Module M2.1 Introducing geometry

Exit module

1 Opening items

1.1 Module introduction

You may very well have met some of the concepts discussed in this module before; in fact it is hardly conceivable that you have not. Nevertheless there are many geometric terms which are in general use among physicists but which are commonly omitted from the standard school syllabus. The main aim of this module is to remedy that deficiency.

Geometry is of importance to all scientists and engineers since it deals with the relationships between points and lines, and with a variety of shapes in two and three dimensions.

In Section 2 of this module we examine straight lines and the angles between them, and then the shapes, such as triangles and rectangles, which can be formed by straight lines. Such concepts are fundamental to the study of physics and engineering, for example:

- It is well known in the study of optics that 'the angle of incidence is equal to the angle of reflection'.
- Parallel lines drawn on a flat surface do not meet, and, if another line is drawn to intersect a pair of parallel lines, certain angles are formed which are equal.
- The simplest shape formed by straight lines is a triangle, but, in spite of being simple, such shapes are very important to the engineer because they form the simplest rigid structure.

In Subsection 3.1 we discuss *congruence*, which, very roughly speaking, means that if you cut two *congruent* shapes from paper, then one will fit exactly over the other. In Subsection 3.2 we introduce *similarity*, and again speaking roughly, this means that one shape is a scaled version of the other (for example, the same shape but twice as big). In Section 4 we examine circles, and, in particular, the important concepts of *radian measure* and *tangent* lines. Finally in Section 5 we discuss the *areas* and *volumes* of various standard shapes.

Before we begin the module we should give you a word of warning. If you glance at Figure 1 you will see that the lines HF and DF appear to be approximately the same length. However in this topic equality means ‘exactly equal’ and not ‘approximately equal’. It is very easy to assume that because two lines in a figure *look* the same length they *are* the same length, and similarly for angles. In this subject you are either told that two lengths are equal or you deduce that they are equal from the given information. There is no other way!

Study comment Having read the introduction you may feel that you are already familiar with the material covered by this module and that you do not need to study it. If so, try the *Fast track questions* given in Subsection 1.2. If not, proceed directly to *Ready to study?* in Subsection 1.3.

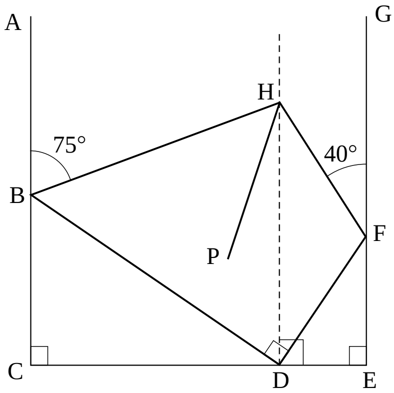


Figure 1 See Questions F1 and F2.

1.2 Fast track questions

Study comment Can you answer the following *Fast track questions*? If you answer the questions successfully you need only glance through the module before looking at the *Module summary* (Subsection 6.1) and the *Achievements* listed in Subsection 6.2. If you are sure that you can meet each of these achievements, try the *Exit test* in Subsection 6.3. If you have difficulty with only one or two of the questions you should follow the guidance given in the answers and read the relevant parts of the module. However, *if you have difficulty with more than two of the Exit questions you are strongly advised to study the whole module.*

Question F1

In Figure 1 the line HP bisects the angle $\mathrm{B\hat{H}F}$. Find the value of angle $\mathrm{P\hat{H}D}$.

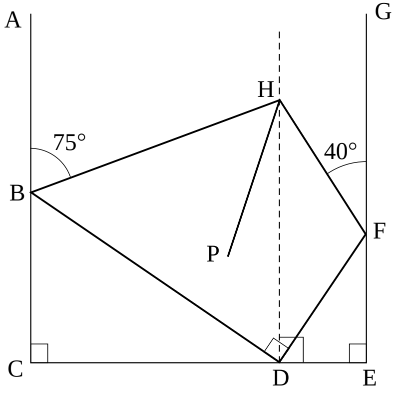


Figure 1 See Questions F1 and F2.

Question F2

Again in Figure 1, $B\hat{C}D = B\hat{D}F = D\hat{E}F = H\hat{D}E = 90°$, and $BD = 15\,\text{m}$, $DF = 5\,\text{m}$, $CE = 15\,\text{m}$, $DE = 3\,\text{m}$.

Show that the triangles BCD and DEF are similar, and hence find the lengths of EF and BC.

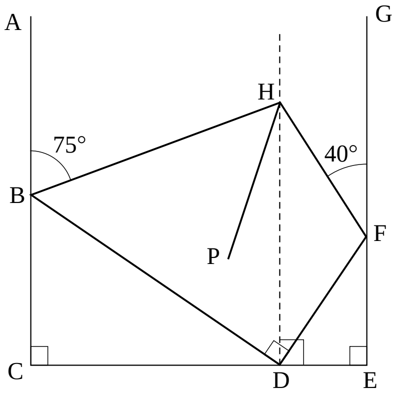


Figure 1 See Questions F1 and F2.

Question F3

In Figure 2, AC = BC and $B\hat{E}C = A\hat{D}C = 90°$. By calculating the area of the triangle ABC using two different methods, or otherwise, show that BE = AD. Hence show that the triangles BEC and ADC are congruent.

Question F4

A steel pipe is 10 m long. Its inside diameter is 40 mm and its outside diameter is 50 mm. Calculate the volume of the steel in m^3.

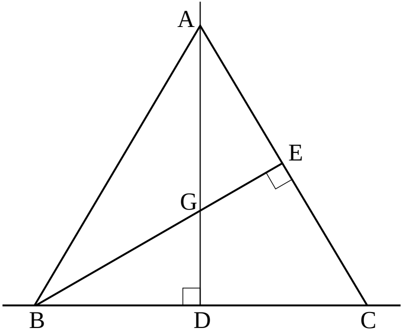


Figure 2 See Question F3.

Study comment

Having seen the *Fast track questions* you may feel that it would be wiser to follow the normal route through the module and to proceed directly to *Ready to study?* in Subsection 1.3.

Alternatively, you may still be sufficiently comfortable with the material covered by the module to proceed directly to the *Closing items*.

1.3 Ready to study?

Study comment In order to study this module you will need to be familiar with the following terms: *angle*, *line*, *two dimensions* and *three dimensions*. If you are uncertain of any of these terms you can review them now by referring to the *Glossary* which will indicate where in *FLAP* they are developed. In addition, you will need to be familiar with *SI units*. It is also assumed that you can carry out basic algebraic and arithmetical manipulations (see *algebra* in the *Glossary*), in particular calculations using *indices*. You will also need a ruler (measuring centimetres), a protractor and a pair of compasses. The following *Ready to study questions* will allow you to establish whether you need to review some of the topics before embarking on this module.

Question R1

(a) If $\dfrac{x}{y} = \dfrac{z}{w}$ and $y = 6$, $z = 15$, $w = 10$, find x.

(b) Given that $\dfrac{x}{y} = \dfrac{y}{z}$, $x = 6$ and $z = 24$, find the possible values of y.

(c) Given that $x^3 = 5$, calculate the value of x^2.

Question R2

Draw a line AB of length 7 cm, then, using a pair of compasses, draw a circle centre A of radius AB and a circle centre B of radius AB. Label the points at which the circles intersect as P and Q, then draw the line PQ and label as M the point where it meets the line AB. Use a protractor to measure the angle between the two lines (PQ and AB), then use a ruler to measure AM and MB in cm. Use your protractor to measure the angle between the lines PA and PB, and then the angle between the lines PA and QA.

2 Euclidean plane geometry ☞

2.1 Lines and angles

The most fundamental geometric concept is that of a **point**, which occupies a position but has no size or dimension. Initially we will be concerned with points that lie on a flat surface (like this sheet of paper), usually called a plane surface or, more simply, a **plane**. We will also investigate the properties of lines drawn on the plane. A line has length as its sole dimension, but it has no thickness. Of course, these are theoretical concepts; in practice, when we plot a point or draw a line we have to give it thickness in order to be able to see it.

A **straight line** is the shortest distance between two points. It is the path traced out by moving from one point to the other in a constant direction. If we wish to be precise then we would have to allow the straight line to extend infinitely away from any point on it and define that part of it between two particular points as a **straight line segment**. Any two points on the plane can be joined by a unique straight line segment which lies entirely in the plane. ☞ We say that two lines **intersect** if they have a point in common, known as the **intercept** (if they have more than one point in common then they must be identical because of the uniqueness property of straight lines). Two lines that do not intersect are said to be **parallel**. That part of a straight line which extends from a given point in one direction only is called a **ray** —you may have heard of the term ‘a ray of light’.

Figure 3 is intended to illustrate a number of important concepts. First, the two lines YD and RV intersect at the point Z; of course the lines are not depicted in full (they are infinite in extent, after all) but by segments. The points C and X also lie on the line YD , and we say that they lie on the line DY **produced**.

There are four angles formed by the two lines YD and RV: $Y\hat{Z}V$, $V\hat{Z}D$, $D\hat{Z}R$ and $R\hat{Z}Y$. The notation is fairly obvious, the caret (ˆ) sign is placed over the letter representing the point of the angle. It is also common practice to denote angles by letters (often Greek letters), so that in this case $Y\hat{Z}R = \theta$ (the Greek letter theta).

