but with $0 \le \phi < 2\pi \qquad$ (Eqn 21)

if you want the solution in the form of Equation 39).

$$y(t) = A\exp(-\gamma t/2)\sin(\omega t + \phi) \qquad \text{(Eqn 39)}$$

○ If $b^2 = 4ac$, the general solution is given by Equation 40.

$$y(t) = (B + Ct)\exp[-bt/(2a)] \qquad \text{(Eqn 40)}$$

Now practise these steps by trying the following question. You must make sure that you have mastered the techniques required in these exercises. You will not be able to make progress with the next section unless you have done so.

Question T11

Find the general solution of each of the following differential equations:

(a) $\dfrac{d^2x}{dt^2} + 6\dfrac{dx}{dt} + 10x = 0$

(b) $2\dfrac{d^2y}{dx^2} + 5\dfrac{dy}{dx} + 3y = 0$

(c) $5\dfrac{d^2y}{dt^2} + 6\dfrac{dy}{dt} + 2y = 0$

(d) $\dfrac{d^2y}{dt^2} + 2\dfrac{dy}{dt} + y = 0$ ❏

The method of comparing b^2 with $4ac$ and then selecting the form of the solution will of course work even if $b = 0$, that is, for the differential equations discussed in Subsections 2.3 and 2.4. Consider, for example,

$$\frac{d^2y}{dt^2} + \omega_0{}^2 y = 0 \qquad \text{(Eqn 15)}$$

Here, $b = 0$, $a = 1$ and $c = \omega_0{}^2$. So $b^2 < 4ac$, and Equation 38

$$\gamma = b/a \qquad \text{(Eqn 38a)}$$

$$\omega = \sqrt{4ac - b^2}\,/(2a) \qquad \text{(Eqn 38b)}$$

tells us that $\gamma = 0$, $\omega = \omega_0$. Thus using Equation 37,

$$y(t) = \exp(-\gamma t/2)[(B\cos(\omega t) + C\sin(\omega t)] \qquad \text{(Eqn 37)}$$

we see that the general solution is $y(t) = B\cos(\omega_0 t) + C\sin(\omega_0 t)$, which is just what we found before, in Equation 16.

$$y(t) = B\cos(\omega_0 t) + C\sin(\omega_0 t) \qquad \text{(Eqn 16)}$$

2.6 Equations of the form $a(d^2y/dt^2) + b(dy/dt) + cy = f(t)$

We are now in a position to set about finding solutions to the general second-order linear inhomogeneous equation with constant coefficients:

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = f(t) \qquad \text{(Eqn 5)}$$

We will first of all show that if we can find any *one* particular solution to Equation 5, then we can *always* find the general solution. We will then show a method that can sometimes be used to find particular solutions.

Let us suppose that we have somehow managed to find a particular solution to Equation 5;

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = f(t) \qquad \text{(Eqn 5)}$$

we will call it $y_p(t)$. We now assert that the *general* solution is obtained by taking the sum of $y_p(t)$ and the general solution to the *homogeneous* equation which is obtained by setting $f(t) = 0$ in Equation 5:

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = 0 \qquad (41)$$

In this context, the general solution to the homogeneous equation (Equation 14)

$$\frac{d^2y}{dt^2} + hy = 0 \qquad \text{(Eqn 14)}$$

is called the **complementary function**; we will denote it by $y_c(t)$. Thus the claim here is that the general solution to Equation 5 is

$$y(t) = y_p(t) + y_c(t) \qquad (42)$$

This result is easy to prove. We simply substitute Equation 42

$$y(t) = y_{\rm p}(t) + y_{\rm c}(t) \qquad \text{(Eqn 42)}$$

into Equation 5.

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = f(t) \qquad \text{(Eqn 5)}$$

