# Module P3.2 Gravitation and orbits

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# 1 Opening items

# **1.1 Module introduction**

In 1687 Newton formulated his *law of universal gravitation* in which he asserted that there is a gravitational interaction between all masses. In this module, we apply Newton's law of gravitation to a variety of situations, beginning in Section 2 with the Cavendish experiment to measure the *universal gravitational constant*, *G*.

In Section 3 we concentrate on gravitational effects due to the Earth and see that, for an object close to the Earth, the gravitational force due to the Earth itself is much greater than that due to any other body. We see how the *weight* of an object on Earth is related to the Earth's *surface gravity* (i.e. the surface *gravitational field*, or *acceleration due to gravity*). We also see that close to the Earth, the gravitational field can be treated as uniform, giving rise to a simple expression for changes in *gravitational potential energy*. However, for an object that travels from the Earth's surface to an infinite distance from the Earth we need to take account of the variation in field strength with distance in order to derive an expression for the *escape velocity*.

In Section 4 we apply the law of gravitation and ideas about circular motion to the orbits of planets and satellites, and hence explain the basis of *Kepler's laws of planetary motion*. We then use ideas about orbital motion to account for ocean *tides* and to explain how astronauts in an orbiting vehicle might detect the presence of a gravitational field despite feelings of *'weightlessness'*.

Sections 2 to 4, are concerned with the gravitational effects of those extended bodies which may be regarded as spheres, but we represent them by point masses located at their geometric centres. In Section 5, we use ideas about *gravitational potential* together with the principle of superposition to justify this procedure. This is a more technical section which assumes a greater knowledge of calculus and trigonometry than the rest of the module, but the method it uses and the result it derives are of importance in many areas of physics.

*Study comment* Having read the introduction you may feel that you are already familiar with the material covered by this module and that you do not need to study it. If so, try the *Fast track questions* given in Subsection 1.2. If not, proceed directly to *Ready to study?* in Subsection 1.3.

# 1.2 Fast track questions

**Study comment** Can you answer the following *Fast track questions*?. If you answer the questions successfully you need only glance through the module before looking at the *Module summary* (Subsection 6.1) and the *Achievements* listed in Subsection 6.2. If you are sure that you can meet each of these achievements, try the *Exit test* in Subsection 6.3. If you have difficulty with only one or two of the questions you should follow the guidance given in the answers and read the relevant parts of the module. However, *if you have difficulty with more than two of the Exit questions you are strongly advised to study the whole module*.

#### **Question F1**

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Calculate the surface gravity (acceleration due to gravity at the surface) of the Moon.

#### **Question F2**

Calculate the radius of the orbit of a geostationary satellite orbiting the Earth.

## **Question F3**

Calculate the escape speed from the surface of the Moon.



#### Study comment

Having seen the *Fast track questions* you may feel that it would be wiser to follow the normal route through the module and to proceed directly to <u>*Ready to study?*</u> in Subsection 1.3.

Alternatively, you may still be sufficiently comfortable with the material covered by the module to proceed directly to the *Closing items*.

# 1.3 Ready to study?

**Study comment** In order to study this module you will need to be familiar with the following terms: <u>acceleration</u>, <u>conservation of energy</u>, <u>density</u>, <u>force</u>, <u>kinetic energy</u>, <u>mass</u>, <u>Newton's laws of motion</u>, <u>position vector</u>, <u>potential energy</u>, <u>torque</u> and <u>work</u>. You will be required to plot <u>graphs</u>, particularly <u>straight-line graphs</u>, and to use them to investigate relationships between <u>variables</u>. You will also need to be familiar with <u>vectors</u>, including the concept of a <u>unit vector</u>. <u>Integration</u> is used at one point in Section 3, but apart from this the use of calculus is confined to Section 5, which also uses <u>implicit differentiation</u> and the <u>cosine rule</u> of trigonometry. All the necessary concepts concerning <u>gravitational field</u>, <u>gravitational potential</u> and <u>circular motion</u> (<u>angular speed</u>, <u>centripetal acceleration</u> and <u>centripetal force</u>) are introduced at the appropriate points in this module, but some of those introductions are rather rapid and you will find it beneficial if you have studied such concepts before; they are all discussed in greater detail elsewhere in *FLAP*. If you are uncertain about any of these terms then you can review them now by reference to the *Glossary*, which also indicates where in *FLAP* they are developed. The following *Ready to study questions* will help you to establish whether you need to review some of the above topics before embarking on this module.

#### Question R1

Evaluate the following <u>integral</u> between the <u>limits</u> given  $\int_{-\infty}^{a} \frac{dx}{x^2}$ .

## **Question R2**

(a) Sketch a graph of y against x for  $y^2 = 2x^3$ .

(b) What would you plot in order to obtain a straight-line graph from a set of (x, y) values that obeyed  $y^2 = 2x^3$ ? What would be the <u>intercept</u> and <u>gradient</u> of your graph?

#### **Question R3**

Calculate the magnitude of the torque (turning effect) produced by the forces shown in Figure 1 about a point midway between the lines of action of the two forces.



Figure 1 See Question R3.

## **Question R4**

The mass of a lead sphere of diameter 100 mm is 5.92 kg. Calculate the density of lead.

(The volume of a sphere of radius r is  $V = \frac{4}{3}\pi r^3$ .)

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# 2 Universal gravitation

# 2.1 Gravitational forces

We are all familiar with the notion of gravitational attraction. We know that the Earth, for example, exerts a 'pull' on objects in its vicinity. In 1687, Isaac Newton (1642–1727) published his explanation of this phenomenon in his *Principia Mathematica*. Having made clear what he meant generally by a 'force', in terms of the acceleration it would cause, he went on to introduce the concept of a *gravitational force*, which is described by his **law of universal gravitation**:

Every particle of matter in the Universe attracts every other particle with a force that is directly proportional to the product of the masses of the particles and inversely proportional to the square of the distance between them.

In modern vector notation, the gravitational force  $\mathbf{F}_{grav}$  on a particle of mass  $m_2$  due to a particle of mass  $m_1$  separated from it by a distance r is given by

$$\boldsymbol{F}_{\text{grav}} = \boldsymbol{F}_{21} = -\frac{Gm_1m_2}{r^2}\,\hat{\boldsymbol{r}}$$
(1a)

where  $G = 6.673 \times 10^{-11} \text{ N m}^2 \text{ kg}^{-2}$  is the <u>universal gravitational constant</u>, and  $\hat{\boldsymbol{r}}$  is a dimensionless *unit vector* (i.e. a vector of magnitude 1)  $\textcircled{}{}$  directed from  $m_1$  to  $m_2$ . The minus sign in Equation 1a signifies that the force is in the opposite direction to  $\hat{\boldsymbol{r}}$ , i.e. that the force on  $m_2$  takes the form of an *attraction* towards  $m_1$ . It follows from Equation 1a that the *magnitude* of the gravitational force is:

$$|\boldsymbol{F}_{\text{grav}}| = F_{\text{grav}} = \frac{Gm_1m_2}{r^2}$$
(1b)

Since the strength of the force is proportional to  $1/r^2$  we say that the gravitational force obeys an *inverse square law*.

It is a big step to go from the force between two point-like particles to the force between two extended bodies, such as that between an apple and the Earth. However, you can probably imagine how such a step might be taken, at least in principle. In order to find the resultant force on the apple (or the oppositely directed force on the Earth) you would have to treat the extended bodies as large collections of particles and then use vector addition to sum all the forces that each particle in one body exerted on every particle in the other body. If you know the appropriate mathematical techniques this is not quite as daunting as it sounds, especially if the bodies are spheres, each with a density  $\rho$  that may vary with the distance r from the centre of the sphere, but which certainly doesn't depend on direction. A spherically symmetric body with density  $\rho(r)$  can be envisaged as something like a spherical onion; it consists of concentric layers that may have different densities, but it does not involve any other kind of density variation. The calculation of the gravitational force exerted by such a body is made simple by the following result:

The gravitational effect of any spherically symmetric body, outside its own surface, can be exactly reproduced by a single particle, with the same mass as that body, located at the body's geometric centre.