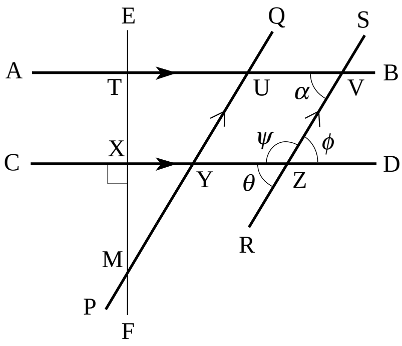


Figure 3 Properties of parallel lines.

An angle is a measure of the rotation from one ray to another, for example from ZV to ZD. To quantify this rotation we measure a full rotation (i.e. one that gets you back to where you started) by 360 units, called **degrees**, written 360°. Hence turning the ray ZV first to ZD, then ZR, then ZY and finally on to its starting point, is a rotation of 360°. When the four angles are equal to each other, and therefore each is equal to 90°, the lines are said to be **perpendicular** to each other or to intersect at **right angles**. This is the case for the lines EF and CD intersecting at X. Notice the symbol (a box) used at X to denote a right angle.

The angles θ and ψ (the Greek letter psi) together form half a complete turn about the point Z, and so their sum must be 180°. This is also the case for the angles ψ and ϕ (phi), and it follows that $\theta = \phi$. The angles θ and ϕ are said to be **vertically opposite**, and we have just shown that vertically opposite angles must always be equal.

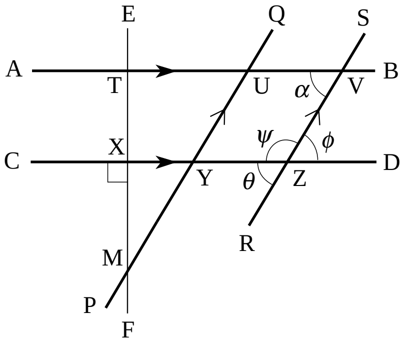


Figure 3 Properties of parallel lines.

❖ In Figure 3, if $\mathrm{U\hat{V}Z} = 50°$ what are the values of the three other angles at V?

Angles less then 90° are called **acute angles**, those greater than 90° and less then 180° are **obtuse angles**, and those greater than 180° and less than 360° are **reflex angles**.

✧ Classify the angles 135°, 60°, 250°, 310°, 90°.

When two angles add up to 180° they are said to be **supplementary**; when they add up to 90° they are **complementary**.

✧ What are the angles that are supplementary to 130°, 60°, 90°? What are the angles that are complementary to 30°, 45°, 60°, 90°?

2.2 Parallel lines

Parallel lines are indicated by arrowheads, as in Figure 3, and notice that different pairs of parallel lines are indicated by arrowheads of different styles. A line which intersects two parallel lines is called a **transversal**, so that (in Figure 3) RS is a transversal to the pair of parallel lines AB and CD. When two parallel lines are intersected by a transversal two sets of equal angles are created.

The angles θ and α (the Greek letter alpha) in Figure 3 are equal; they are **corresponding angles** — they occur on the same side of the transversal.

The angles ϕ and α are equal; they are **alternate angles** — they occur on opposite sides of the transversal.

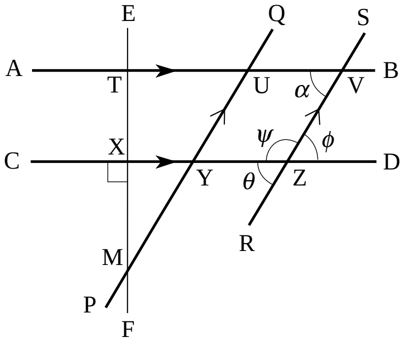


Figure 3 Properties of parallel lines.

✧

In Figure 3 PQ and RS are parallel lines and AB is a transversal.

(a) Identify the corresponding angle to α that occurs at point U.

(b) Identify which angle at V is vertically opposite to α.

(c) Give an example of two angles at U that are supplementary.

(d) Identify which angle at U is alternate to α.

2.3 Triangles

A **triangle** is a closed figure which has three sides, each of which is a straight line segment. A triangle is usually labelled with a capital letter at each of the corners which are more properly known as the vertices (the plural of vertex). We speak of the triangle ABC, for example.

Figure 4 shows three types of triangle (for the moment you can ignore the dotted lines and labelled angles).

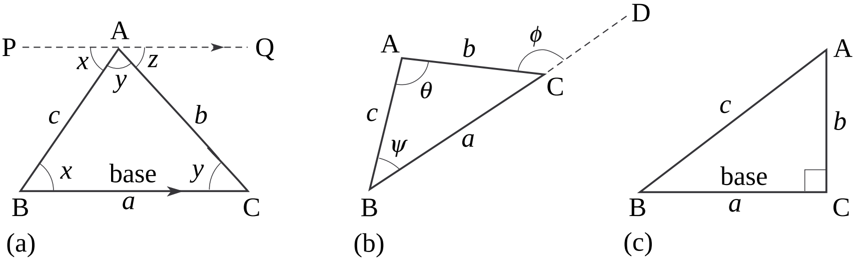


Figure 4 Various types of triangle.

In each of the three triangles the angles $\mathrm{B\hat{A}C}$, $\mathrm{A\hat{C}B}$ and $\mathrm{C\hat{B}A}$ are known as the **interior angles** of the triangle (and sometimes referred to as the angles $\hat{\mathrm{A}}$, $\hat{\mathrm{B}}$ and $\hat{\mathrm{C}}$, or even more briefly as the angles A, B and C).

In Figure 4a all the interior angles are acute.

In Figure 4b one of the angles, A, is obtuse.

In Figure 4c one of the angles, C, is 90°, i.e. a right angle.

Notice that we have also introduced the notation of a lower-case letter to represent each side of a triangle, the side a is opposite the angle A, and so on.

In the case of Figures 4a and 4c it is natural to refer to the side BC as the base of the triangle, however any side may be regarded as the *base* — it just depends on how you want to view it.

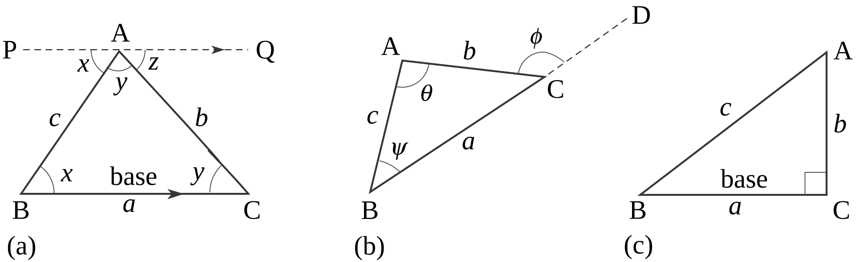


Figure 4 Various types of triangle.

If all three sides of a triangle are equal (i.e. their lengths are equal) then the triangle is known as an **equilateral triangle** and it is a consequence of this definition that the angles are equal. It is also the case that if the angles are equal then so are the sides.

If two sides of the triangle are equal the triangle is said to be **isosceles**, and the interior angles at either end of the third side are equal. A triangle for which one of the interior angles is a right angle is called a **right-angled triangle** and the side of the triangle opposite to the right angle (i.e. the longest side) is called the **hypotenuse**. A triangle whose three sides are unequal is called **scalene**.

An important result is that the sum of the interior angles of *any* triangle is 180°. ☞

In Figure 4a the dotted line PQ is drawn through the vertex A and parallel to the opposite side BC. In the figure we have marked a pair of equal alternate angles x and a pair of equal alternate angles z. The sum of the angles (at A) on a straight line is 180° and hence $x + y + z = 180°$, which proves the required result.

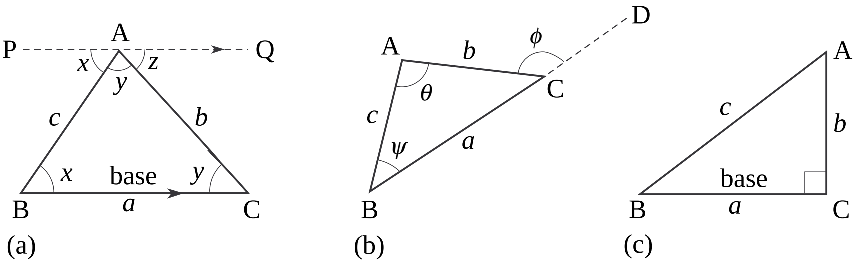


Figure 4 Various types of triangle.

The angle ϕ in Figure 4b is called an **exterior angle** of the triangle.

❖ By drawing a ray from C in Figure 4b, parallel to BA, show that $\phi = \theta + \psi$.

There is an alternative proof of the fact that the interior angles of a triangle sum to 180° which is worth mentioning since the same method can be applied very easily to figures with more than three sides.

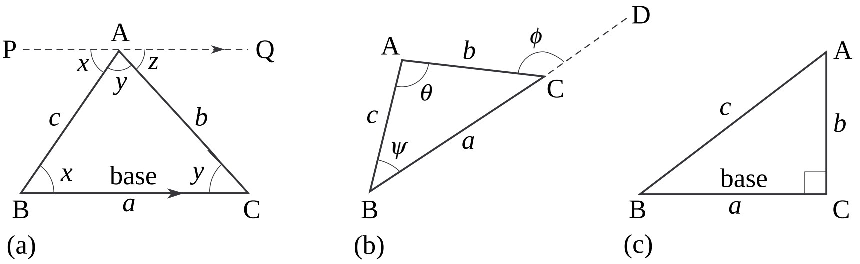


Figure 4 Various types of triangle.