The left-hand side becomes

$$a\frac{d^2}{dt^2}(y_{\rm p} + y_{\rm c}) + b\frac{d}{dt}(y_{\rm p} + y_{\rm c}) + c(y_{\rm p} + y_{\rm c})$$

which is equal to

$$a\frac{d^2y_{\rm p}}{dt^2} + b\frac{dy_{\rm p}}{dt} + cy_{\rm p} + a\frac{d^2y_{\rm c}}{dt^2} + b\frac{dy_{\rm c}}{dt} + cy_{\rm c} \qquad (43)$$

but by assumption, $y_{\rm p}$ is a solution to Equation 5; that is

$$a\frac{d^2y_{\rm p}}{dt^2} + b\frac{dy_{\rm p}}{dt} + cy_{\rm p} = f(t)$$

and y_c is the general solution to Equation 41,

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = 0 \qquad \text{(Eqn 41)}$$

so that

$$a\frac{d^2y_c}{dt^2} + b\frac{dy_c}{dt} + cy_c = 0$$

So we see from Equation 43

$$a\frac{d^2y_p}{dt^2} + b\frac{dy_p}{dt} + cy_p + a\frac{d^2y_c}{dt^2} + b\frac{dy_c}{dt} + cy_c \qquad \text{(Eqn 43)}$$

that, with the substitution $y = y_p + y_c$, the left-hand side of Equation 5

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = f(t) \qquad \text{(Eqn 5)}$$

is equal to $f(t) + 0$, i.e. equal to $f(t)$. Thus Equation 42

$$y(t) = y_p(t) + y_c(t) \qquad \text{(Eqn 42)}$$

is a solution to Equation 5; it is the general solution since y_c contains two arbitrary constants.

You should read through the following example carefully because it typifies the method which we apply to all such equations.

Example 3 Find the general solution to the differential equation

$$\frac{d^2y}{dt^2} + 4y = t \tag{44}$$

☞

Given that $y = t/4$ is a particular solution.

Solution To find the general solution, we must add the complementary function to the particular solution. The complementary function is the general solution of the homogeneous equation that we obtain by setting the right-hand side of Equation 44 to zero,

$$\frac{d^2y}{dt^2} + 4y = t \tag{Eqn 44}$$

$$\frac{d^2y}{dt^2} + 4y = 0$$

This equation is of the form given in Equation 15,

$$\frac{d^2y}{dt^2} + \omega_0^2 y = 0 \tag{Eqn 15}$$

with $\omega_0 = 2$. Its general solution is given by Equation 16

$$y(t) = B\cos(\omega_0 t) + C\sin(\omega_0 t) \tag{Eqn 16}$$

to be $B\cos(2t) + C\sin(2t)$. So the general solution to Equation 44

is $y = t/4 + B\cos(2t) + C\sin(2t)$. ❑

☞

Question T12

Find the general solution to the differential equation

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} - 4y = 2\,\mathrm{e}^{-3t}$$

given that $y = -\dfrac{\mathrm{e}^{-3t}}{2}$ is a particular solution. ❏

Finding a particular solution

So we now know how to find a *general* solution to a linear inhomogeneous equation if we are *given* a *particular* solution. But how can we *find* a particular solution? There are several methods for doing this, and we will explain here only the simplest. It is sometimes known as the **method of undetermined coefficients**, but as you will see, it consists of little more than intelligent guesswork. The idea is that, guided by the form of $f(t)$, we should *assume* a particular form for $y_p(t)$ which contains some undetermined constants, and simply substitute this into the differential equation. If the form we have chosen is correct, we will be able to determine the constants appearing in our trial solution from the requirement that we must obtain an *identity* on making the substitution. If we have not chosen the correct form, then we will find that it is not a solution, and we must think again! The method can best be explained by an example.

Example 4 Find a particular solution to the differential equation

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 5y = t + 2$$

Solution There is a function of the form $Ht + K$ on the right-hand side of this equation. A function of this sort (a *polynomial*) has the property that when it is differentiated any number of times, no *new* functions of t appear in its derivatives—indeed, the results are just constants (or zero). This suggests that the differential equation might be satisfied by a function of this sort, with H and K suitably chosen. We may as well try it, anyway.
If we put $y = Ht + K$ into the equation, we obtain

$$4H + 5(Ht + K) = t + 2$$

i.e. $5Ht + (4H + 5K) = t + 2$

This equation must be an *identity* if $y = Ht + K$ is a solution. Thus the coefficients of t on both sides must be the same, as must the constant terms. This gives us two equations to be solved for H and K:

$$5H = 1 \quad \text{and} \quad 4H + 5K = 2$$

These are easily solved, to give $H = 1/5$, $K = 6/25$. So we have found a particular solution; it is
$y = t/5 + 6/25$. ❏