This result may look at first like 'common sense', but its justification is not obvious — it took Newton almost 20 years to produce a satisfactory proof, and it is sometimes referred to as <u>Newton's theorem</u>. The theorem will be proved in Section 5, but in the meantime we will simply accept that it is true and make use of it.

*Study comment* Section 5 uses ideas that are discussed in Sections 2 to 4, and is mathematically the most demanding section of this module, so we advise you to study the rest of the module first. However, if you really don't like using unjustified results, and you are confident that you have a good grasp of all the mathematical prerequisites listed in Subsection 1.3, you may study Section 5 next and then return to Sections 2 to 4 later.

# 2.2 Determining the gravitational constant G

The value of *G*, the universal gravitational constant, must be determined experimentally by measuring the gravitational force between known masses —currently there is no way to deduce its value theoretically. The gravitational forces between objects that we can handle in the laboratory are extremely weak (for example, the force between two 5 kg masses separated by 0.1 m has a magnitude of approximately  $10^{-7}$  N) so they can be measured only with very sensitive apparatus. It is not surprising therefore that *G* is the least accurately known of all the fundamental constants.

Recent determinations all use the same basic technique as the experiment performed in 1798 by Henry Cavendish. *At* the heart of the **Cavendish's experiment** was a *torsion balance* of the kind shown schematically in Figure 2. In its simplest form such a balance consists of two small spheres of equal mass attached to the ends of a horizontal support rod. The rod is suspended at its centre by a fine vertical wire (or fibre), so it can turn in the horizontal plane but in doing so it will twist the wire and thus produce a torque that opposes the rotation.



**Figure 2** Schematic diagram of a Cavendish torsion balance. (From this side view, one of the large spheres is in front of a small sphere, the second is an equal distance behind the other small sphere.)

The degree of twisting can be monitored by observing the behaviour of a beam of light that comes from a fixed source and is reflected from a small mirror attached to the centre of the support rod. At the start of a typical experiment the balance is shielded from draughts, vibrations and other external influences and allowed to settle into an equilibrium state in which the wire is untwisted. Once this has been achieved two



Figure 3 Use of a light beam to measure the rotation of the torsion balance.

large spherical masses are placed close to the small spheres, in the symmetrical arrangement shown in Figure 2. The gravitational forces between these large spheres and the torsion balance will cause the balance to turn in the horizontal plane until the increasing twist of the wire prevents further rotation. Left to itself, the system will eventually settle into a new equilibrium state as indicated in Figure 3.



**Figure 2** Schematic diagram of a Cavendish torsion balance. (From this side view, one of the large spheres is in front of a small sphere, the second is an equal distance behind the other small sphere.)

The angle of rotation between the two equilibrium positions can be determined from the change in direction of the reflected light beam, and this angle together with the properties of the balance and the masses of the large spheres can be used to determine the value of G, as explained below.

When the large spheres are in position each of them will exert a horizontal gravitational force of magnitude  $F_{\text{grav}}$  on the nearest small sphere. Assuming these are the only significant forces in the horizontal plane and that the spheres are positioned to ensure that the final position is one in which the gravitational forces are at right angles to the support rod, we can say that the final gravitational torque tending to rotate the balance is of magnitude  $F_{\text{grav}}L$ , where L is the length of the support rod. As the support wire twists in response to the gravitational torque it will exert a restoring torque that tends to untwist the wire. In this sort of experiment the magnitude  $\Gamma$  of the restoring torque  $\Im$  will be proportional to the angle of rotation of the wire  $\theta$ , so the quantity  $c = \Gamma/\theta$  will be a constant and will be a characteristic property of the wire. The angle at which the balance eventually comes to rest will be such that the magnitudes of the gravitational torque and the restoring torque are equal, so it will be given by

$$F_{\rm grav}L = c\theta$$

(2)

If the mass of each small sphere is  $m_1$ , the mass of each large sphere  $m_2$ , and the distance between the centres of the small and large spheres r, then we can use Equation 1b

$$|\boldsymbol{F}_{\text{grav}}| = F_{\text{grav}} = \frac{Gm_1m_2}{r^2}$$
(Eqn 1b)

to calculate the value of  $F_{\text{grav}}$  in this case  $\underline{}$ , giving

$$\frac{Gm_1m_2}{r^2}L = c\theta \tag{3}$$

Now,  $m_1, m_2, L$  and c are determined by the equipment,  $\theta$  can be measured and r (the final distance between the centre of either of the small spheres and the centre of its neighbouring large sphere) can be determined by simple geometry. Thus, using Equation 3, we can determine G from the experiment.

## Question T1

What practical steps could you take to make  $\theta$  as large as possible?  $\Box$ 

Modern accurate determinations of *G*, while based on the same principle of measurement, require clever modifications and extremely elaborate precautions to reduce sources of error such as vibration.



# **3** Gravity and the Earth

# 3.1 Gravitational forces at the Earth's surface

*Study comment* All the main results in this subsection are also discussed in greater detail elsewhere in the fields block. See the *Glossary* for details.

Let us return to our apple on the surface of the Earth. We have seen how, using Newton's theorem, we might calculate the force of attraction between the apple and the Earth, but according to Newton's law of universal gravitation each object in the Universe is simultaneously attracted by all other objects in the Universe. How will other nearby objects affect the apple?

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Estimate the magnitudes of (a) the gravitational force between an apple of mass 300 g and a spherical boulder of mass  $10^3$  kg when their centres are separated by 10 m, and (b) the gravitational force between the apple and the Earth.



The gravitational force due to the Earth is  $10^{10}$  times larger than that due to the boulder, even though the centre of the Earth is much more distant — the Earth, though, has much the greater mass. If it is large masses which dominate the gravitational attraction, then shouldn't the Moon and the Sun have considerable influence too?

• Compare the magnitudes of (a) the gravitational forces on the apple due to the Moon and the Earth, and

(b) the magnitudes of the forces on the apple due to the Sun and the Earth.  $\underline{}^{\underline{}}$ 

So the gravitational force due to the Moon is  $3 \times 10^5$  times *weaker* than that due to the Earth and even that due to the Sun is nearly 2000 times smaller than that due to the Earth despite the Sun's being nearly  $10^6$  times more massive. Although every object on the surface of the Earth feels a gravitational force that is the vector sum of an infinite number of forces from the whole Universe, in practically all everyday situations the only significant gravitational force on a small object near the Earth's surface is that due to the Earth itself.

# 3.2 The Earth's gravity close to its surface

In everyday life — and in laboratory experiments — we often need to deal with the effects of the Earth's gravity close to its surface. In this subsection we look at various ways of describing these effects and, in doing so, summarize some concepts and definitions connected with gravitation.

## The gravitational field and acceleration due to gravity

One way to describe the gravitational effect of the Earth is via its gravitational field. The gravitational field at any point is a vector quantity which provides a quantitative measure of the gravitational force that would act on a point test mass *m* located at that point. (By a *test mass* we mean a mass sufficiently small that it will not significantly disturb the gravitational field we are using it to measure.) If the point we are interested in is defined by a position vector  $\mathbf{r}$ , and if the gravitational force on a test mass *m* located at that point is  $\mathbf{F}_{grav}(\text{on }m \text{ at }\mathbf{r})$ , then we can define the gravitational field  $\mathbf{g}(\mathbf{r})$  at that point by

$$\boldsymbol{g}(\boldsymbol{r}) = \frac{\boldsymbol{F}_{\text{grav}}(\text{on } m \text{ at } \boldsymbol{r})}{m}$$
(4)

In other words, the gravitational field at any point is equal to the gravitational force per unit mass that would act on a point test mass at that point.