Imagine that you are standing at B in Figure 4b and that you walk along the side BC until you reach C. You now turn anticlockwise through the angle ϕ until you are facing A; in other words you turn through an angle (180° – C) ☞. Now continue along CA until you reach A, where you turn anticlockwise through an angle (180° – A). Finally you proceed along AB until you reach B, then turn anticlockwise through an angle (180° – B) so that you have returned to your starting point and you are facing in your original direction, which means that you have made a complete turn of 360°. However this complete turn was composed of three exterior angles, so that

$$(180° - A) + (180° - B) + (180° - C) = 360° \text{ and therefore } A + B + C = 180°.$$

Question T1

Use an argument similar to the above to show that the sum of the interior angles of a four-sided figure (such as a square) is 360°. ❑

Pythagoras's theorem

The sides of a right-angled triangle are related in a particularly simple fashion, and the following result, using the notation of Figure 4c, is known as **Pythagoras's theorem**

$$c^2 = a^2 + b^2 \qquad (1)$$

where c is the longest side, i.e. the hypotenuse. We state the theorem without proof. ☞

✦ Given that $a = 0.8\,\text{cm}$ and $c = 1.7\,\text{cm}$ in Figure 4c, calculate the value of b.

It should be noted that any triangle with sides of lengths a, b and c that satisfies Pythagoras's theorem *must* be a right-angled triangle. Thus, even if the only thing you know about a triangle is that its sides are of lengths 3 units, 4 units and 5 units you can still be sure that it is a right-angled triangle simply because $3^2 + 4^2 = 5^2$. Other well known sets of numbers that satisfy $a^2 + b^2 = c^2$ are 5, 12, 13 and 6, 8, 10.

❖ Is a triangle with sides 9 m, 12 m, and 15 m a right-angled triangle?

2.4 Polygons

Quadrilaterals

Any closed geometric shape whose sides are straight line segments is called a **polygon**. We have already met a three-sided polygon — a triangle. Four-sided polygons are known as **quadrilaterals**. When all the sides of a polygon are equal it is, not surprisingly, called an **equilateral polygon**; if all the internal angles are equal it is **equiangular**; and a polygon that is both equilateral and equiangular is said to be **regular**. ☞ Thus, for example, a regular quadrilateral is just a **square**. The **perimeter** of a polygon is the sum of the lengths of the sides.

A **parallelogram** is a special case of a quadrilateral in which the opposite sides are parallel (as, for example, UVZY in Figure 3).

◆ Imagine a line drawn from U to Z in Figure 3. Is the triangle UVZ necessarily isosceles?

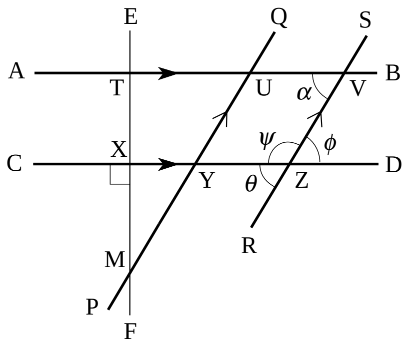


Figure 3 Properties of parallel lines.

A parallelogram with all sides equal in length is called a **rhombus**, popularly known as diamond-shaped.

If the internal angles of a parallelogram are each 90° then the quadrilateral is a **rectangle**, if the sides are also equal in length then the rectangle is a *square*.

A quadrilateral in which one pair of opposite sides is parallel is known as a **trapezium** (see Figure 5).

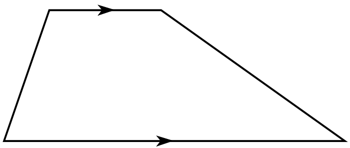


Figure 5 A trapezium.

◆ Is every rectangle a square? Is every square a rectangle? Is every rectangle a rhombus? Is every square a rhombus?

General polygons

Although triangles and quadrilaterals are the most commonly occurring polygons, the following result is sometimes useful since it can be applied to polygons with any number of sides. ☞ As one would expect, the corners of a polygon are known as **vertices** and the angle formed at a vertex which lies in the interior of the polygon is called an *interior angle*. The method that we used to show that the sum of the interior angles of a quadrilateral is 360° can easily be adapted to show that:

> The sum of the interior angles of an n sided polygon is $(2n - 4) \times 90°$, i.e. $(2n - 4)$ right angles.

Figure 6 shows a regular pentagon and a regular hexagon.

❖ What is the size of an interior angle in (a) a regular pentagon, and (b) a regular hexagon?

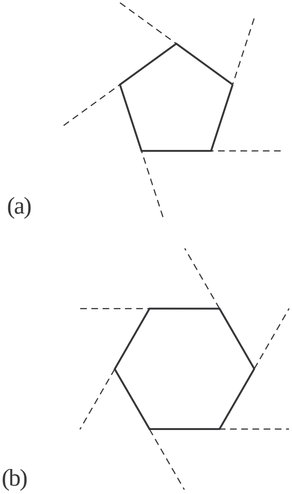


Figure 6 (a) A regular pentagon, and (b) a regular hexagon.

3 Congruence and similarity

3.1 Congruent shapes

Geometric figures which are the same shape and size are said to be **congruent**. If we were to cut congruent triangles out of a piece of paper, we would be able to fit one exactly on top of the other, though we might have to turn one over or rotate it to do so. It is very important to be able to decide when triangles are congruent and the main aim of this section is to discuss this question.

In order to decide whether triangles are congruent, we need to know what is the minimum information to specify completely a triangle of unique size and shape. All triangles drawn with such a specification will then be copies of each other, and so congruent. There are four possible ways to specify a triangle of unique size and shape. These are as follows:

Minimal specifications of a triangle

(a) State the length of all three sides. (SSS)

(b) State the lengths of two sides and the angle between them. (SAS)

(c) State the length of one side and the angles at each end of it. (ASA)

(d) Specify a right angle, the length of the hypotenuse and the length of one other side. (RHS)

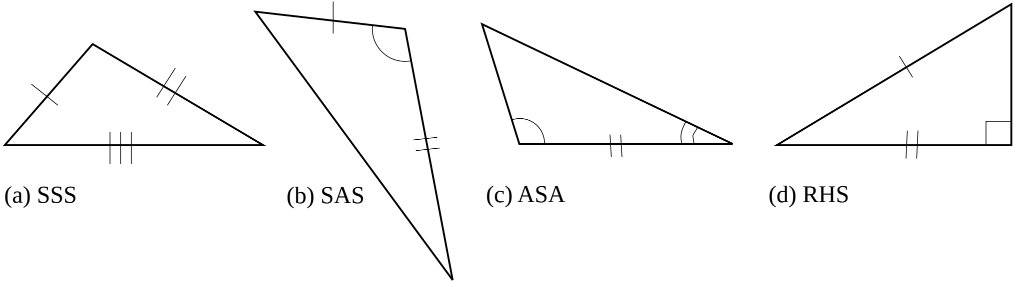


Figure 7 Minimal specifications for a triangle. The cross bars and angles indicate the sides and angles referred to in the specifications.

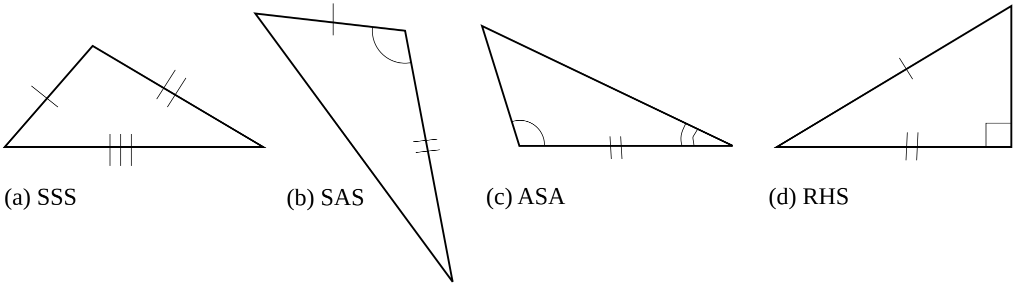


Figure 7 Minimal specifications for a triangle. The cross bars and angles indicate the sides and angles referred to in the specifications.

These specifications are shown diagramatically in Figure 7 together with abbreviations which may help you to remember them. The bars across the lines are often a useful device for indicating lines of equal length and in Figure 7 they indicate the lines referred to in the specifications above. We can now try to construct a triangle from each of these four specifications.

(a) Suppose that we are told that the lengths of the sides of a triangle are 7.4 cm, 6.0 cm and 3.8 cm. We first draw a line AB with the length of one of the sides, say 7.4 cm, as in Figure 8a.
Then we draw a *circle*, ☞ with its centre at A, with the length of another side, say 6.0 cm, as its radius. Finally we draw a second circle with B as centre and the length of the third side, 3.8 cm, as its radius. If the circles intersects at a point C_1, then the triangle ABC_1 has sides of the required lengths. (The other intersection, C_2, would do equally well: triangle ABC_1 is a mirror-image in the line AB of triangle ABC_2.)