If we were to try to generalize the reasoning in Example 4, it would go something like this. The particular solution must be such that when we substitute it into the left-hand side of the equation, $f(t)$ is recovered. If $f(t)$ is a function whose derivatives are all of the same *form* as $f(t)$ itself (such as a polynomial, an exponential function or a sinusoidal function), we may be able to achieve this by trying as a particular solution a function which is also of the form of $f(t)$, but contains some as yet unknown constants (the 'undetermined coefficients'), the values of which we will determine when we make the substitution. This leads to the following rules for finding particular solutions, for certain forms of $f(t)$:

Rules for finding particular solutions, for certain forms of $f(t)$:

1 If $f(t)$ is a *polynomial of degree m*: try a particular solution that is also a polynomial of degree m

$$y_{\rm p}(t) = Ht^m + Kt^{m-1} + \dots + N$$

but containing undetermined coefficients $H, K \dots N$

2 If $f(t)$ is an exponential, Ce^{kt}: try a particular solution that is also an exponential,

$$y_{\rm p}(t) = He^{kt}$$

where H is an undetermined coefficient. ☞

3 If $f(t)$ is a sinusoidal function, $f(t) = C\sin(kt) + D\cos(kt)$: try a particular solution that is also a sinusoidal function

$$y_{\rm p}(t) = H\sin(kt) + K\cos(kt)$$

where H, K are undetermined coefficients.

You can now use Rules 1–3 to answer the following question.

Question T13

Find a particular solution to each of the following equations

(a) $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + 3y = 2e^{-x}$

(b) $\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + 4y = 2\cos x - 11\sin x$ ❏

The method of undetermined coefficients will work for some more complicated forms of $f(t)$, but we have said enough here for you to be able to find particular solutions to many second-order inhomogeneous equations of physical interest. (It is worth pointing out, though, that if $f(t)$ is given by a *sum* of two or more functions of the sorts mentioned in Rules 1–3, then the particular solution is simply the sum of the particular solutions corresponding to each of these functions.) At the end of the next subsection, in Question T14, you can put together the methods you have learnt so far to find general solutions to inhomogeneous equations. First, however, we will consider an example of great importance in physics.

2.7 A worked example: damped driven harmonic motion

We are now in a position to solve Equation 1, introduced in Subsection 1.1 ☞;

$$m\frac{d^2y}{dt^2} = -ky - b\frac{dy}{dx} + F_0\sin(\Omega t) \qquad \text{(Eqn 1)}$$

rewritten slightly, it is

$$\boxed{m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = F_0\sin(\Omega t) \qquad (45)}$$ ☞

An equation of this sort applies whenever an oscillating object is subjected to a *damping* force $\left(-b\dfrac{dy}{dt}\right)$ as well as an external *driving* force, $F_0\sin(\Omega\, t)$ with angular frequency Ω. ☞

It is said to describe *forced, damped oscillations* or *damped, driven oscillations*.

In this subsection, we will find the general solution to Equation 45 for the case $b^2 < 4mk$.

The complementary function

We will first find the complementary function, i.e. the general solution to

$$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = 0$$

We will solve this using the steps listed at the end of Subsection 2.5. Since we have decided to consider the case $b^2 < 4mk$, the general solution is given by Equation 38,

with $\quad \gamma = b/2 \quad$ and $\quad \omega = \sqrt{4ac - b^2}\,/(2a)$ (Eqn 38a and b)

with

$$\gamma = b/m \quad \text{and} \quad \omega = \sqrt{4mk - b^2}\Big/(2m)$$

So the complementary function is

$$y_{\rm c}(t) = \exp[-bt/(2m)][B\cos(\omega t) + C\sin(\omega t)] \tag{46}$$

A particular solution

We will now find a particular solution. We have a sinusoidal function on the right-hand side of Equation 45,

$$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = F_0 \sin(\Omega t) \qquad \text{(Eqn 45)}$$

so, in accordance with Rule 3 at the end of Subsection 2.6, we will try a particular solution of the form $y_{\rm p}(t) = H\cos(\Omega t) + K\sin(\Omega t)$. ☞

On substituting this into Equation 45, we find

$$-m\Omega^2[H\cos(\Omega t) + K\sin(\Omega t)] + b\Omega[-H(\Omega t) + K\cos(\Omega t)] + k[H\cos(\Omega t) + K\sin(\Omega t)] = F_0 \sin(\Omega t)$$

which can be rewritten in the form

$$[(k - m\Omega^2)H + b\Omega K]\cos(\Omega t) + [(k - m\Omega^2)K - b\Omega H]\sin(\Omega t) = F_0 \sin(\Omega t)$$