Treating the Earth as a spherically symmetric body we can use Equations 1a and 4

$$\boldsymbol{F}_{\text{grav}} = \boldsymbol{F}_{21} = -\frac{Gm_1m_2}{r^2}\,\hat{\boldsymbol{r}}$$
(Eqn 1a)  
$$\boldsymbol{g}(\boldsymbol{r}) = \frac{\boldsymbol{F}_{\text{grav}}\left(\text{on } m \text{ at } \boldsymbol{r}\right)}{m}$$
(Eqn 4)

to write down an expression for its gravitational field. In this case, the assumed spherical symmetry of the Earth will give rise to a gravitational field which is always directed towards the centre of the Earth and which has a magnitude that only varies with the distance from the centre of the Earth. If we take the centre of the Earth as our origin of coordinates, any point with position vector  $\mathbf{r}$  will be at a distance  $r = |\mathbf{r}|$  from the centre and we can write the gravitational field as

$$\boldsymbol{g}(\boldsymbol{r}) = -\frac{GM_{\rm E}}{r^2}\,\hat{\boldsymbol{r}} \tag{5}$$

where  $M_E$  is the mass of the Earth,  $\hat{\boldsymbol{r}} = \boldsymbol{r}/|\boldsymbol{r}|$  is a unit vector parallel to  $\boldsymbol{r}$ , and the minus sign indicates that the field, like the force, points in the opposite direction to  $\hat{\boldsymbol{r}}$ , i.e. towards the centre of the Earth.

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It follows from Equation 5

$$\boldsymbol{g}(\boldsymbol{r}) = -\frac{GM_{\rm E}}{r^2}\,\hat{\boldsymbol{r}} \tag{Eqn 5}$$

that the magnitude of the field at the surface of the Earth, where  $r = R_E$ , will be

$$g(R_{\rm E}) = \frac{GM_{\rm E}}{R_{\rm E}^2} \tag{6}$$

♦ What is the surface field strength predicted by Equation 6? Take care to include appropriate units of measurement.

Now imagine a small object somewhere in the Earth's gravitational field subject only to gravitational forces. Such an object is said to be in <u>free fall</u>. Since there is an unbalanced force acting on the object it will accelerate and, according to Newton's second law of motion, its acceleration must be equal to  $\mathbf{F}_{grav}/m$ . However this is just the quantity that defines the gravitational field. We can therefore identify the free fall acceleration of the object with the gravitational field at its location.

At the surface of the Earth, the free fall acceleration is usually referred to as the <u>acceleration due to gravity</u>, or the <u>surface gravity</u>, and its magnitude is denoted by g. It follows from Equation 6

$$g(R_{\rm E}) = \frac{GM_{\rm E}}{R_{\rm E}^2} \tag{Eqn 6}$$

that if the Earth was accurately spherically symmetric we would find that

$$g = \frac{GM_{\rm E}}{R_{\rm E}^2} \tag{7}$$

• Confirm for yourself that the units of acceleration (m s<sup>-2</sup>) are identical to those of gravitational field strength (N kg<sup>-1</sup>).

Of course, the Earth is not really spherically symmetric, so measured values of g vary from place to place, though departures from the value predicted by Equation 7 are never very great. Accurate measurements of g can be based on observations of the time t that a freely falling object, starting from rest, takes to cover a known distance s (since  $s = g t^2/2$ ), or on the period T of a freely swinging simple pendulum of length l (since  $T = 2\pi\sqrt{l/g}$ ).

 $g/m \, s^{-2} = 9.806\,16 - 0.025\,928\cos(2\lambda) + 0.000\,069\cos^2(2\lambda) - 0.000\,003h$ 

The sensitivity of g to local geology and the experimentally precise methods available make *gravimetric* surveying (the measurement of g over some specified area) an important technique in geophysics. Apart from being used to seek out various kinds of mineral deposit and other geological structures, short-term variations in g around volcanoes have even been used to assist in the prediction of eruptions. Despite the practical importance of variations in g we will ignore them throughout the rest of this module, simply using  $g = 9.81 \text{ m s}^{-2}$  whenever we need a numerical value.

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Now that we have a value for the magnitude of the field at all points on the Earth's surface, we can say that the surface gravitational field is

 $g = -(9.81 \,\mathrm{m\,s^{-2}})\hat{r}$ 

In practice, we are often concerned with relatively restricted parts of the Earth's surface that are sufficiently small to be considered a flat plane. In such cases the unit vector  $-\hat{r}$ , directed towards the centre of the Earth, may be replaced by a 'downward' directed unit vector  $-\hat{z}$  perpendicular to the flat plane, giving

 $\boldsymbol{g} = -(9.81\,\mathrm{m\,s^{-2}})\hat{\boldsymbol{z}}$ 

The ideas of gravitational field and acceleration due to gravity are closely related to that of <u>weight</u>. At the surface of the Earth, the weight of an object is usually taken to mean the gravitational force that acts on it, in which case we may write

W = mg

(8)

where *m* is the mass of the object and *g* is the acceleration due to gravity. Some authors prefer to modify this slightly by allowing for the reduction in the effective value of *g* caused by the Earth's rotation. We will have more to say about the significance of such a modification in the next subsection. Equation 8 enables us to find the gravitational force on an object at the Earth's surface without explicitly using values of  $M_{\rm E}$ ,  $R_{\rm E}$  or *G*.

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## **Question T2**

What is the weight of a 60 kg man on the surface of the Earth?  $\Box$ 

Knowing g and G it is also possible to calculate the mass of the Earth and hence its average density — it is often said that Cavendish's determination of G made it possible to 'weigh the Earth'. (There is some truth in this, but Newton had previously estimated a value for  $M_E$  that was within 0.5% of the currently accepted value! Newton used his estimate together with measured values of g and  $R_E$  to determine G.)

### **Question T3**

Derive an approximate expression for the Earth's average density  $\rho$  in terms of G, g and  $R_{\rm E}$ .



## The gravitational potential and gravitational potential energy

An object of mass *m* in a gravitational field  $\mathbf{g}(\mathbf{r})$  will be subject to a gravitational force  $\mathbf{F} = m\mathbf{g}(\mathbf{r})$ . If such an object is displaced from one point to another within the field, a gravitational force will act on it throughout the displacement and may consequently do *work* as a result of the displacement. In any particular region of the Earth's surface, where the gravitational field may be represented by the fixed vector  $-g\hat{\mathbf{z}}$ , the work done by the gravitational force when the object is displaced by an amount  $\Delta z$  in the  $\hat{\mathbf{z}}$  direction will be  $\Delta W = -mg\Delta z$ , where  $\Delta z$  will be positive if the object is raised, and negative if it is lowered.

• The 60 kg man (of Question T2) climbs a vertical height of 10 m. What is the work done by the gravitational force on the man as a result of the movement?

As a result of the work done when an object moves through a gravitational field, there may be a corresponding change in the **gravitational potential energy** that the object possesses by virtue of its position. In general the change in the gravitational potential energy will be equal to *minus* the work done by the *gravitational force*, since the gravitational potential energy is a measure of the ability of the object to do work by virtue of its position.

For an object of mass m close to the surface it follows that

 $\Delta E_{\text{grav}} = mg\Delta z$  for  $\Delta z \ll R_{\text{E}}$ 

♦ What is the change in the gravitational potential energy of the 60 kg person as a result of his 10 m climb?\_

#### **Question T4**

Calculate the changes in gravitational potential energy when a 5.0kg bag of potatoes is (a) lifted from a supermarket floor and placed on a trolley 0.40 m above the floor, (b) wheeled 30 m across to the checkout, (c) taken in the trolley to an underground car park with a floor 6.0 m below the supermarket floor. In each case, say whether  $E_{\text{grav}}$  increases or decreases.

The relation  $\Delta E_{\text{grav}} = mg\Delta z$  can be used to calculate potential energy changes in situations where  $\Delta z$  is a few kilometres or less. Note that it only describes *changes* in gravitational potential energy—it says nothing about what the potential energy  $E_{\text{grav}}$  actually *is* at any point. We are therefore free to define  $E_{\text{grav}} = 0$  at any convenient reference point, e.g. at the Earth's surface, or at the floor of a laboratory. Once we have made the choice, though, the value of  $E_{\text{grav}}$  at all other points follows from the expression for  $\Delta z$ .