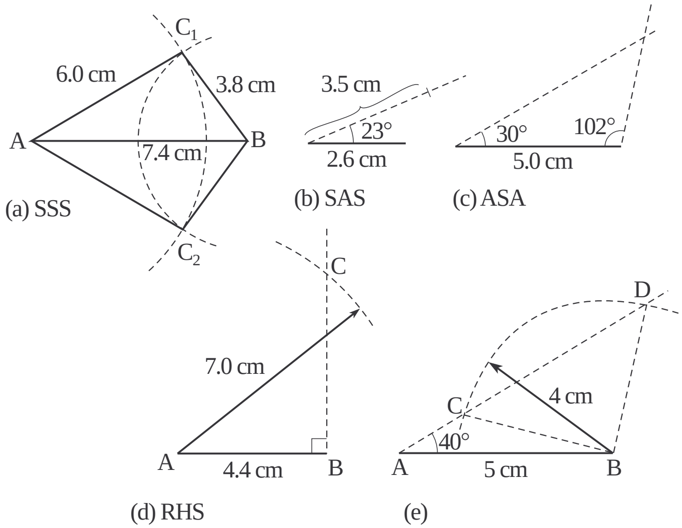


Figure 8 Constructing triangles using the minimal specifications. (These figures are not to scale.)

(b) Suppose that we are told that the lengths of two of the sides of a triangle are 2.6 cm and 3.5 cm, and that the angle between them is 23°. First we draw a line segment of one of the known lengths, say 2.6 cm, as in Figure 8b. Next we use a protractor to draw a straight line through one end of this line segment and at the known angle, 23°, to it. We make the length of the second line the second known length 3.5 cm. Finally we join the end of the second line to the end of the first, giving the required triangle.

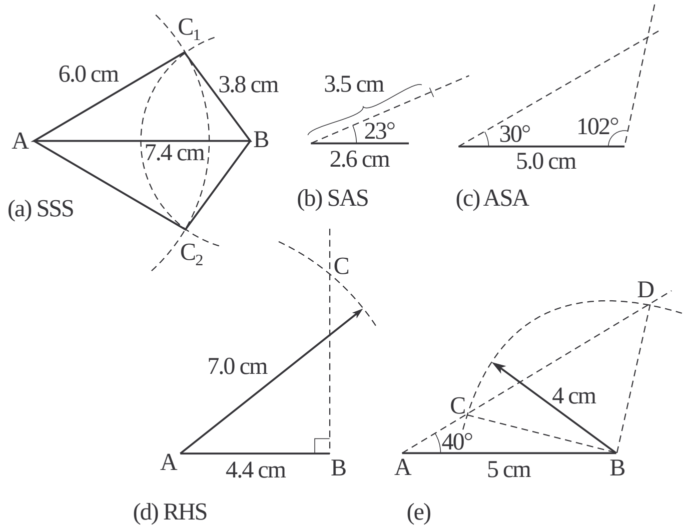


Figure 8 Constructing triangles using the minimal specifications. (These figures are not to scale.)

(c) Suppose that we are given the length of one side, say 5.0 cm, and the angles at either end of this line, say 30° and 102°. We first draw a line segment of the known length, 5.0 cm, which immediately gives us two vertices of the triangle, as in Figure 8c. Then from each vertex we draw a line at the specified angle to the line segment. The intersection of these two lines gives us the third vertex.

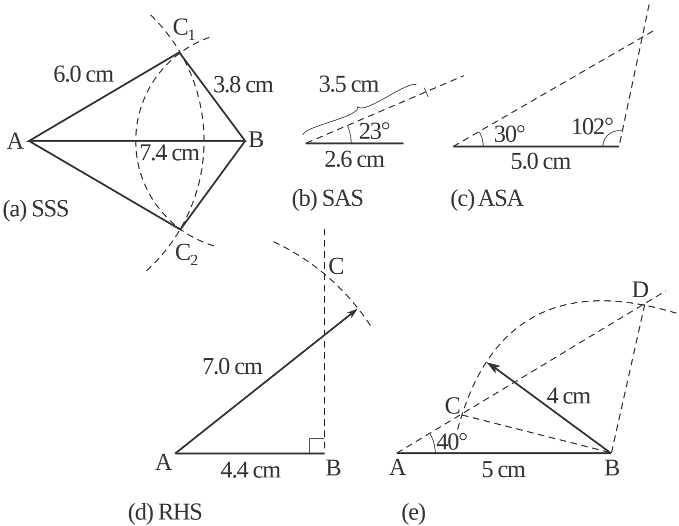


Figure 8 Constructing triangles using the minimal specifications. (These figures are not to scale.)

(d) Suppose that we are told that one of the angles in the triangle is a right angle and we are given the lengths of the hypotenuse and one other side, say 7.0 cm and 4.4 cm, respectively. We first draw a line segment the length of the given side other than the hypotenuse, AB in Figure 8d. Then we draw a line at right angles to it at one end, B say; finally, we draw a circle with its centre at the other end, A, and of radius equal to the length of the hypotenuse. The point at which this circle intersects the second line, C, is the third vertex of the triangle.

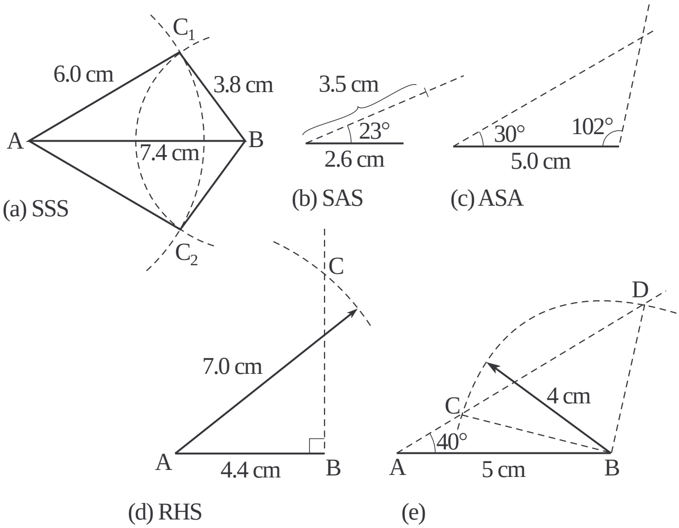


Figure 8 Constructing triangles using the minimal specifications. (These figures are not to scale.)

Three pieces of information are not always sufficient to specify a triangle completely. If we are given the lengths of two sides and an angle other than the one between the sides, it is possible that we will be able to draw two (non-congruent) triangles each meeting the given specification. For example, suppose we are told that a triangle has two sides of length 4 cm and 5 cm, and that the angle at the end of the 5 cm line (not between it and the 4 cm line) is 40°. We can construct two possible triangles (ABC and ABD), but they are not congruent with each other, as illustrated in Figure 8e.

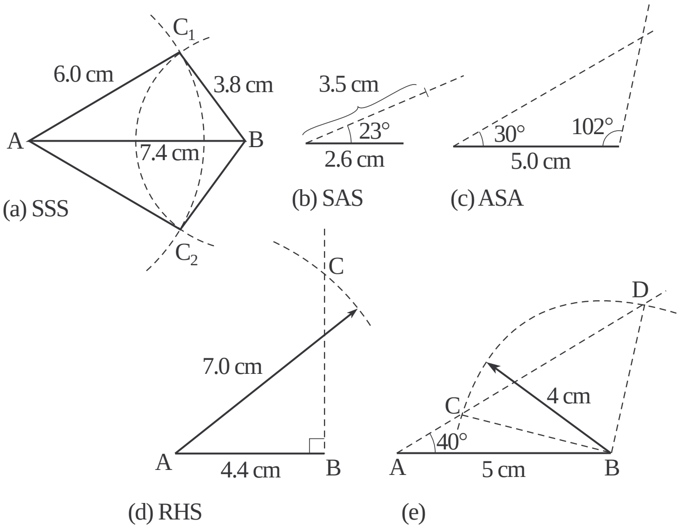


Figure 8 Constructing triangles using the minimal specifications. (These figures are not to scale.)

✦ Would the three interior angles completely specify a unique triangle?

Triangles like these which are of the same shape but of different sizes are called *similar triangles* and we will be discussing them in detail in the next subsection. Of course once the length of a particular side is chosen, you then have a side and the angle at each end of it specified, and so you can draw a unique triangle as in case (c) above.

Having determined the minimal amount of information needed to uniquely specify a triangle (the conditions SSS, SAS, ASA and RHS of the previous box), we can now present a simple test for the congruency of two triangles.

Test for congruent triangles

Two triangles must be congruent if the information contained in any of the conditions SSS, SAS, ASA or RHS is the same for both.

It should be noted that although the above test only requires *one* of the minimal specifications to be common to both triangles it must follow that if the triangles are congruent then common values can be found for all three of the general specifications (SSS, SAS, ASA) and for the fourth specification (RHS) if it applies.

Bisectors

Isosceles triangles have some useful properties arising from their symmetry.

In the triangle ABC of Figure 8a, AB = AC. The line AD is drawn so that $B\hat{A}D = C\hat{A}D = \frac{1}{2}(B\hat{A}C)$.

This line is called the **bisector** of the angle $B\hat{A}C$. The term is also applied to line segments, so that a point lying at the mid-point of a line segment is sometimes said to bisect the line segment.