If this is to be an identity, the coefficients of $\cos(\Omega t)$ and $\sin(\Omega t)$ on each side of the equation must be equal. So we obtain two equations (for H and K)

$$(k - m\Omega^2)H + b\Omega K = 0 \quad \text{and} \quad (k - m\Omega^2)K - b\Omega H = F_0$$

On solving these equations for H and K, we find (after some algebra! ☞)

$$H = \frac{-(b\Omega)F_0}{(k - m\Omega^2)^2 + (b\Omega)^2} \quad \text{and} \quad K = \frac{(k - m\Omega^2)F_0}{(k - m\Omega^2)^2 + (b\Omega)^2}$$

Thus the particular solution is the rather complicated looking expression

$$y_{\rm p}(t) = \frac{-(b\Omega)F_0}{(k - m\Omega^2)^2 + (b\Omega)^2}\cos(\Omega t) + \frac{(k - m\Omega^2)F_0}{(k - m\Omega^2)^2 + (b\Omega)^2}\sin(\Omega t) \qquad (47)$$

☞

We can simplify this greatly if we use Equations 20 and 21

$$A = \sqrt{B^2 + C^2} \qquad \text{(positive square root)} \qquad \text{(Eqn 20)}$$

$$\frac{B}{C} = \frac{A\sin\phi}{A\cos\phi} = \tan\phi \quad \text{i.e. } \phi = \arctan(B/C) \text{ but with } 0 \le \phi < 2\pi \qquad \text{(Eqn 21)}$$

to write $y_{\rm p}$ in the form

$$y_{\rm p}(t) = A\sin(\Omega t + \phi)$$

Applying Equations 20 and 21, we find (after some more algebra)

$$A = \frac{F_0}{\sqrt{(k - m\Omega^2)^2 + (b\Omega)^2}} \tag{48a}$$

and $$\phi = \arctan\left(\frac{-b\Omega}{k - m\Omega^2}\right) \quad \text{but with } 0 \le \phi \le 2\pi \tag{48b}$$

The general solution

Thus the general solution of Equation 45,

$$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = F_0\sin(\Omega t) \qquad \text{(Eqn 45)}$$

given by the sum of the complementary function (Equation 46)

$$y_{\rm c}(t) = \exp[-bt/(2m)][B\cos(\omega t) + C\sin(\omega t)] \qquad \text{(Eqn 46)}$$

and the particular solution (Equation 47),

$$y_{\rm p}(t) = \frac{-(b\Omega)F_0}{(k-m\Omega^2)^2 + (b\Omega)^2}\cos(\Omega t) + \frac{(k-m\Omega^2)F_0}{(k-m\Omega^2)^2 + (b\Omega)^2}\sin(\Omega t) \qquad \text{(Eqn 47)}$$

is

$$y(t) = \exp[-bt/(2m)][B\cos(\omega t) + C\sin(\omega t)] + A\sin(\Omega t + \phi) \tag{49}$$

where $\omega = \sqrt{4mk - b^2}\Big/(2m)$

and A and ϕ are given in Equations 48a and b.

$$A = \frac{F_0}{\sqrt{(k - m\Omega^2)^2 + (b\Omega)^2}} \qquad \text{(Eqn 48a)}$$

and $$\phi = \arctan\left(\frac{-b\Omega}{k - m\Omega^2}\right) \quad \text{but with } 0 \le \phi \le 2\pi \qquad \text{(Eqn 48b)}$$

Interpretation

We can see from the form of the complementary function in Equation 46

$$y_c(t) = \exp[-bt/(2m)][B\cos(\omega t) + C\sin(\omega t)] \tag{Eqn 46}$$

(or Equation 49)

$$y(t) = \exp[-bt/(2m)][B\cos(\omega t) + C\sin(\omega t)] + A\sin(\Omega t + \phi) \tag{Eqn 49}$$

that, whatever the values of the arbitrary constants B and C, this part of the solution will eventually tend to zero (because of the presence of the decaying exponential factor, $\exp[-bt/(2m)]$). All that remains after a long time, therefore, is the particular solution, which represents the response of the oscillating mass to the applied force. So it is this part of the solution which is of greatest interest to us. Equation 47

$$y_p(t) = \frac{-(b\Omega)F_0}{(k - m\Omega^2)^2 + (b\Omega)^2}\cos(\Omega t) + \frac{(k - m\Omega^2)F_0}{(k - m\Omega^2)^2 + (b\Omega)^2}\sin(\Omega t) \tag{Eqn 47}$$

shows that it corresponds to sinusoidal oscillations, of the same angular frequency Ω as the applied force.