If we choose to measure positive z vertically upwards (as we have done so far) from ground level, and we choose some point with z = 0 to be the point at which  $E_{grav} = 0$ , we can say that at a height z above the surface of the Earth

 $E_{\rm grav}(z) = mgz$ 

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A related quantity that is often used when characterizing the Earth's gravitational field is the **gravitational potential**. Just as the gravitational field is defined as the gravitational force per unit mass, so the gravitational potential is the gravitational potential energy per unit mass. Thus, in the case of the gravitational field near the surface of the Earth, the gravitational potential is given by

$$W_{\rm grav}(z) = \frac{E_{\rm grav}(z)}{m}$$

It follows that the value of the gravitational potential close to the surface of the Earth is

 $V_{\rm grav}(z) = g z \tag{9}$ 

Note that the gravitational force on an object of mass *m* close to the Earth, and the corresponding gravitational potential energy depend on the properties of the Earth *and* the mass of the object, *m*. On the other hand, the gravitational field and the corresponding gravitational potential depend only on the Earth and are independent of the mass of the object.

## **Question T5**

A ball of mass 500 g is thrown vertically upwards with a speed of  $7 \text{ m s}^{-1}$ . If all its initial kinetic energy  $\leq 2^{-1}$  is converted into gravitational potential energy, how far does it rise? Do you actually need to know the mass of the ball in order to answer this question?



# **3.3** The Earth's gravity at large distances

We saw in the last subsection that if we treat the Earth as a spherically symmetric body of radius  $R_E$  then its gravitational field at any distance *r* is given by

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$$\boldsymbol{g}(\boldsymbol{r}) = -\frac{GM_E}{r^2}\,\hat{\boldsymbol{r}} \tag{Eqn 5}$$

provided  $r \ge R_{\rm E}$ . It follows from the definition of the gravitational field (the gravitational force per unit mass that would act on a point test mass located at r) that the gravitational force on a mass m at a point with position vector r is

$$\boldsymbol{F}_{\text{grav}}(\boldsymbol{r}) = -\frac{GM_{\text{E}}m}{r^2}\,\hat{\boldsymbol{r}}$$
(10)

If we are dealing with objects that move more than a few kilometres from the Earth's surface we must take account of changes in gravitational force with distance implied by these formulae when calculating the gravitational potential and the gravitational potential energy.

In particular, it follows from Equation 10

$$\boldsymbol{F}_{\text{grav}}(\boldsymbol{r}) = -\frac{GM_{\text{E}}m}{r^2}\,\hat{\boldsymbol{r}}$$
(Eqn 10)

that the radial component of the gravitational force at a distance r from the centre of the Earth, irrespective of direction, will be

$$F_r(r) = -\frac{GM_{\rm E}m}{r^2} \tag{11}$$

and the work done by the gravitational force when an object of mass *m* moves from a point with position vector  $\mathbf{r}_1$ , at which  $r = r_1$ , to some other point  $\mathbf{r}_2$ , at which  $r = r_2$ , will be given by the integral

$$W = \int_{r_1}^{r_2} F_r(r) dr$$
 (12)

and the corresponding change in the gravitational potential energy of the object will be

$$\Delta E_{\text{grav}} = -W = -\int_{r_1}^{r_2} F_r(r) dr$$

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So, using Equation 11,

$$F_r(r) = -\frac{GM_{\rm E}m}{r^2}$$

we can write

$$\Delta E_{\rm grav} = \int_{r_1}^{r_2} \frac{GM_{\rm E}m}{r^2} dr$$

Evaluating the integral we obtain

$$\Delta E_{\rm grav} = -GM_{\rm E}m \left(\frac{1}{r_2} - \frac{1}{r_1}\right) \tag{13}$$

When dealing with the force specified by Equation 10

$$\boldsymbol{F}_{\text{grav}}(\boldsymbol{r}) = -\frac{GM_{\text{E}}m}{r^2}\,\hat{\boldsymbol{r}}$$
(Eqn 10)

it is convenient to adopt the convention that  $E_{\text{grav}} = 0$  where  $F_r = 0$ , i.e. at an infinite distance from Earth, where  $r = \infty$ .

(Eqn 11)

The change in gravitational potential energy  $\Delta E_{\text{grav}}$  in moving an object of mass *m* from  $r_1 = \infty$  to  $r_2 = r$  will then be equal to the full value of the gravitational potential energy at *r* and we may write

$$E_{\rm grav}(r) = -\frac{GM_{\rm E}m}{r} \tag{14}$$

Note that according to the convention we have adopted in this subsection, at the surface of the Earth, where  $r = R_E$ , the gravitational potential energy is given by

$$E_{\rm grav}(R_{\rm E}) = -\frac{GM_{\rm E}m}{R_{\rm E}}$$
(15)

The fact that this is a negative quantity is of no direct significance, it is merely a consequence of our decision to let  $E_{\text{grav}} = 0$  at  $r = \infty$ . Only *differences* in gravitational potential energy are physically meaningful, and they may be positive or negative.

With the aid of Equation 14 we can work out the minimum amount of energy required to enable a space probe to escape entirely from the Earth's gravitational field. This minimum amount of energy will be equal to the change in the gravi-tational potential energy of the probe as it travels from  $r = R_E$  to  $r = \infty$   $\leq \infty$ .

So, if the probe has mass *m*, the required energy will be  $E_{\text{grav}}(\infty) - E_{\text{grav}}(R_{\text{E}})$ , but since  $E_{\text{grav}}(\infty) = 0$  the required energy is just  $-E_{\text{grav}}(R_{\text{E}}) = GM_{\text{E}}m/R_{\text{E}}$ .

This 'escape energy' as it is sometimes called can be used to compute the <u>escape speed</u> of the probe, i.e. the minimum speed at which the probe must be launched from the Earth (neglecting air resistance and any other non-gravitational effects) if it is to escape to infinity. Using the same approach as in <u>Question T5</u>, we simply equate the required escape energy with the initial kinetic energy of the probe,  $\frac{1}{2}mv_e^2$ .

Thus, 
$$\frac{1}{2}mv_{\rm e}^2 = \frac{GM_{\rm E}m}{R_{\rm E}}$$
 (16)

so, escape speed 
$$v_{\rm e} = \sqrt{\frac{2GM_{\rm E}}{R_{\rm E}}}$$
 (17)

Note that this is independent of the mass of the probe!

## **Question T6**

Evaluate the escape speed from the Earth.  $\underline{\textcircled{2}}$  Ignoring the atmosphere and any other non-gravitational influences, does the direction in which a space probe is launched have any influence on its escape speed?  $\Box$ 

If we know the escape speed from a planet's surface we can work out whether the planet is able to retain an atmosphere. The speed of molecules depends on the temperature  $\underline{\checkmark}$ . The *average* speed of hydrogen molecules at 300 K, for example, is  $1.8 \text{ km s}^{-1}$ , whereas for oxygen molecules it is  $0.4 \text{ km s}^{-1}$ . The actual molecular speeds are distributed above and below these values, so there will be some molecules of each species which are moving fast enough to escape from the Earth and, since the average speed for hydrogen is greater than that for oxygen, hydrogen molecules will escape at a much greater rate. The escape speed from the Moon is only  $2.4 \text{ km s}^{-1}$  and the rate of escape of gas molecules has been such that it has completely lost its original atmosphere.

Using Equation 14

$$E_{\rm grav}(r) = -\frac{GM_{\rm E}m}{r}$$
(Eqn 14)

we can also evaluate the Earth's gravitational potential (i.e. the gravitational potential energy per unit mass at any point in the field). According to the general definition of the gravitational potential

$$V_{\text{grav}}(\boldsymbol{r}) = \frac{E_{\text{grav}}(\text{of } m \text{ at } \boldsymbol{r})}{m}$$
(18)

In the case of the Earth our assumption of spherical symmetry ensures that the gravitational potential will vary with distance from the Earth, but will be independent of direction. We may therefore write

$$V_{\text{grav}}(\boldsymbol{r}) = \frac{E_{\text{grav}}(r)}{m}$$
  
so, from Equation 14  $V_{\text{grav}}(\boldsymbol{r}) = -\frac{GM_{\text{E}}}{r}$  (19)

Since  $V_{grav}$  depends only on the field, and not on the mass of the object under consideration, we can use it to characterize a gravitational field. As you will see in Section 5, it can sometimes be easier to work with  $V_{grav}(r)$  rather than the gravitational field g(r).
### **Question T7**

Calculate the magnitudes of the following:

(a) the gravitational force between the Earth and a satellite of mass  $m_{\rm S}$ = 4.0 × 10<sup>4</sup> kg orbiting the Earth at a distance  $R_{\rm S}$  = 4.2 × 10<sup>7</sup> m from the centre of the Earth

(b) the Earth's gravitational field strength at that distance

(c) the change in its gravitational potential energy when the satellite is put into orbit.