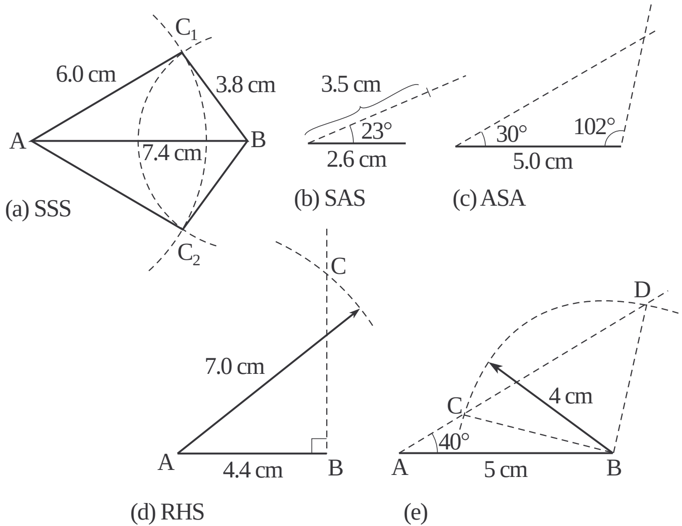
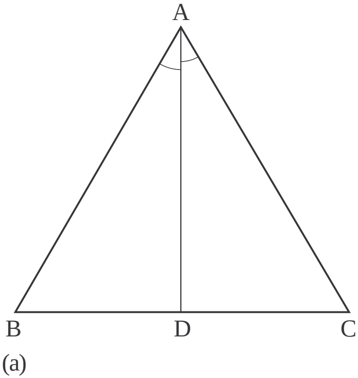


Figure 8 Constructing triangles using the minimal specifications. (These figures are not to scale.)

Question T2

By finding a pair of congruent triangles in Figure 9a, show that:

(a) $A\hat{B}C = A\hat{C}B$, (b) $A\hat{D}B = A\hat{D}C = 90°$, (c) $BD = CD$. ❑



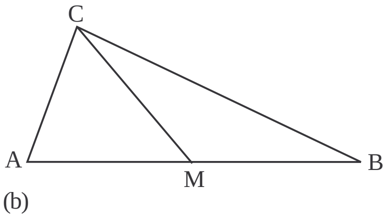


Figure 9 (a) See Question T2. (b) See Question T4.

The properties established in Question T2 are quite general for isosceles triangles and may be summarized as follows.

In any isosceles triangle:

(a) The angles opposite the equal sides are themselves equal.

(b) The line bisecting the angle between the equal sides also bisects the third side, and is perpendicular to it.

Question T3

Show that the angles in an equilateral triangle are of equal size, and find their magnitude. ❏

A line which bisects a line segment and intersects it at right angles is called a **perpendicular bisector** of the line segment. Thus the line AD in Figure 9a is a perpendicular bisector of the side BC.

A **median** is a line drawn from one vertex of a triangle to the mid-point of the opposite side. Although it is not the case for all triangles, for an isosceles triangle the median, perpendicular bisector and the angle bisector are identical.

Question T4

In Figure 9b the line CM = AM and CM is a median so that AM = MB. Show that the angle $A\hat{C}B$ is 90°. ❏

3.2 Similarity, ratios and scales

If two roads meet at an angle of 30°, we expect the same angle to occur on an Ordnance Survey map of the area, in other words we expect maps and plans to preserve the true angles. However, it would be ridiculous to construct a road map with the same dimensions as the original site, so map-makers and designers use the concept of **scaling**. Often a short distance on the map or plan represents a much longer distance in reality, but sometimes the reverse is true and a relatively large distance on the plan may represent a minute length in the real object (as, for example, in the plan of a circuit in a microchip). On one popular series of Ordnance Survey maps, 1 cm on the map represents 50 000 cm or 0.5 km on the ground so that 2 cm represent 1 km.

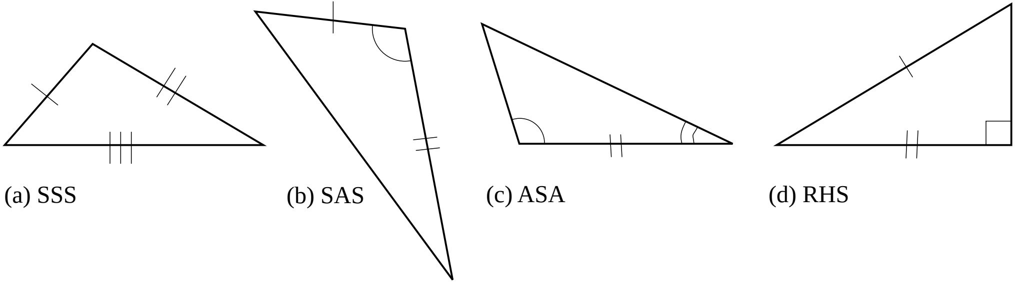


Figure 7 Minimal specifications for a triangle. The cross bars and angles indicate the sides and angles referred to in the specifications.

The essence of scaling is that it preserves shape. You have already come across this idea in Subsection 3.1. When two (or more) triangles are the same shape they are said to be **similar triangles**. The triangles shown in Figures 7b and 7c are actually similar (although it requires a rotation and a reflection of one of them to put them into the same orientation).

If you look at Figure 3 you should be able to see that triangles MXY and MTU are also similar. You should note the convention used when discussing similar triangles, the order of the letters indicates the correspondence of the vertices. It is also true that any two *congruent* triangles are also *similar*. ☞

An important property of similar triangles is that corresponding sides are in the same ratio, i.e. they are *scaled* by the same factor. ☞

Suppose that in an Ordnance Survey map the scale is 2 cm to represent 1 km on the ground. We can express this scale as the ratio of the distance on the map to the corresponding distance on the ground, this ratio is then 1 : 50 000. In other words, actual distances are reduced by a factor of 50 000 before they are drawn on the map. The essential point is that only one scale is used for *all* distances in *all* directions (even curved paths).

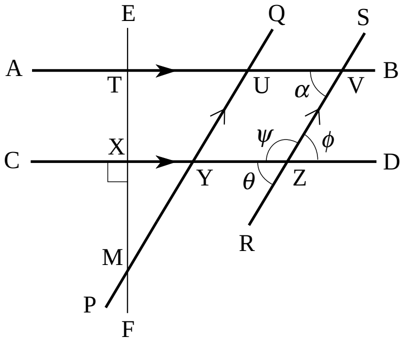


Figure 3 Properties of parallel lines.

If A, B and C in Figure 10 represent the positions of three towns, and P, Q and R are their respective positions on the map, then

$$PQ = \frac{AB}{50\,000} \qquad QR = \frac{BC}{50\,000} \qquad RP = \frac{CA}{50\,000}$$

It follows that

$$\frac{PQ}{AB} = \frac{QR}{BC} = \frac{RP}{CA}$$

In any two similar triangles the ratios of corresponding sides are equal, and this result is true whatever the scale.

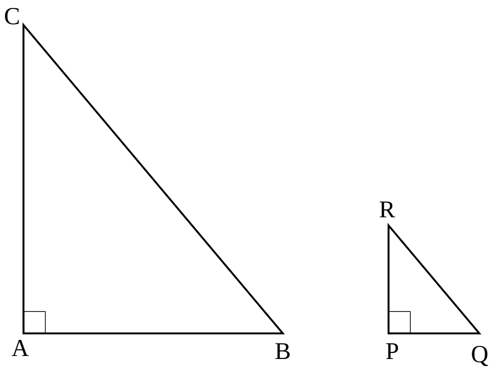


Figure 10 Similar triangles.

Question T5

Drivers are expected to be able to read car number plates at a distance of about 25 metres. The letters on the number plates are 8 cm high. You wish to put up a road sign which drivers travelling at $30\,\mathrm{m\,s^{-1}}$ will be able to read with ease. If it takes 3 s to read what is on the sign, and the driver must have completed reading it 100 m before the car reaches the sign (so that the driver can respond to what she/he reads), how large should you make the lettering on the road sign? This is illustrated in Figure 11. ❏

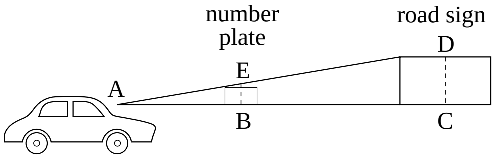


Figure 11 See Question T5.

Similar triangles and shadows

Now suppose that in Figure 10, AC represents a tree while AB represents the shadow cast by the tree; RP is a measuring pole of length 1 m and PQ represents its shadow. Suppose that we measure the length of the shadows and find them to be AB = 3.0 m and PQ = 0.75 m, we can then use the fact that ABC and PQR are similar triangles to calculate the height of the tree. Since the triangles are similar,

$$\frac{\mathrm{CA}}{\mathrm{RP}} = \frac{\mathrm{AB}}{\mathrm{PQ}}$$

Hence $$\mathrm{CA} = \frac{\mathrm{AB}}{\mathrm{PQ}} \times \mathrm{RP} = \frac{(3\,\mathrm{m})}{(0.75\,\mathrm{m})} \times (1\,\mathrm{m}) = 4\,\mathrm{m}$$

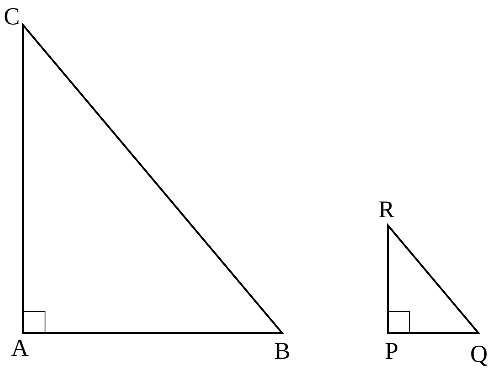


Figure 10 Similar triangles.

