Module P4.1 DC circuits and currents

Exit module

1 Opening items

1.1 Module introduction

The development of electrical and electronic products in our society requires a sophisticated knowledge of circuits and devices: indeed some knowledge of circuitry is called for in the most elementary maintenance of plugs, fuses and connecting wires. Many measurements of physical quantities are made nowadays by sensors or transducers that result in current or voltage signals being developed in a detection circuit. Moreover, the electrical properties of materials give important clues to the underlying nature of the materials themselves. For all these reasons, the behaviour of electrical circuits is an important area of physics.

In this module we will be concerned only with *d.c. circuits*. We begin in Section 2 with a discussion of *current* in *conductors* and the factors restricting the current, which lead to the definition of *resistance* and *Ohm's law*. It is shown how the resistance of a material sample depends on its dimensions, its *resistivity* (or *conductivity*) and its temperature. This is interwoven with discussions of *electric potential energy* and *voltage*, electrical heating and *power*. Important laws of circuitry (*superposition* and *Kirchhoff's laws*) are introduced in Section 3 and used to analyse a simple *Wheatstone bridge circuit*. In Section 4 the technique of *equivalent circuits* is applied to *resistors in series* and *in parallel*, and to *voltage generators* with *output resistance*. Finally, *Thévenin's theorem* is used to show how a bridge circuit may be employed to monitor changes in resistance.

This module aims to help you to think about the laws of electrical circuitry in ways that will continue to be valid when you study more sophisticated circuits.

Study comment Having read the introduction you may feel that you are already familiar with the material covered by this module and that you do not need to study it. If so, try the *Fast track questions* given in Subsection 1.2. If not, proceed directly to *Ready to study?* in Subsection 1.3.

1.2 Fast track questions

Study comment Can you answer the following *Fast track questions*?. If you answer the questions successfully you need only glance through the module before looking at the *Module summary* (Subsection 5.1) and the *Achievements* listed in Subsection 5.2. If you are sure that you can meet each of these achievements, try the *Exit test* in Subsection 5.3. If you have difficulty with only one or two of the questions you should follow the guidance given in the answers and read the relevant parts of the module. However, *if you have difficulty with more than two of the Exit questions you are strongly advised to study the whole module*.

Question F1

For the circuit of Figure 1, (a) use the principle of superposition to find the current in the $6\,\Omega$ resistor and (b) find the power dissipated in the $10\,\Omega$ resistor.

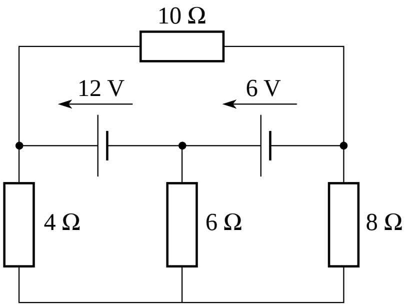


Figure 1 See Question F1.

Question F2

A moving-coil galvanometer has a resistance of $50\,\Omega$ and gives a full-scale deflection for a current of $250\,\mu$A. Calculate the values of the shunt resistance or the series resistance to convert it into (a) an ammeter to measure currents up to 5 A and (b) a voltmeter to measure potential differences up to 100 V.

Study comment

Having seen the *Fast track questions* you may feel that it would be wiser to follow the normal route through the module and to proceed directly to *Ready to study?* in Subsection 1.3.

Alternatively, you may still be sufficiently comfortable with the material covered by the module to proceed directly to the *Closing items*.

1.3 Ready to study?

Study comment To begin the study of this module you will need to understand the following terms: *charge*, *conservation of energy*, *electric field*, *electron*, *electrostatic force*, *ion*, *kinetic energy* and *potential energy*. You will also need to know the everyday meanings of the terms *gravity*, *temperature* and *weight*. You should be familiar with the following mathematical terms: *average* (i.e. *mean*), *constant of proportionality*, *fraction*, *gradient* (of a *graph*), *inversely proportional*, *percentage*, *proportional*, *ratio*, *reciprocal* and *sum*. If you are uncertain about any of these terms then you can review them now by reference to the *Glossary*, which also indicates where in *FLAP* they are developed. The module also assumes that you can *evaluate* simple *algebraic expressions*, *rearrange* simple *equations* and *solve* elementary *simultaneous equations*. *Calculus notation* is used to summarize one definition, but you do not need to be familiar with the techniques of calculus in order to study this module.

The following *Ready to study questions* will help you to establish whether you need to review some of the above topics before embarking on this module.

Question R1

The charge on an electron is $-e = -1.602 \times 10^{-19}$ coulomb.