However, there is a constant phase difference ϕ between the oscillations of the mass and oscillations of the driving force. Mathematically, ϕ represents the amount by which the movement of the mass *leads* the force, but physically it makes more sense to say that the motion *lags* behind the force by an amount $2\pi - \phi$. Notice (from Equations 48a and b)

$$A = \frac{F_0}{\sqrt{(k - m\Omega^2)^2 + (b\Omega)^2}} \qquad \text{(Eqn 48a)}$$

and $$\phi = \arctan\left(\frac{-b\Omega}{k - m\Omega^2}\right) \quad \text{but with } 0 \le \phi \le 2\pi \qquad \text{(Eqn 48b)}$$

that the amplitude A of the oscillations depends not only on F_0, but also on Ω; in other words, applied forces of different frequency may invoke oscillations of the mass of very different magnitude.

Figure 8 shows A as a function of Ω, for a certain choice of the parameters m, k, b and F_0. You can see that it has a pronounced maximum at a value of Ω close to the *natural angular frequency* $\omega_0 = \sqrt{k/m}$.

This large response for a particular value of the angular frequency of the applied force is the well-known phenomenon of **resonance**.

☞

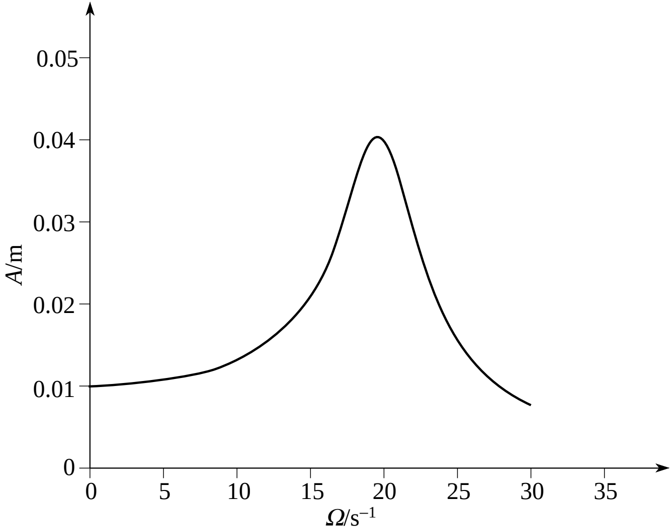

Figure 8 Graph of A as a function of Ω, in the case $m = 0.2\,\mathrm{kg}$, $b = 1.0\,\mathrm{N\,s\,m^{-1}}$, $k = 80\,\mathrm{N\,m^{-1}}$, $F_0 = 0.8\,\mathrm{N}$.

The above example involved some rather unpleasant algebra (because we wished to discuss a general case), however, finding the general solution to a specific inhomogeneous equation is generally straightforward enough — provided, of course, that $f(t)$ is one of the functions mentioned in Rules 1–3 at the end of Subsection 2.6 — the method is summarized here:

1 Find the complementary function, using the methods of Subsection 2.5

2 Find the particular solution, using one of Rules 1–3 in Subsection 2.6

3 Add the two together.

You can practise these steps by trying the following question.

Question T14

Find the general solution to the equation

$$\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 9y = 1 + 3t \quad ❏$$

3 Closing items

3.1 Module summary

1 A *linear differential equation with* *constant coefficients* has the form:

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = f(t) \qquad \text{(Eqn 5)}$$

If $f(t)$ in Equation 5 is set to zero it is a *linear homogeneous differential equation*. If $f(t)$ is not zero in Equation 5 it is a *linear inhomogeneous differential equation*

2 Equations of the form

$$\frac{d^2y}{dt^2} = f(t) \qquad \text{(Eqn 9)}$$

can be solved by direct integration.