(*Hint*: Treat the Earth as a point mass situated at its centre.)  $\Box$ 

# 4 Gravity and orbits

By combining Newton's laws of motion with his law of universal gravitation, it is possible to work out the path that a body such as a planet will follow as it moves under the gravitational influence of another body such as the Sun. Such investigations show that these paths, or <u>orbits</u> as they are called, are members of a family of curves called <u>conic sections</u>. As Figure 4 indicates, this family includes <u>circles</u>, <u>ellipses</u>, <u>parabolas</u> and <u>hyperbolas</u>. In this section we examine some of these orbits in greater detail, beginning with simple circular orbits.

Figure 4 The conic sections obtained by slicing a double cone.



## 4.1 Orbital periods and circular orbits

Perfectly circular orbits are quite rare in nature, certainly amongst natural astronomical bodies such as planets and their moons. However, many bodies, including the Earth, have orbits that closely approximate circles. Moreover, circular orbits are easy to describe and investigate. For both of these reasons circular orbits provide a good starting point for a more general study of orbits and deserve close attention.

If a body moves in a circle of radius *r* with constant speed *v*, and takes a time *T* to complete each orbit, then we say that *T* is its <u>orbital period</u>, and its <u>angular speed</u>  $\omega$  (measured in rad s<sup>-1</sup>) is given by  $\leq 2$ 

$$\omega = \frac{2\pi \text{ radians}}{T}$$

Since the total distance travelled in time T is equal to the circumference of the circle we can say that

$$vT = 2\pi r$$
  
so  $\frac{v}{r} = \frac{2\pi}{T}$ 

and hence  $v = r\omega$ 

In order to keep the body moving in a circle it must always be subject to an acceleration towards the centre of the circle. This centrally directed acceleration is called the <u>centripetal acceleration</u> and has magnitude

$$a = r\omega^2 = v\omega = \frac{v^2}{r}$$

The centrally directed force needed to maintain this acceleration is called the <u>centripetal force</u>. If the circling body has mass *m*, the magnitude of this force will be

$$F = mr\omega^2 = mv\omega = m\frac{v^2}{r}$$

If the body is held in its circular orbit by the gravitational attraction of some other body of mass M, then the centripetal force is provided by the gravitational force on the orbiting body and we may write

$$F = F_{\text{grav}}$$
  
i.e.  $mr\omega^2 = \frac{GMm}{r^2}$  (20)

Dividing both sides by m and rearranging the equation we obtain

$$\frac{1}{\omega^2} = \frac{r^3}{GM} \tag{21}$$

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Recalling that the orbital period of the circular motion is  $T = 2\pi/\omega$ , we can write

$$T^2 = \frac{4\pi^2}{GM}r^3 \tag{22}$$

Notice that T does not depend on the mass m of the orbiting object, so the mass M can be calculated from measurements of T and r.

### **Question T8**

The radius of the Earth's orbit around the Sun is  $1.496 \times 10^{11}$  m. Calculate the mass of the Sun 2.

#### **Question T9**

Many communications satellites orbit the Earth with a period of exactly 1 day so that they can always remain in a fixed position relative to the rotating Earth. Such an orbit is known as a 'geostationary' orbit. Calculate the orbital speed of such a satellite.  $\Box$ 





# 4.2 Kepler's laws and elliptical orbits

Newton's investigations of orbital motion under gravity provided an explanation of <u>Kepler's laws of planetary motion</u>. Johannes Kepler (1571–1630) had arrived at these laws early in the 17th century by careful analysis of observational data on planetary motion collected by the Danish astronomer Tycho Brahe (1546–1601).

*Kepler's first law*: The orbits of the planets in the solar system are ellipses with the Sun at one *focus* (see Figure 5a).



**Figure 5a** Illustration of Kepler's laws. (a) A point P that moves in such a way that the sum of its distances from two fixed points  $F_1$  and  $F_2$  is constant, will follow an ellipse.  $F_1$  and  $F_2$  are the focii of the ellipse and *a* is its semimajor axis and *b* is the semi-minor axis. The planets move in elliptical orbits with the Sun at one focus.

An ellipse may be obtained by plotting all the points (x, y) that satisfy the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Such an ellipse has a <u>semi-major axis</u> of length a, and <u>foci</u>  $F_1$  and  $F_2$ 

located at the points (*ae*, 0) and (-*ae*, 0), where  $e = \frac{1}{a}\sqrt{a^2 - b^2}$  is the

<u>eccentricity</u> of the ellipse. A circle may be regarded as a limiting case of an ellipse, in which a = b and e = 0.

Of the nine planets that orbit the Sun, only two (Mercury and Pluto) have eccentricities greater than 0.10 (b/a < 0.995), all the others (especially Venus, Earth and Neptune) have very nearly circular orbits.



**Figure 5a** Illustration of Kepler's laws. (a) A point P that moves in such a way that the sum of its distances from two fixed points  $F_1$  and  $F_2$  is constant, will follow an ellipse.  $F_1$  and  $F_2$  are the focii of the ellipse and *a* is its semimajor axis and *b* is the semi-minor axis. The planets move in elliptical orbits with the Sun at one focus.

*Kepler's second law*: A radial line from the Sun to a planet sweeps out equal areas in equal intervals of time (see Figure 5b).

This law implies that a planet moves more quickly as it gets closer to the Sun. In the context of Newtonian mechanics, Kepler's second law may be viewed as a consequence of the *conservation of angular momentum*, a basic physical principle that is discussed elsewhere in *FLAP*.



(b) if  $t_2 - t_1 = t_4 - t_3$ then  $A_1 = A_2$ 

**Figure 5b** Illustration of Kepler's laws. (b) A radial line from a planet to the Sun sweeps out equal areas in equal amounts of time. *Kepler's third law*: The square of the orbital period of each planet is proportional to the cube of its semi-major axis, i.e.  $T^2 = ka^3$  (see Figure 5c).

This law is a generalization of Equation 22 to elliptical orbits.

$$T^2 = \frac{4\pi^2}{GM}r^3 \tag{Eqn 22}$$

We can therefore identify the constant of proportionality *k* with the  $4\pi^2/(GM)$  that appeared in that earlier equation, at least to the extent that we are justified in neglecting the masses of the planets relative to that of the Sun

Thus, 
$$T^2 = \left(\frac{4\pi^2}{GM}\right)a^3$$
 (23)



**Figure 5c** Illustration of Kepler's laws. (c) The square of a planet's orbital period is proportional to the cube of its semi-major axis.

Planet	Mean distance from the Sun, $a/10^6$ km	Orbital period, T/years	
Mercury	57.9	0.241	
Venus	108	0.615	
Earth	150	1.00	
Mars	228	1.88	
Jupiter	778	11.9	
Saturn	1430	29.5	
	Planet Mercury Venus Earth Mars Jupiter Saturn	PlanetMean distance from the Sun, a/106 kmMercury57.9Venus108Earth150Mars228Jupiter778Saturn1430	

**Table 1**Orbital data on the six inner planets.

(The following question may help.)

As we have already said.

If  $T^2$  is proportional to  $a^3$ , how can we present the data on a graph to produce a straight line? ♦



#### Question T10

**Table 1**Orbital data on the six inner planets.

(a) Plot a graph of <i>T</i> versus $a^{3/2}$ for the six inner planets. Do the data agree with Kepler's third law?	Planet	Mean distance from the Sun, $a/10^6$ km	Orbital period, T/years	
	Mercury	57.9	0.241	
	Venus	108	0.615	
	Earth	150	1.00	
(b) What would be the orbital period of an asteroid (a small rocky	Mars	228	1.88	
	Jupiter	778	11.9	
	Saturn	1430	29.5	

body orbiting the Sun) in an elliptical orbit with a semi-major axis of  $300 \times 10^6$  km ?