Question T6

Suppose that the pole RP (in Figure 10) is 2 m long, its shadow is 1 m long and the shadow of the tree is 3 m long. How tall is the tree? ❏

Question T7

A vertical tree is growing on a hill that slopes upwards away from the sun at an inclination of 10° to the horizontal. The shadow of a pole 2 m long (held vertically on the hill) at the time of measurement is 1.5 m long and the shadow of the tree is 4.5 m long. Draw a diagram representing the situation and show that the properties of similar triangles still apply. What is the height of the tree?
(Assume that the tops of the shadows from the pole and the tree fall in the same place.) ❏

Sometimes it is not obvious when two triangles are similar. In Figure 12 ABC is a right-angled triangle with the right angle at B. We have drawn BD perpendicular to AC, and labelled the angles at A and C as α and β, so $\alpha + \beta = 90°$.

✦ Can you see any similar triangles in Figure 12?

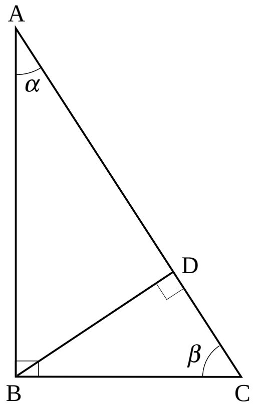


Figure 12 Similar triangles within a triangle.

4 Circles

4.1 Parts of a circle

Various terms are used to describe the geometry of circles, and it is necessary to commit them to memory since they occur frequently.

A **circle** is the set of points in a plane which are the same distance (i.e. **equidistant**) from a given point (in the same plane) called the **centre** of the circle and the distance is called the **radius** of the circle. The set of points with a given property is sometimes called the **locus**, so that in this case we might say that a circle is the locus of all points equidistant from a given point (i.e. the centre). ☞ The perimeter of the circle is known as the **circumference** of the circle. A straight line segment which passes through the centre of the circle and divides the interior of the circle into two equal parts is called a **diameter** (which is equal in length to twice the radius).

In Figure 13a C is the centre of the circle, ACB is a diameter and CP is a radius. The circumference of a circle plus its interior are sometimes known as a **disc**. The two halves of the circumference of the circle on either side of a diameter are called **semicircles**; two diameters which intersect at right angles (at the centre of the circle) divide the disc into four **quadrants** and when the two diameters are vertical and horizontal, as in Figure 13b the quadrants are labelled as shown.

A portion of the circumference of the circle between two points, such as D and E in Figure 13a, is called an **arc** of the circle. When DE is not a diameter, the longer route along the circumference from D to E is called a **major arc** and the shorter route is called a **minor arc**. The straight line segment DE which connects two points on the circumference is called a **chord**. When the chord is not a diameter, it divides the disc into two unequal regions, called **segments**. The smaller region (the shaded portion from D to E) is called a **minor segment**, the larger region is called a **major segment**.

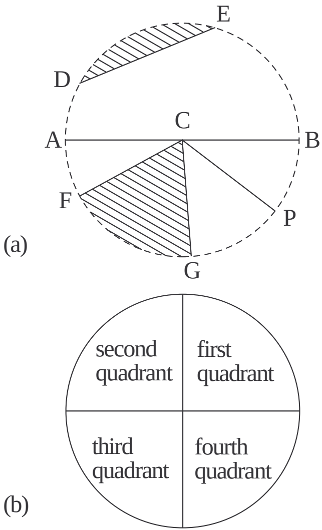


Figure 13 Features of a circle.

The region inside a disc cut off by two radii, such as the shaded region FCG in Figure 13a, is called a **sector**, as is the remainder of the disc when this shaded region is removed. The minor arc FG of the circle is said to **subtend** an acute angle $\mathrm{F\hat{C}G}$ at the centre of the circle, whereas the major arc FG of the circle *subtends* an obtuse or reflex angle $\mathrm{F\hat{C}G}$ at the centre.

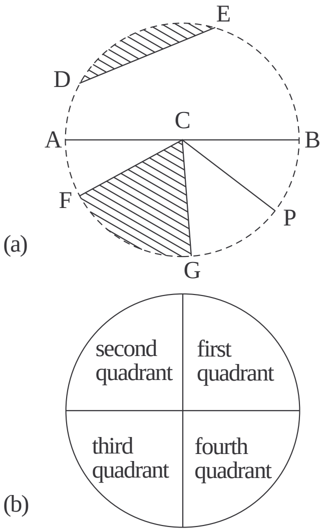


Figure 13 Features of a circle.

Two or more circles with the same centre but different radii are said to be **concentric** (see Figure 14b) and the region bounded by two concentric circles is called an **annulus**.

✥ What term is used for a *segment* formed by an *arc* of a *circle* that *subtends* a right angle at the *centre* of a circle?

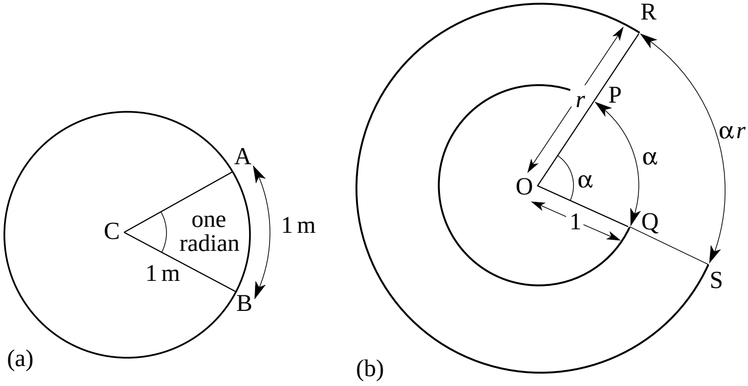


Figure 14 The radian measure.

4.2 Properties of circles

The ratio of the circumference of a circle to its diameter is the well-known constant π (π is the Greek letter pi), which has an approximate value of 3.141 59 (though the less precise fractional value of 22/7 is often used).

So we can write $c = \pi d$

where c and d are the circumference and diameter of the circle, respectively.

The circumference of a circle of radius r is $2\pi r$	(2)

since the diameter is twice the radius.

So far in this module we have measured angles in degrees, a method which goes back to ancient times. In spite of its age, and its common use, this is not in fact a ‘natural’ system of measurement since there is nothing natural about choosing 360 rather than some other number ☞.

An alternative scale, based on the lengths of arcs formed on a circle of unit radius (i.e. a unit circle), turns out to be far more appropriate in many circumstances (in particular in the context of *calculus*). Figure 14a represents a circle of radius 1 m, and the arc AB has been chosen so that it is exactly 1 m in length. The angle subtended by the arc at C is defined to be one **radian**. Since angles are preserved by scaling, there is nothing special about the choice of metres as our units of length, and in fact it is more usual to define a *radian* in terms of a circle of unit radius, without specifying what that unit of length might be.

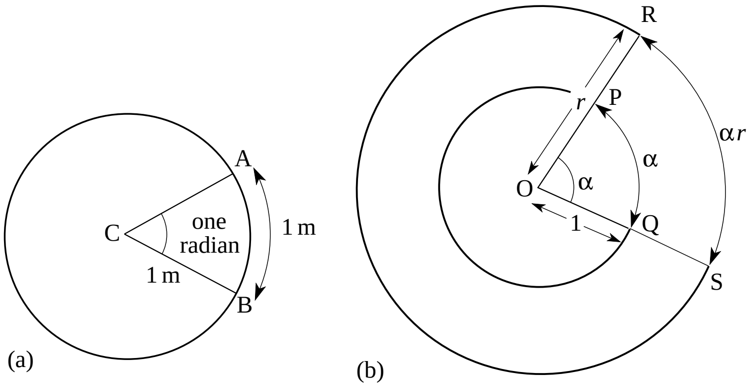


Figure 14 The radian measure.

In Figure 14b we have not specified the units of length, we have simply referred to them as ‘units’. Thus in Figure 14b the circle is of radius 1 unit. An arc, PQ say in Figure 14b, of length α units subtends an angle α radians at the centre O of the circle, as does an arc RS of length αr units on a circle of radius r.

Thus we have RS $= \alpha r$ so that $\alpha = \text{RS}/r$, and since both RS and r have dimensions of length, it follows that angles measured in radians are dimensionless quantities. You should remember that:

> An arc of a circle of radius r, subtending an angle α (in radians) at the centre of the circle, has length αr so that, in Figure 14b,
>
> $$\text{RS} = \alpha r \tag{3}$$

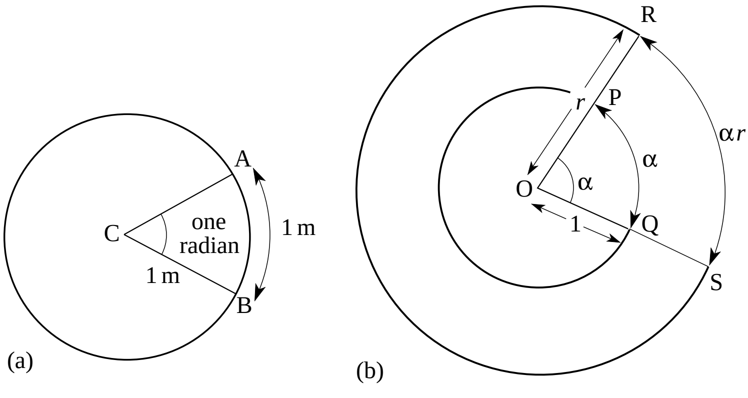


Figure 14 The radian measure.