3 The general solution of the SHM (simple harmonic motion) equation

$$\frac{d^2y}{dt^2} + \omega_0^2 y = 0 \qquad \text{(Eqn 15)}$$

can be written either as

$$y(t) = B\cos(\omega_0 t) + C\sin(\omega_0 t) \qquad \text{(Eqn 16)}$$

or $\quad y(t) = A\sin(\omega_0 t + \phi)$ (Eqn 19)

corresponding to sinusoidal oscillations of period $2\pi/\omega_0$. The values A and ϕ are determined from values of B and C (or vice versa) by Equations 17, 20 and 21.

$$B = A\sin\phi \quad \text{and } C = A\cos\phi \qquad \text{(Eqns 17a, b)}$$

$$A = \sqrt{B^2 + C^2} \qquad \text{(positive square root)} \qquad \text{(Eqn 20)}$$

$$\frac{B}{C} = \frac{A\sin\phi}{A\cos\phi} = \tan\phi \quad \text{i.e. } \phi = \arctan(B/C) \text{ but with } 0 \le \phi < 2\pi \qquad \text{(Eqn 21)}$$

4 The general solution of an equation of the form

$$\frac{d^2y}{dt^2} - \lambda^2 y = 0 \qquad \text{(Eqn 23)}$$

is $y(t) = Be^{\lambda t} + Ce^{-\lambda t}$ (Eqn 26)

5 The nature of the solution of the (damped harmonic motion) equation

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = 0 \qquad \text{(Eqn 29)}$$

depends upon the sign of $b^2 - 4ac$.

The case where $b^2 > 4ac$ corresponds to heavy damping, as illustrated in Figure 6 (with the solution given by Equation 33).

$$y(t) = B\exp(p_1 t) + C\exp(p_2 t) \qquad \text{(Eqn 33)}$$

where $p_1 = \dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$ and $p_2 = \dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$

The case where $b^2 < 4ac$ corresponds to light damping, as illustrated in Figure 7 (with the solution given by Equation 37, or alternatively Equation 39).

$$y(t) = \exp(-\gamma t/2)[(B\cos(\omega t) + C\sin(\omega t)] \qquad \text{(Eqn 37)}$$

$$y(t) = A\exp(-\gamma t/2)\sin(\omega t + \phi) \qquad \text{(Eqn 39)}$$

The case where $b^2 = 4ac$ corresponds to critical damping (with the solution given by Equation 40).

$$y(t) = (B + Ct)\exp[-bt/(2a)] \qquad \text{(Eqn 40)}$$

6 To solve the (damped, driven harmonic motion) equation

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + c = f(t)$$

we first obtain the *complementary function* by finding a general solution of the equation

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + c = 0$$

The *general solution* of $a\dfrac{d^2y}{dt^2} + b\dfrac{dy}{dt} + c = f(t)$ can then be found, if we know one *particular solution*, by adding the complementary function to that particular solution.

Particular solutions can be found by the method of *undetermined coefficients* if $f(t)$ is a *polynomial*, an *exponential* function, or a *sinusoidal* function.

In the case of a sinusoidal function $f(t)$, the amplitude of the solution (for large values of t) is dependent upon the angular frequency of f. In some cases this amplitude may be very large for a particular choice of the applied angular frequency, leading to the phenomenon of *resonance*.

3.2 Achievements

Having completed this module, you should be able to:

A1 Define the terms that are emboldened and flagged in the margins of the module.

A2 Recognize linear differential equations with constant coefficients and distinguish between homogeneous and inhomogeneous linear differential equations.

A3 Solve equations of the form $\dfrac{d^2y}{dt^2} = f(t)$ by direct integration.

A4 Recognize the SHM equation whenever it arises, identify the parameter ω that determines the period of oscillations and solve the equation.

A5 Convert between a particular solution of the form $y = B\cos(\omega t) + C\sin(\omega t)$ and one of the form $y = A\sin(\omega t + \phi)$ and vice versa.

A6 Solve an equation of the form $\dfrac{d^2y}{dt^2} - \lambda^2 y = 0$.

A7 Solve an equation of the form $a\dfrac{d^2y}{dt^2} + b\dfrac{dy}{dt} + cy = 0$, and describe the forms of the solution in the cases $b^2 > 4ac$, $b^2 < 4ac$ and $b^2 = 4ac$.

A8 Use the method of undetermined coefficients to find particular solutions of an equation of the form

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = f(t) \qquad \text{(Eqn 5)}$$

in the cases where $f(t)$ is a polynomial, an exponential, or a sinusoidal function.

A9 Find the complementary function of an equation of the type given in Equation 5 and *use* it with the particular solution to *obtain* the general solution.