Newton was able to show that Kepler's laws agreed with the expected behaviour of two bodies interacting in the way described by the inverse square law of gravitation. This was a great success for Newton's general approach, and it was made even more impressive by the quantitative agreement that it provided with a range of astronomical measurements. However, as Newton was well aware, the planets are not isolated bodies; the motion of each is influenced by all the other bodies in the Solar System as well as by the Sun.

As you might expect, these additional interactions are relatively small, but they give rise to observable departures from the simple behaviour described by Kepler's laws. Moreover, these departures from Keplerian behaviour are, in principle at least, calculable on the basis of Newton's laws of motion and the law of gravitation.

Following the publication of Newton's *Principia* scientists and mathematicians were able to make increasingly accurate analyses of orbital motion and thereby uncovered many new phenomena. Perhaps the most notable was the discovery of Neptune, the eighth planet from the Sun. The seventh planet, Uranus, was discovered observationally by William Herschel (1738–1822) in 1781, but small unexplained irregularities in its motion suggested that it might be acted upon by some other planet yet further from the Sun. John Couch Adams (1819–1892) and Urbain Le Verrier (1811–1877), working independently in England and in France, calculated the position that such a planet might occupy, and in 1846 Neptune was found only about a degree from its predicted position!

# 4.3 Energy and parabolic orbits

Figure 6 shows the effect of launching a body horizontally from the top of a tall vertical tower at a variety of different speeds, neglecting nongravitational effects such as air resistance and treating the Earth as a point mass concentrated at its geometric centre.



As you can see, the orbits are all conic sections, the precise shapes of which are determined by the speed of launch.

If  $v_c$  is the speed required to attain a circular orbit, any lesser speed will produce an elliptical orbit with the centre of the Earth at the farthest focus of the ellipse.



Launch speeds increasingly greater than  $v_c$  will first produce elliptical orbits with the Earth's centre at the nearer focus, then (at speed  $v_p$ ) a parabolic orbit, and finally a range of hyperbolic orbits.

The different speeds shown in Figure 6 have an important influence on the energy associated with each orbit.



The gravitational field is a *conservative* field, so if we ignore all non-gravitational effects the mechanical energy of each orbit (the sum of the kinetic and potential energies at any point) will be constant. Since the bodies are being launched from the same position they all have the same potential energy at that point, so their energies differ by amounts that are determined by their respective speeds. It follows that the increasing launch speeds indicate increasing orbital energies.

• In the case of the circular orbit, write down an expression for the sum of the kinetic and potential energies of a body of mass *m*, and then use it to show that the total energy of the circling body is  $E = -\frac{GmM_E}{2r}$ , where  $M_E$  is the mass of the Earth and *r* is the distance of the launch point from the centre of the Earth.

Note that in a circular orbit the kinetic energy (which is always positive) has exactly half the magnitude of the potential energy (which is negative), and the total energy is negative. This ensures that a body in a circular path can never escape to infinity where its potential energy would be zero and its total energy non-negative.

In fact, we can see from Figure 6 that the lowest energy orbit that does permit escape to infinity is the parabolic orbit. This must be the orbit with a total energy of zero. The closed elliptical orbits must therefore all have negative total energy, and the open hyperbolic orbits must have all positive total energy.



The zero energy parabolic orbit marks an important dividing line between open and closed orbits.

• What is the speed  $v_p$  that corresponds to the parabolic orbit?





### 4.4 Weightlessness and tidal effects

Astronauts in an orbiting space shuttle float around the vehicle and feel 'weightless', even though they are close to the Earth and must therefore be affected by its gravitational field. How can this be?

In Subsection 3.2 we noted that measurements made by observers located at fixed points on the rotating Earth have to allow for a reduction in the effective value of g, the acceleration due to gravity. This apparent reduction is actually caused by the rotation of the observers and their experimental equipment. A similar effect is at work in an orbiting spacecraft. The crucial point is that both the observers on the turning Earth and the observers in an orbiting spacecraft are *accelerating*. Such observers, when using Newton's laws of motion, must always make allowances for the effects of their own acceleration. In the case of observers on the Earth these effects are small and can often be ignored, but for orbiting observers the effects are much larger. The orbital motion has no effect on the mass of an object, and in a low Earth orbit, 100 to 150 km above the Earth's surface, the gravitational force on an object is very similar to its surface weight, but the *effective* weight, as observed by the occupants of the spacecraft, will be zero. It is in this sense that orbiting objects may be said to be <u>weightless</u>, though it would probably be better to describe them as being in 'free fall' under gravity.

In reality, orbiting astronauts are not quite in free fall and they can, in principle at least, tell that they are in the Earth's gravitational field even without looking out of the window. In the above discussion we treated the spacecraft as though it were a point mass orbiting at a precise distance from the Earth, but it is actually extended and so some parts are closer to the Earth than others. The Earth's gravitational field decreases with distance so the parts of the vehicle nearest the Earth will be in a slightly stronger gravitational field than those parts that are further away. Within the vehicle, small objects (e.g. dust grains) at the centre of the craft will be subjected to the gravitational force needed to keep them in the orbit they are following, but grains on the Earthward side of the craft will experience a slightly stronger field than is necessary to keep them in their somewhat smaller 'orbit', and grains that are on the side of the ship away from the Earth will experience a weaker force than necessary. The result of this is that grains on the Earthward side tend to move towards the Earth while grains on the away side move away. The situation of three such grains forced to follow circular paths of different radii at a common



**Figure 7** Dust grains in an orbiting space vehicle. The grains are assumed to be moving at a common angular speed, but are at differing distances from the centre of the Earth.

angular speed is shown in Figure 7. The central grain is held in a true circular orbit by the gravitational field that acts upon it, but the circular paths of the other two grains are not true orbits since the gravitational force that acts on them is either too great or too small to provide the necessary centripetal force.

An astronaut would be able to deduce, from the behaviour of the dust, that his or her vehicle was in a gravitational field.

You may be thinking that the movement of dust in a space shuttle is a very esoteric effect, but the tides commonly observed in the Earth's oceans have a very similar cause. Ocean tides are complicated by factors such as local geography and ocean depth, but their basic cause is the variation in the strength of the gravitational fields of the Moon and the Sun across the diameter of the Earth. Ocean water is not rigidly attached to the Earth, so it can move towards or away from these bodies just as the dust grains in a space shuttle can move towards or away from the Earth (Figure 8). The gravitational fields (together with geographical factors) determine the 'shape' of the oceans, and the solid Earth rotates within that deformed body of water, causing the high tides and low tides that are actually observed.

### **Ouestion T11**

Suggest a reason why the Moon has a greater influence on the Earth's ocean tides than does the Sun (despite the fact that the gravitational field due to the Sun is stronger, at the Earth, than that due to the Moon).  $\Box$ 





# 5 Gravity and extended bodies

Throughout this module, we have used the fact that the gravitational effects outside a spherically symmetric body are identical to those of a single particle, of the same mass located at its centre. In this final section, we will derive that result and also consider the gravitational field inside a spherical body.

*Study comment* This section is more mathematically demanding than the rest of the module. The results that it contains are an important part of the module but the techniques used to obtain those results are not the subject of any the achievements listed at the end of the module. Do not allow any difficulties you may have following the detailed arguments to cause unnecessary delays. If you encounter problems, consult your tutor.

# 5.1 Approaching the task

What we are trying to find is an expression for the gravitational field at some point P outside a spherical body (Figure 9). One way of approaching the problem would be to treat the whole sphere as a collection of tiny elements, each with its own individual mass  $\Delta m_i$ . We could then work out the gravitational field at P due to each tiny element and add them all together using *vector* addition.



**Figure 9** A sphere of radius R with a point P at a distance r from its centre.

However, this calculation gets rather cumbersome, so instead of working with gravitational fields and getting involved in vector additions, it is easier to work in terms of gravitational potentials since these are *scalar* quantities. The resultant potential at any point due to a collection of particles is simply the *algebraic* sum of all the contributions. The gravitational field at P may be determined fairly straightforwardly once we have an expression for the gravitational potential, so this is the approach we will adopt.