If we extend the arc in Figure 14b so that RS is actually the circumference of the circle (and R coincides with S) we saw, in Equation 2, that its length will be $2\pi r$. Therefore RS $= \alpha r = 2\pi r$ and α for this 'arc' is a full turn, i.e. 360°.

Therefore a complete turn of 360° corresponds to 2π radians, which we write as

$$360 \text{ degrees} = 2\pi \text{ radians}$$

From this we can deduce other relationships, for example,

$180° = \pi$ radians, $90° = (\pi/2)$ radians,

$60° = (\pi/3)$ radians, $45° = (\pi/4)$ radians, $30° = (\pi/6)$ radians.

1 radian is equivalent to $\left(\dfrac{180}{\pi}\right)^{\circ}$ which is approximately 57.3°.

✦ Express $3\pi/2$ radians in degrees and 135° in radians.

Question T8

In Figure 15 a belt ABCDA passes round two pulleys as shown. Find the length of the belt. ❏

4.3 Tangents

In Figure 16 the line TP intersects the circle at just one point. Such a line is called a **tangent** to the circle and T is the point of contact where the tangent touches the circle. The tangent is perpendicular to the radius at T. The radius OT joins the point of contact to the centre of the circle. In Figure 15 the line AB is a tangent to both circles, in which case we say that it is a **common tangent**.

From any point outside the circle, say P in Figure 16, it is possible to draw two tangents to the circle PT and PS, and the lengths of PT and PS are equal.

✦ Show that the lengths of PT and PS are equal.

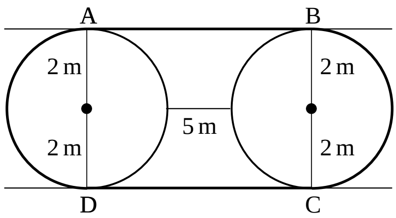


Figure 15 See Question T8.

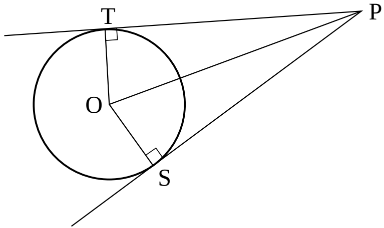


Figure 16 Tangents to a circle.

5 Areas and volumes

5.1 Areas of standard shapes

The areas of quadrilaterals and triangles

The problem of finding areas of plane shapes is a common one. The area of the rectangle shown in Figure 17a is the product ab of the lengths of two adjacent (i.e. neighbouring) sides. Provided that we are given the sizes, it is easy to calculate the area of a shape such as that shown in Figure 17b by dividing it into suitable rectangles.

Figure 17c shows a rectangle ABCD with area ah. The triangles AED and BFC are congruent and so of equal area, and thus the area of the parallelogram EFCD is also ah. ☞ With DC as the base of the parallelogram, the length h is known as the **perpendicular height**, so that:

The area of a parallelogram = base × perpendicular height

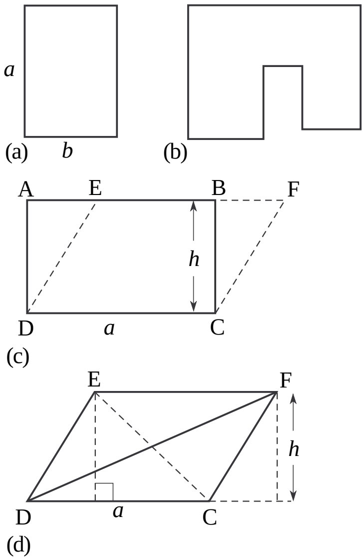


Figure 17 Rectangles, parallelograms and triangles.

Figure 17d shows the same parallelogram as in Figure 17c, and the diagonal of the parallelogram EC divides the parallelogram into two parts of equal area (as does the diagonal DF). It follows that

area of triangle EDC = area of triangle DFC = $ah/2$ ☞

and in general ☞:

The area of a triangle = $\frac{1}{2}$ base × perpendicular height

Another result that is sometimes of use relates the area of a triangle to the lengths of its sides. We quote this result without proof

The area of a triangle = $\sqrt{s(s-a)(s-b)(s-c)}$

where $s = \frac{1}{2}(a + b + c)$

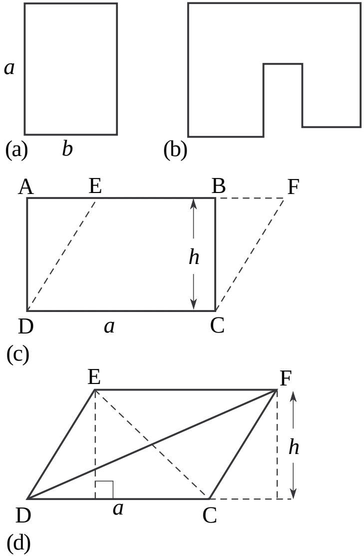


Figure 17 Rectangles, parallelograms and triangles.

A trapezium may be regarded as the sum of a parallelogram and a triangle. It follows that its area is given by

The area of a trapezium = $\frac{1}{2}$ sum of parallel side lengths $\times$ perpendicular height

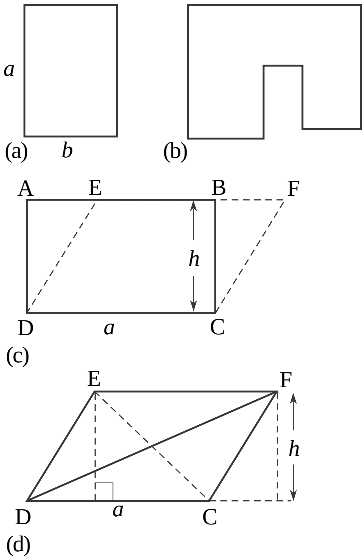


Figure 17 Rectangles, parallelograms and triangles.

Area enclosed by a circle

We state without proof that the area enclosed by a circle of radius r is πr^2.

This area is commonly (loosely) referred to as 'the area of the circle', though strictly we should say 'the area enclosed by the circle' or 'the area of the disc'.

$$\boxed{\text{The area of a circle of (radius } r) = \pi r^2 \qquad (4)}$$

Area of a sector

Consider the shaded sector FCG shown in Figure 13a and suppose that the radius of the circle is r and that $\mathrm{F\hat{C}G} = \theta$ (radians). The ratio of the area of the sector to the area of the circle is the same as the ratio of the angle θ to the angle of a complete turn, i.e. 2π.

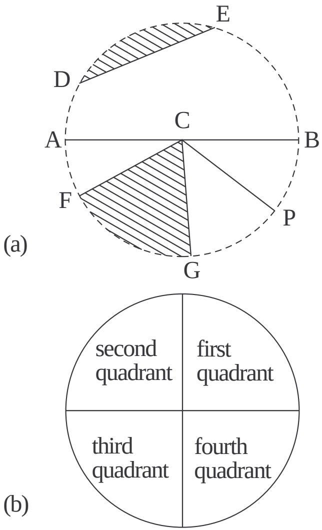


Figure 13 Features of a circle.

In other words,

$$\frac{\text{area of sector}}{\pi r^2} = \frac{\theta}{2\pi} \qquad \text{(Remember that } \theta \text{ \textit{must} be measured in radians.)}$$

Hence:

$$\boxed{\text{The area of sector } = \tfrac{1}{2}\,\theta r^2 \qquad (5)}$$

Area of a segment

We can find the area of a segment by finding the area of a sector and then subtracting the area of a triangle. ☞

Question T9

In Figure 13a suppose that a minor arc DE is of length 2π metres and that the radius of the circle is 4 metres.

(a) Find the angle θ subtended by the arc DE at C in both degrees and radians.

(b) Find the area of the triangle DEC.

(c) Find the area of the sector DEC.

(d) Find the area of the (larger) segment DAFGPBE.

(e) Find the area of the segment from D to E (the hatched area in Figure 13a).

Leave your answers in terms of π where appropriate. ❑

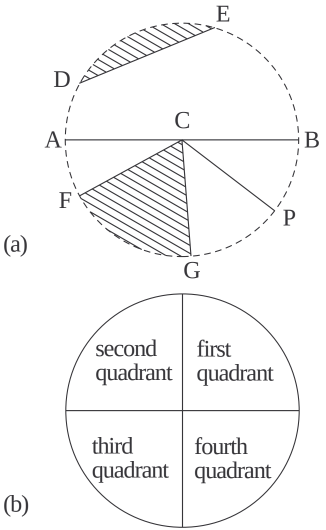


Figure 13 Features of a circle.

Area of the surface of a cylinder

The ends of a cylinder are circles. The curved surface of a cylinder can be opened out to form a rectangle as shown in Figure 18 so its area is the height of the cylinder times its circumference: that is, $2\pi R \times L$.

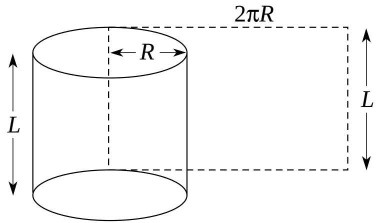


Figure 18 Surface area of a cylinder.

5.2 Volumes of standard solids

Prisms

Just as it is useful to be able to find the area of geometrical shapes, so it is useful to know the volume of various solid shapes. One of the simplest solid shapes is a box, known more formally as a **cuboid** or a **rectangular block**.

If the sides of the cuboid are of length a, b and c, its volume is $a \times b \times c$.