Study comment You may now wish to take the *Exit test* for this module which tests these Achievements. If you prefer to study the module further before taking this test then return to the *Module contents* to review some of the topics.

3.3 Exit test

Study comment Having completed this module, you should be able to answer the following questions, each of which tests one or more of the *Achievements*.

Question E1

(*A2* and *A3*) The differential equation

$$\frac{d^2x}{dt^2} = ge^{-kt}$$

where g is the magnitude of the acceleration due to gravity and k is a positive constant, may be used to model the motion of a parachutist falling under gravity. The x-axis is chosen to be the downward vertical, with $x = 0$ when $t = 0$.

Is this equation linear, linear with constant coefficients, or homogeneous?

Find the particular solution if the parachutist starts from rest.

Question E2

(*A4*) If the Earth is represented by a sphere of uniform density, the gravitational force on a particle of mass m inside the Earth at a distance r from the centre is of magnitude mgr/R, and is directed towards the centre of the Earth, where $R = 6400\,\mathrm{km}$ is the (approximate) radius of the Earth and g is the magnitude of the acceleration due to gravity at the Earth's surface. If it were possible to pass a straight tube through the centre of the Earth, show that a particle placed in the tube would execute simple harmonic motion, and find the period of the motion. (Take g as $10\,\mathrm{m\,s^{-2}}$.)

Question E3

(*A4* and *A5*) The current in the circuit shown in Figure 1 satisfies the differential equation

$$L\frac{d^2I}{dt^2} + \frac{I}{C} = 0$$

If $L = 0.02\,\text{mH}$, $C = 3.2 \times 10^{-10}\,\text{F}$, and if $I = 2.5\,\text{mA}$ and $dI/dt = -3.75 \times 10^4\,\text{A s}^{-1}$ at $t = 0$, find the particular solution for I, both in the form $B\cos(\omega_0 t) + C\sin(\omega_0 t)$, and in the form $A\sin(\omega_0 t + \phi)$.

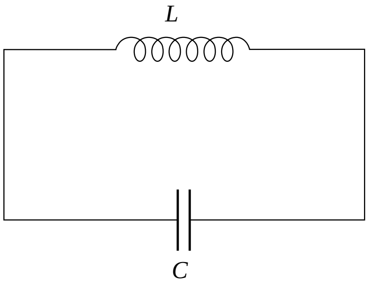

Figure 1 See Question E3.

Question E4

(*A2*, *A4* and *A6*) In a certain medium, the z-component of the electric field, E_z, varies with x, the distance from the boundary of the medium, according to the equation

$$\frac{d^2E_z}{dx^2} + \frac{\Omega^2}{c^2}E_z - \mu^2 E_z = 0$$

where Ω, c and μ are positive constants. ☞ Find the general solution to this equation in the three cases: (a) $\Omega > \mu c$, (b) $\Omega = \mu c$, (c) $\Omega < \mu c$.

Question E5

(*A7*) In the circuit shown in Figure 5, L has the value 0.02 mH and C has the value 3.2×10^{-10} F. The current in the circuit satisfies Equation 28.

$$L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{I}{C} = 0 \qquad \text{(Eqn 28)}$$

For what values of R will the current display lightly damped, oscillatory behaviour?

Find the general solution to Equation 28 (a) if $R = 300\,\Omega$, (b) if $R = 1300\,\Omega$, (c) if $R = 500\,\Omega$.

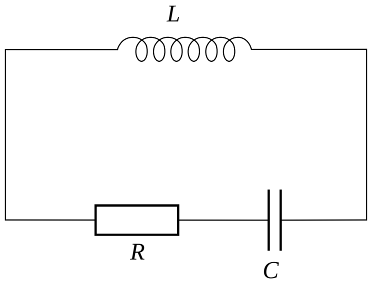

Figure 5 A circuit containing an inductance L, a resistor R and a capacitor C, connected in series.

Question E6

(*A8* and *A9*) (a) Explain what is meant by the term *complementary function*.

(b) Find the general solution to the differential equation:

$$\frac{d^2y}{dt^2} + \omega_0^2 y = A\cos(\Omega t)$$

where ω, A, Ω are constants and $\Omega^2 \neq {\omega_0}^2$.

Study comment This is the final *Exit test* question. When you have completed the *Exit test* go back to Subsection 1.2 and try the *Fast track questions* if you have not already done so.

If you have completed **both** the *Fast track questions* and the *Exit test*, then you have finished the module and may leave it here.