(a) An atom becomes ionized by losing two of its electrons. What is the charge of the resulting ion?

(b) How many electrons are required to make up a charge of $-1.000\,\mathrm{C}$?

Question R2

An electron is released from rest between two oppositely charged metal plates (see, for example, Figure 4) in the absence of any non-electrical forces.

(a) Which way will the electron move?

(b) Where would the electron have the greatest potential energy?

Question R3

Suppose $a \propto b$ and $c \propto 1/a$:

(a) Write down a single proportionality involving a, b and c with a as the subject.

(b) How could you use a graph to find the constant of proportionality relating values of a and c?

Question R4

Solve the following simultaneous equations:

$$x + y = z \qquad 2x + 3y = 2z - 1 \qquad 3x - y = z + 1$$

2 Current, voltage and resistance

2.1 Current

Electric **current** is the rate at which charge is transferred and is generally represented by the symbol I. The SI unit of current is the **ampere** (often abbreviated to amp) which has the unit symbol A. 1 ampere represents a charge flow of 1 coulomb per second, so $1\,\mathrm{A} = 1\,\mathrm{C\,s^{-1}}$. ☞

Thus, in Figure 2, if a net charge q ☞ is transferred from a point A to a point B in a time t, the *average current* $\langle I \rangle$ from A to B is given by

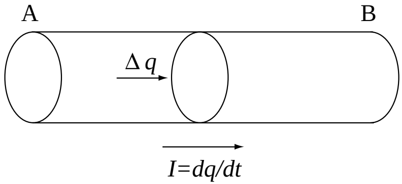


Figure 2 The current along a wire is defined by the net rate at which charge passes through a plane perpendicular to the axis of the wire.

$$\text{average current} = \frac{\text{net charge transferred}}{\text{time taken}}$$

i.e. $$\langle I \rangle = q/t \tag{1}$$

For a current that varies with time we can define an *instantaneous* value of the current. For charges moving along a wire this is the rate (at a given instant of time) at which charge passes through a plane perpendicular to the axis of the wire. If a net charge Δq ☞ crosses such a plane (see Figure 2) in a time interval Δt, the *instantaneous current* from A to B may be expressed using the notation of calculus as

$$\text{instantaneous current} = \lim_{\Delta t \to 0}\left(\frac{\Delta q}{\Delta t}\right)$$

i.e. $$I = \frac{dq}{dt} \tag{2}$$ ☞

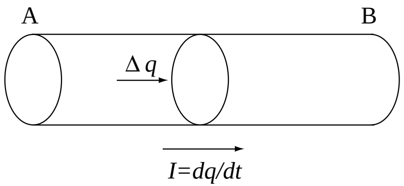


Figure 2 The current along a wire is defined by the net rate at which charge passes through a plane perpendicular to the axis of the wire.

In general, both the direction and rate of flow of charge may vary with time. If the flow is always in one direction, it is said to be a **direct current** (**d.c.**). In this module, we shall be concerned only with conditions where neither the direction nor the rate of flow varies with time. ☞ In such situations the average current and the instantaneous current are equal.

Broadly speaking, there are two types of charge-carrying particles that move around easily: *ions* and *electrons*. However, in this module we will be dealing with currents in metal wires where only electrons, which have become detached from metal *atoms*, are free to move through the metal. It is the movement of these 'free' electrons that will be responsible for the current flowing through the metal. The positive ions that are left behind when atoms lose electrons can only oscillate about fixed positions in a regular array known as a **lattice**. ☞

Electrons are usually bound to atoms by quite strong forces, so how can they move around so freely within a metal? A full answer to this question involves a very sophisticated study of the behaviour of electrons in a regular array of metal atoms, but at a much cruder level you can simply imagine an electron from each atom 'stepping to the side', i.e. moving, to an adjacent atom in the array. If all the free electrons in a piece of metal made such a move simultaneously the only ones that would experience a restoring force after the move would be those at the ends of the piece of metal, since those are the only ones that would experience a *change* of environment. If the metal were part of a continuous **circuit** (i.e. a closed path within which charged particles may flow) there would be no ends and hence no restoring forces. Under such circumstances you can picture the electrons moving very freely through a uniform sea of positively charged ions that has almost no net effect on the current.

The charge on an electron is denoted by $-e$, where $e = 1.602\,177 \times 10^{-19}$ C. This is an extremely small charge and, even for currents of a few picoamps, ☞ the flow rate of electrons in a wire is many millions per second.