In fact, a cuboid is a special case of a class of solids called *prisms*. A triangular **prism** is a solid for which the cross sections cut parallel to a certain direction are congruent triangles (oriented the same way), as shown by the hatched area in Figure 19. Every slice through the prism parallel to the base will produce a triangle identical to the shaded triangle. A *cuboid* is a prism for which the cross sections parallel to the base are identical quadrilaterals, and a cylinder is a prism for which the cross sections perpendicular to its axis are identical circles (see Figure 18). The *perpendicular height* of the prism is the length labelled b in Figure 19, and the hatched triangle is known as the **cross-sectional area** of the prism.

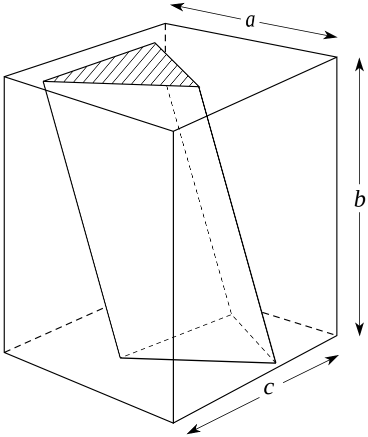


Figure 19 A cuboid and a prism.

These terms can be easily generalized to prisms of any cross-sectional shape, and in general:

The volume of a prism = cross-sectional area × perpendicular height

❖ Find the volume of a cylinder with perpendicular height 1.5 m and whose cross section is a circle of radius 10 cm.

Question T10

A prism ☞ has perpendicular height 5 m and its cross section is an annulus formed by circles of radius 1 m and 2 m. What is the volume of the prism? ❏

Spheres

A **sphere** is a three-dimensional surface whose points are at an equal distance (the radius) from a fixed point (the centre of the sphere). *Calculus* ☞ provides the easiest way of finding the area and volume of a sphere, and we will simply quote the results:

The surface area of a sphere of radius r is $4\pi r^2$ and its volume is $\frac{4}{3}\pi r^3$ (6)

❖ The surface area of a sphere is $10\,\text{m}^2$. What is its volume?

Question T11

A given sphere has twice the volume of another. What is the ratio of their surface areas? ❏

6 Closing items

6.1 Module summary

1 Much of this module is devoted to the various terms that are used to describe *geometric figures*.

2 When a line *intersects* a pair of *parallel lines* various angles between the lines are equal, in particular *corresponding* and *alternate* angles.

3 The sum of the *interior angles* of a *polygon* with n sides is $(2n - 4)$ *right angles*.

4 *Similar* figures (in particular triangles) are the same shape, and corresponding lengths in two similar figures are in the same ratio. *Congruent* figures are the same shape and the same size. The four minimal specifications of a unique triangle are as follows:

(a) State the length of all three sides. (SSS)

(b) State the lengths of two sides and the angle between them. (SAS)

(c) State the length of one side and the angles at each end of it. (ASA)

(d) Specify a right angle, the length of the hypotenuse and the length of one other side. (RHS)

If any of these specifications is common to two triangles, then those triangles must be congruent.

5 The sides a, b and c of a right-angled triangle are related by *Pythagoras's theorem*

$$c^2 = a^2 + b^2 \qquad \text{(Eqn 1)}$$

where c is the hypotenuse of the triangle.

6 The *circumference* of a circle of radius r is $2\pi r$.

7 One *radian* is the angle subtended at the centre of a unit circle by an *arc* of unit length. The length of an arc of a circle of *radius* r which *subtends* an angle θ (in radians) at the centre is $r\theta$.

$$360 \text{ degrees} = 2\pi \text{ radians}$$

8 A *tangent* is a line that meets a circle at one point. A right angle is formed by the tangent and a radius drawn to the point of contact.

9 The area of a *parallelogram* is the base times the *perpendicular height*, and the area of a triangle = $\frac{1}{2}$ base $\times$ perpendicular height.

10 The area of a *circle* of radius r is πr^2 and the area of a sector *subtending* an angle θ at the centre is $\frac{1}{2}\theta r^2$.

11 The volume of a *prism* is the *cross-sectional area* times the *perpendicular height*.

12 The surface area of a *sphere* of radius r is $4\pi r^2$ and its volume is $\frac{4}{3}\pi r^3$.

6.2 Achievements

Having completed this module, you should be able to:

A1 Define the terms that are emboldened and flagged in the margins of the module.

A2 Identify the different types of angles formed when lines intersect and calculate their values.

A3 Identify which angles formed by a line intersecting two parallel lines are equal.

A4 Classify the different types of quadrilateral.

A5 Understand and apply the simple properties of polygons.

A6 Classify the different types of triangle and apply the test for the congruence of two triangles using the four minimal specifications of a triangle.

A7 Recognize when two triangles are similar and compare the lengths of corresponding sides.

A8 Recognize the main features of a circle, including tangents, arcs, segments and sectors.

A9 Evaluate areas and volumes of standard shapes.

A10 Apply Pythagoras's theorem and use it to identify right-angled triangles.

Study comment You may now wish to take the *Exit test* for this module which tests these Achievements. If you prefer to study the module further before taking this test then return to the *Module contents* to review some of the topics.

6.3 Exit test

Study comment Having completed this module, you should be able to answer the following questions each of which tests one or more of the Achievements.

Question E1

(*A2* and *A3*) Find the angle $B\hat{C}D$ in Figures 20a and 20b.

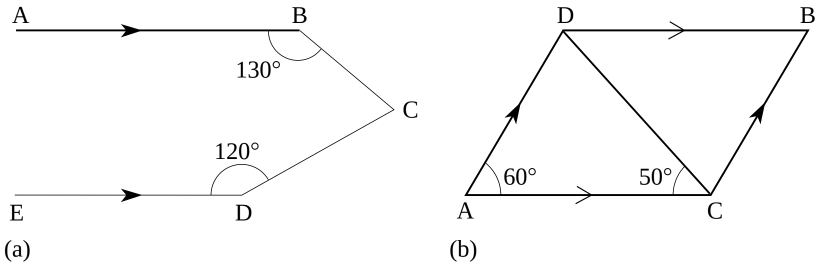


Figure 20 See Question E1.

Question E2

(*A3* and *A5*)   Suppose that you are told that in Figure 9b $\mathrm{A\hat{C}B} = 90^\circ$ and that AM = MC. Prove that M is the mid-point of AB.

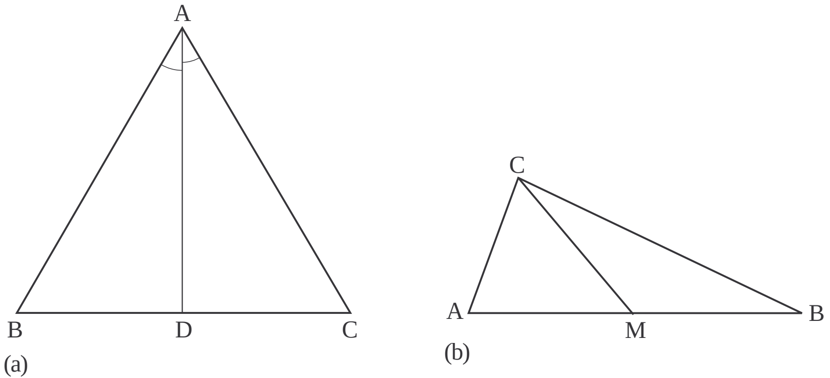


Figure 9   (a) See Question T2. (b) See Question T4.

Question E3

(*A3* and *A7*) In Figure 21 ABCD is a parallelogram. A line through A meets DC at E and meets BC produced at F.

Show that triangles ADE and FCE are similar.

If DE = 15 mm, EC = 10 mm and FC = 5 mm find AD.

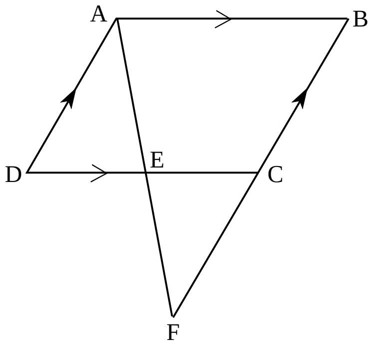


Figure 21 See Question E3.

Question E4

(*A7*, *A*8 and *A10*) In Figure 22 AB is a tangent to the circle of radius a and centre C. DB represents the perpendicular height of the triangle ABC with AC as its base. The lengths CD and DA are x and y, respectively. Use an argument based on similar triangles to show that:

$$\frac{x}{a} = \frac{a}{x+y} \quad \text{and} \quad \frac{y}{c} = \frac{c}{x+y}$$

Show that $(\mathrm{AC})^2 = a^2 + c^2$.

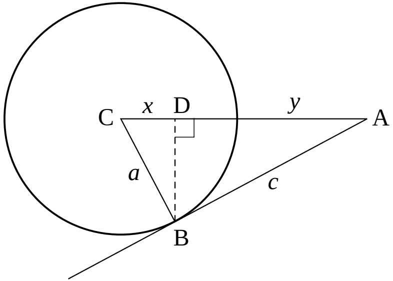


Figure 22 See Question E4.

Question E5

(*A9*) A washer is in the form of a cuboid with sides of length 1.5 cm and thickness 4 mm with a circular hole of radius 0.6 cm drilled through its centre. Calculate the volume of the washer.

Study comment This is the final *Exit test* question. When you have completed the *Exit test* go back to Subsection 1.2 and try the *Fast track questions* if you have not already done so.

If you have completed **both** the *Fast track questions* and the *Exit test*, then you have finished the module and may leave it here.