The problem of finding the gravitational potential due to a general spherically symmetric body is still quite demanding, so we will start by considering the potential due to a thin spherical shell of uniform composition.

# 5.2 Gravitational field due to a thin spherical shell

A spherical shell can be divided into narrow rings in such a way that all the points on each ring are effectively at the same distance from P. This is indicated in Figure 10, where all the points on the ring through X and Y are at a distance s from P. The radius of the ring, NX, is  $R \sin \alpha$  and the width of the ring, measured along the surface of the shell, is  $R\Delta\alpha$ . It follows that the area of the ring is  $(2\pi R \sin \alpha) \times$  $(R\Delta\alpha)$ .  $\leq$  Consequently, if the mass per unit area of the shell's surface is  $\sigma$ ,  $\leq \sigma$  we can say that the mass of the ring will be  $2\pi\sigma R^2 \sin(\alpha)\Delta\alpha$ . Since all the points on the ring are at the same distance from P it then follows from our usual expression for the gravitational potential due to a point mass that the gravitational potential produced at P by the whole ring will be

$$\Delta V = -\frac{G2\pi\sigma R^2 \sin\left(\alpha\right)\Delta\alpha}{s}$$



**Figure 10** A narrow ring passing through the points XY on the surface of a thin spherical shell of radius *R*. The width of the ring subtends a small angle  $\Delta \alpha$  at the centre of the sphere.

(24)

In principle we can use this expression to compare the potential due to any two rings, but doing so is complicated because any other ring will correspond to different values of the two related variables  $\alpha$  and s. In order to simplify the expression we need to eliminate one or other of these variables. To do so we can use the *cosine rule* which tells us that

 $s^2 = R^2 + r^2 - 2Rr\cos\alpha$ 

If this is then differentiated (implicitly)  $\leq s$  we obtain the rate of change of s with respect to  $\alpha$ 

$$2s\frac{ds}{d\alpha} = 2Rr\sin\alpha$$
  
so 
$$\frac{ds}{d\alpha} = \frac{Rr\sin\alpha}{s}$$

and it therefore follows from this that the increase in s that corresponds to a small increase  $\Delta \alpha$  in  $\alpha$  is

$$\Delta s = \frac{ds}{d\alpha} \Delta \alpha = \frac{Rr\sin(\alpha)\Delta\alpha}{s}$$

Substituting th

Substituting this 
$$\Delta s = \frac{ds}{d\alpha} \Delta \alpha = \frac{Rr \sin(\alpha)\Delta\alpha}{s}$$
into Equation 24 
$$\Delta V = -\frac{G2\pi\sigma R^2 \sin(\alpha)\Delta\alpha}{s}$$
(Eqn 24)

we find

$$\Delta V = -\frac{2G\pi\sigma R\Delta s}{r} \tag{25}$$

The material of the sphere provides rings at all values of s between r - R and r + R. We can add their contributions together to find the total gravitational potential V(r) at P by performing the following integral

$$V(r) = -\frac{2G\sigma\pi R}{r} \int_{r-R}^{r+R} ds$$

$$V(r) = -\frac{2G\sigma\pi R}{r} [s]_{r-R}^{r+R} = -\frac{2G\sigma\pi R}{r} [(r+R) - (r-R)]$$

$$V(r) = -\frac{2G\sigma\pi R}{r} 2R = -\frac{4\pi R^2 \sigma G}{r}$$
(26)

FLAP P3.2 Gravitation and orbits COPYRIGHT © 1998 THE OPEN UNIVERSITY \$570 V1.1 Now, the surface area of the shell is  $4\pi R^2$ , so its total mass *M* is given by,

$$M = 4\pi R^2 \sigma \tag{27}$$

Hence, the gravitational potential due to a uniform spherical shell of mass M at an external point at a distance r from the shell's centre will be

$$V(r) = -\frac{GM}{r} \tag{28}$$

Note that this is independent of the radius of the shell.

You should recognize this expression — it is identical to the expression for the gravitational potential at a distance r from a point mass M. We therefore come to the perhaps surprising conclusion that:

The gravitational effect of a hollow sphere whose mass M is uniformly spread over its surface is the *same* as if all its mass were concentrated in a point at the centre of the sphere.

This important result means we can use the expression for the gravitational field g(r) at a distance r from a point mass (see Equation 5)

$$\boldsymbol{g}(\boldsymbol{r}) = -\frac{GM_{\rm E}}{r^2}\,\hat{\boldsymbol{r}} \tag{Eqn 5}$$

to represent the gravitational field outside a uniform spherical shell. At a point with position vector  $\boldsymbol{r}$ 

$$\boldsymbol{g}(\boldsymbol{r}) = -\frac{GM}{r^2}\hat{\boldsymbol{r}}$$
(29)

We now know about gravitational effects outside a hollow sphere - but what happens inside?

• Can you deduce what the field will be exactly at the centre of the shell, point O?



Suppose we now move off centre to a point Q, to the right of O (see Figure 11). Now although there's more mass on the left-hand side of Q, it is generally further away than the mass on the right-hand side, so again the forces cancel and the field at Q is zero. It is possible to show, by summing the contributions from each mass element, that:

There is no	gravitational	field	inside	any	spherically	symmetric	hollow
sphere.	<u> </u>						

This does not mean that the shell shields the space inside it from gravitational fields due to other masses; it means simply that its own field is zero, i.e. a mass inside the shell doesn't feel any net force due to the mass of the surrounding shell.

### **Question T12**



**Figure 11** Point Q inside a uniform solid sphere, distance *r* from the centre O.

Sketch the variation of the gravitational field due to a shell of radius R as a function of the distance r from the centre O of the shell. Include distances r both less than and greater than R.

### **5.3** Gravitational field due to a solid sphere

We are now in a position to return to a solid sphere, which we can think of as a series of shells one inside the other, like the layers of an onion. For each, the field at a point P outside the sphere is as though the shell's mass were at the centre O.

The gravitational effect of any spherically symmetric body of total mass M, at any point outside that body, is identical to the effect of a point mass Mlocated at the geometric centre of the sphere. æ

R Figure 11 Point Q inside a uniform solid sphere, distance r

from the centre O.

What is the gravitational field *inside* a uniform solid sphere? We will assume that the sphere is of uniform density, and think about point Q in Figure 11. Point Q is inside all the shells making up the solid sphere with radii between r

and R so there will be no effect due to these shells. For the mass inside Q, i.e. within a sphere of radius r, its effect is as though it were all concentrated at the centre. Since the density of the sphere is uniform, the mass m of the sphere of radius *r* is given by

$$m = \frac{Mr^3}{R^3}$$

(30)



It follows that the field at Q is the field at a point r, a distance r from a point mass m:

i.e. 
$$\boldsymbol{g}(\boldsymbol{r}) = -\frac{Gm}{r^2}\hat{\boldsymbol{r}}$$

so 
$$\boldsymbol{g}(\boldsymbol{r}) = -\frac{G}{r^2} \left(\frac{Mr^3}{R^3}\right) \hat{\boldsymbol{r}}$$

i.e. inside a uniform solid sphere

$$\boldsymbol{g}(\boldsymbol{r}) = -\frac{GMr}{R^3}\,\hat{\boldsymbol{r}} \tag{31}$$



**Figure 11** Point Q inside a uniform solid sphere, distance *r* from the centre O.

### **Question T13**

Sketch a graph showing the magnitude of the gravitational field as a function of distance from the centre of a uniform solid sphere of radius R for points inside and outside the sphere.  $\Box$ 

In this section we have only considered one simple solid object — a sphere. However, in principle, we can find the gravitational effects of an object of *any* shape by dividing it up into pieces that are simpler to deal with, then summing their contributions to the total gravitational potential. The maths can sometimes get very complicated, but the principle is always the same. In the case of the sphere we were able to use its particular symmetry to deduce the field that corresponded to the potential. In general there will be no such symmetry, but it is always the case that the field can be deduced from the potential, generally by investigating its rate of change in various directions.