Because the current in metals is carried by electrons, which have negative charge, there is a slight complication in defining the direction of current. The negative sign of the electron's charge is the result of an arbitrary choice made by Franklin ☞ in the eighteenth century, between 'two types of electricity'. The consequence is that what we describe as a flow of (positive) current in a certain direction consists (in metals) of electrons flowing in the opposite direction. ☞ We conventionally define the direction of *positive* current as the direction in which positive charge would flow, but bear in mind that the electrons are actually moving in the opposite direction — it is only when we are considering microscopic aspects of current that this becomes important. The possibility that the current from A to B may be positive or negative is a direct consequence of Equation 2.

$$I = \frac{dq}{dt} \qquad \text{(Eqn 2)}$$

Reversing either the sign of the charge being transferred or its direction of flow would reverse the sign of the current. However, in the absence of any clear indication of the direction of positive current flow (such as an arrow on a diagram or the words 'from A to B') it is the convention to use the term 'current' to describe the (positive) *magnitude* of the rate of charge transfer. Thus, unless a direction is explicitly indicated, currents should always be positive quantities.

Question T1

If 4.80×10^{10} electrons pass through a plane perpendicular to the axis of a wire in 5 s, what is the average current in the wire? ❏

2.2 Voltage: why currents flow

In an isolated wire there is no net charge flow. The ions are in their lattice positions and their only movements are very small vibrations, the size (amplitude) of which depends upon the temperature of the wire. The free electrons will also have a certain amount of kinetic energy associated with their temperature, but this thermal motion will be randomly directed like that of molecules in a gas and certainly won't cause a current. So how is it possible to arrange for a net movement of electrons through the wire?

The simplest answer follows from considering the energetics of the system. Electrons will flow from A to B in a wire if, by so doing, the *potential energy* of the system is reduced. You may find helpful the analogy between the flow of water in a pipe (Figure 3) and the flow of electrons in a wire.

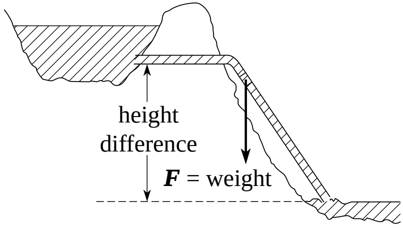


Figure 3 Water in a pipe flows from places of high gravitational potential energy to places of low gravitational potential energy.

Water flows through the pipe when there is a height difference, and therefore a difference in *gravitational potential energy*, between the ends of the pipe. It is the weight of the water — the *gravitational force* acting on it, arising from the *gravitational field* — that pushes the water along the pipe.

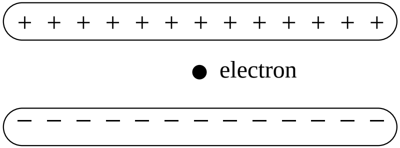


Figure 4 An electron between charged plates.

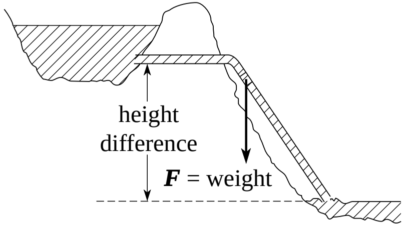


Figure 3 Water in a pipe flows from places of high gravitational potential energy to places of low gravitational potential energy.

For the flow of electrons in a wire, it is the **electric potential energy** which is important. The idea of electric potential energy (sometimes just called electrical energy) is illustrated by the situation shown in Figure 4. An electron, free to move, placed between charged plates will accelerate towards the positive plate, just as water will flow down the pipe in Figure 3. The electric potential energy of the system is reduced as the electron gains kinetic energy and moves towards the positive plate. In terms of forces, the electron has been moved by an *electrostatic force* arising from the *electric field* between the plates. The source of the electric field could be a *battery* (the plates would be charged by connecting one to each battery terminal), in which case the battery would also be the source of the electron's kinetic energy — there would be a net transfer of energy from the battery to the electron as it moved. ☞

If the terminals of a battery were joined by a wire, the battery would produce an electric field within the wire and the free electrons in the wire would move towards the positive terminal (Figure 5), rather like the water moving downhill in the pipe. Again, there would be a net transfer of energy from the battery to the electrons.

The change in the electric potential energy of a particle as it moves within an electric field is proportional to the charge of that particle. Thus, if we divide the change in a particle's electric potential energy by its charge we get a new quantity, the *change in electric potential energy per unit charge*, that is independent of the particle's charge and therefore tells us more about the electrical environment (i.e. the electric field) in which the particle is located.