#### Study comment

If you have studied Section 5 before completing Sections 2 to 4 then you should return now to Section 2.2. Otherwise you can now move on to Closing items.





# 6 Closing items

## 6.1 Module summary

1 According to the law of universal gravitation, the gravitational force  $\mathbf{F}_{grav}$  on a particle of mass  $m_2$  due to a particle of mass  $m_1$  separated from it by a distance r is given by

$$\boldsymbol{F}_{\text{grav}} = \boldsymbol{F}_{21} = -\frac{Gm_1m_2}{r^2}\,\hat{\boldsymbol{r}}$$
(Eqn 1)

where  $G = 6.673 \times 10^{-11} \text{ N m}^2 \text{ kg}^{-2}$  is the <u>universal gravitational constant</u>, and  $\hat{\boldsymbol{r}}$  is a <u>unit vector</u> directed from  $m_1$  to  $m_2$ .

- 2 According to <u>Newton's theorem</u>: the gravitational effect of any <u>spherically symmetric</u> body, outside its own surface, can be exactly reproduced by a single particle, with the same mass as that body, located at the body's geometric centre.
- 3 The *gravitational field* at a point r is the gravitational force per unit mass that would act on a point test mass at that point

$$\boldsymbol{g}(\boldsymbol{r}) = \frac{\boldsymbol{F}_{\text{grav}}(\text{on } m \text{ at } \boldsymbol{r})}{m}$$

(Eqn 4)

4 The gravitational field of a spherically symmetric Earth at an external point a distance r from its centre would be

$$\boldsymbol{g}(\boldsymbol{r}) = -\frac{GM_{\rm E}}{r^2}\,\hat{\boldsymbol{r}} \tag{Eqn 5}$$

where  $M_{\rm E}$  is the mass of the Earth. Over a localized region close to the surface of the Earth the field may be approximated by

$$\boldsymbol{g} = -g\hat{\boldsymbol{z}}$$

where g is the local magnitude of the <u>acceleration due to gravity</u> (9.81 m s<sup>-2</sup> in the UK). For the spherically symmetric Earth

$$g = \frac{GM_{\rm E}}{R_{\rm E}^2} \tag{Eqn 7}$$

- 5 The gravitational potential energy of a body is the energy that it possess by virtue of its position in a gravitational field.
- 6 The gravitational potential energy of a body of mass m at an external point a distance r from the centre of a spherically symmetric Earth would be

$$E_{\text{grav}}(r) = -\frac{GM_{\text{E}}m}{r} \quad (\text{with } E_{\text{grav}} = 0 \text{ at } r = \infty)$$
 (Eqn 14)

Over a localized region close to the surface of the Earth this may be approximated by

$$E_{\text{grav}}(z) = mgz$$
 (with  $E_{\text{grav}} = 0$  at  $z = 0$ )

7 The <u>gravitational potential</u> at a point r is the gravitational potential energy per unit mass at that point.

$$V_{\text{grav}}(\boldsymbol{r}) = \frac{E_{\text{grav}}(\text{of } m \text{ at } \boldsymbol{r})}{m}$$
(Eqn 18)

so for a spherically symmetric Earth

$$V_{\rm grav}(\boldsymbol{r}) = -\frac{GM_{\rm E}}{r} \tag{Eqn 19}$$

and over a localized part of the surface

$$V_{\rm grav}(z) = g z \tag{Eqn 9}$$

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- 8 The <u>escape speed</u> for a planet (or satellite) is the least speed at which a body must be launched from its surface in order to escape to an infinite distance. The body can only escape if its initial kinetic energy is at least as great as the increase in gravitational potential energy associated with moving from the surface to an infinite distance.
- 9 When a body moves about a fixed point under the influence of a force that satisfies an inverse square law (such as the gravitational force due to a much more massive body) its <u>orbit</u> is a <u>conic section</u>; an <u>ellipse</u>, a <u>circle</u>, a <u>parabola</u> or a <u>hyperbola</u>, according to the energy of the body.
- 10 According to <u>*Kepler's laws of planetary motion*</u>:
  - (1) Planets orbit the Sun in ellipses with the Sun at one *focus*.
  - (2) A radial line from the Sun to a planet sweeps out equal areas in equal intervals of time.

(3) The square of the orbital period T of each planet is proportional to the cube of the semi-major axis a of the ellipse.

- 11 Astronauts in an orbiting vehicle experience *weightlessness* because they, along with the contents of their space vehicle, are falling freely under gravity.
- 12 Ocean *tides* arise because of the difference in strength of the gravitational fields of the Moon and of the Sun across the Earth's diameter.
- 13 The gravitational field inside a spherically symmetric hollow sphere is zero.
# 6.2 Achievements

Having completed this module, you should be able to:

- A1 Define the terms that are emboldened and flagged in the margins of the module.
- A2 Describe and explain the principles of the Cavendish experiment for measuring the universal gravitational constant *G*.
- A3 Calculate the surface gravity of a spherically symmetric body (such as a planet), and determine the gravitational field of that body.
- A4 Calculate the change in potential energy when an object is raised a small distance from the surface of the Earth.
- A5 Calculate the gravitational potential energy of an object due to a spherically symmetric body (such as a planet), and determine the gravitational potential of that body.
- A6 Apply the principle of conservation of energy to a body moving in a gravitational field, and relate the shape of an orbit to its energy in simple cases.
- A7 Explain what is meant by escape speed, and calculate the escape speed from the surface of a planet or satellite.

- A8 Describe the orbital motion of planets around the Sun and of satellites around their 'parent' body and apply the equations of uniform circular motion to planetary motion.
- A9 Quote the general relationship between the orbital period and the mean radius of the orbit (Kepler's third law).
- A10 Explain why astronauts in an orbiting vehicle experience weightlessness.
- A11 Explain the origin of the Earth's ocean tides.
- A12 Calculate (not derive) the gravitational force between two bodies, if the bodies are point particles, hollow spheres or solid spheres.
- A13 Calculate (not derive) the gravitational field due to a uniform sphere both inside and outside its surface, and state the value of the gravitational field produced inside a hollow spherically symmetric sphere.

*Study comment* You may now wish to take the *Exit test* for this module which tests these Achievements. If you prefer to study the module further before taking this test then return to the *Module contents* to review some of the topics.

# 6.3 Exit test

*Study comment* Having completed this module, you should be able to answer the following questions, each of which tests one or more of the *Achievements*.

## **Question E1**

(A2 and A12) In a Cavendish torsion balance experiment to measure G the distance between the centres of the large and small spheres is reduced from 100 mm to 70 mm. What effect does this have on the angle of rotation of the torsion fibre?



### **Question E2**

(A3 and A8) A small satellite of Saturn has a mass of  $3.8 \times 10^{19}$  kg and a diameter of 500 km. Calculate the maximum angular speed with which it can rotate if loose rocks sitting on its surface at its equator are not to fly off.

# **Question E3**

(A4, A5 and A6) Calculate the gravitational potential energy of a 5 kg mass on the surface of the Earth, taking the zero of potential energy to be at infinity and compare this with the change in potential energy when the mass is lifted a height of 500 m above the surface of the Earth.

What would be the minimum launch speed of a missile required to reach a height of 500 m above the Earth's surface?

## **Question E4**

(A8 and A9) What orbital period does Kepler's third law predict for the planet Uranus, whose mean orbital radius (semi-major axis) is 19.14 times the Earth's orbital radius?



# **Question E5**

(*A11*) Why is the tidal range (the difference between the depth of water at high and low tides) greatest when the Moon is full or new, and least when the Moon is mid way between these phases?

### **Question E6**

(A13) The gravitational field at the surface of a uniform solid sphere of mass M, radius R is -g. What is the value of the field at a distance X below the surface of the sphere if X is equal to R/10? *Study comment* This is the final *Exit test* question. When you have completed the *Exit test* go back to Subsection 1.2 and try the *Fast track questions* if you have not already done so.

If you have completed **both** the *Fast track questions* and the *Exit test*, then you have finished the module and may leave it here.