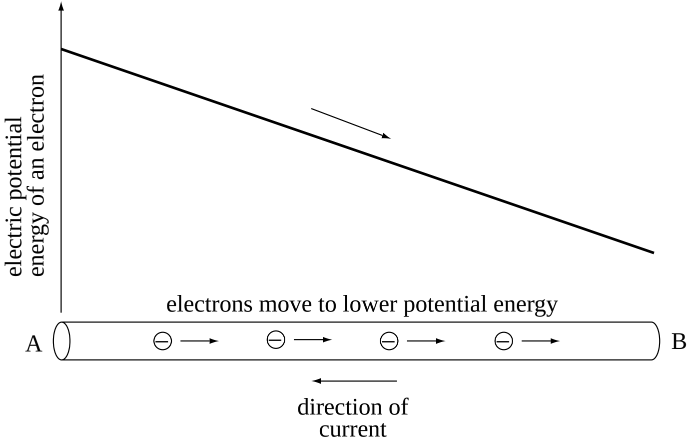


Figure 5 Electrons move from places of high electric potential energy to places of low electric potential energy.

✦ What are the SI units of ‘change in electric potential energy per unit charge’?

The change in electric potential energy per unit charge between two points is known as the **electric potential difference** or, more commonly, the **potential difference (p.d.)**, between those points. The SI unit of potential difference is the **volt** (unit symbol V), where $1\,\mathrm{V} = 1\,\mathrm{J\,C^{-1}}$. The electric potential difference is therefore also known as the **voltage difference**, and can be represented by the symbol ΔV. Thus, the change in electric potential energy $\Delta E_{\rm el}$ when a charge q moves through a potential difference of ΔV is given by

$$\Delta E_{\rm el} = q\Delta V \qquad (3)$$

The electric potential energy per unit charge at a given point is called the **electric potential** (often just ‘the potential’) at that point and is denoted by the symbol V. Electric potential at a point has (like p.d.) the volt as its SI unit, and is often referred to simply as ‘the **voltage**’ at that point.

Note that it is only the *difference* in electric potential between two points that is physically significant since it is that difference that tells us the *change* in electric potential energy when unit charge moves between those points. We are therefore free to choose *any* convenient reference point as the place where a charge has zero electric potential, and then define the electric potential at any other point relative to that reference point.

Also note that potential is defined in terms of the electric potential energy of unit *positive* charge. If a free positive charge moved spontaneously from some point A to our chosen point of zero potential, it would be losing electric potential energy as it did so — point A must therefore be at a higher (i.e. more positive) potential than the chosen zero point. If, though, a free positive charge moved spontaneously *from* a point of zero potential to some other point B, then the potential at B must be negative, otherwise such a movement would represent a spontaneous *increase* in electric potential energy. ☞

If a point in a circuit is **earthed** (literally, connected to the Earth by a wire) then that point is generally taken to be the point of zero electric potential, often referred to as **earth potential**. If a circuit is not earthed, then the negative terminal of a battery is generally taken as having a potential of 0 V.

Question T2

A battery has a potential difference of 12 V between its terminals. How much electrical energy is released when there is a spontaneous net charge transfer of 5 C between the terminals? ❏

Question T3

Suppose it was decided to define the *positive* terminal of a 12 V battery as being at zero potential. What would be the potential at the negative terminal? ❏

A note on symbols and conventions We have used the subscript 'el' to make it clear that we are dealing with *electrical* energy changes. However, when dealing with voltage and voltage difference, no ambiguity arises if the subscript is dropped, so we have omitted it. We have also been careful to distinguish the voltage difference *between* two points (ΔV) from the voltage *at* a point (V). However, many texts do not make the distinction so explicit, they often refer just to 'the voltage' and use the symbol V to mean the voltage difference between two points. In the remainder of this module, we too will generally omit the Δ when referring to a voltage difference. However, in common with many other texts, we will sometimes use $V_{\rm A}$ to represent the potential at a point A, and $V_{\rm AB}$ for the potential difference between the points A and B (so, $V_{\rm AB} = V_{\rm B} - V_{\rm A}$). ☞

2.3 Ohm's law and resistance

Within certain limits, the rate at which water passes through a pipe is proportional to the difference in gravitational potential (or height) between the ends of the pipe. In 1827 a German physicist, Georg Simon Ohm (1787–1854), found an equivalent relationship for charge flow in metal wires: the current I through a wire is proportional to the electric potential difference V between its ends: $I \propto V$.

This relationship is usually written in the form

$$I = \frac{V}{R} \qquad \textit{Ohm's law} \qquad (4)$$

and is known as **Ohm's law**. The constant of proportionality is written as $(1/R)$ where R is called the **resistance**. The unit of resistance, the **ohm**, has the symbol Ω (Greek letter omega). Equation 4 may be used to define the ohm: $1\,\Omega = 1\,\mathrm{V\,A^{-1}}$.

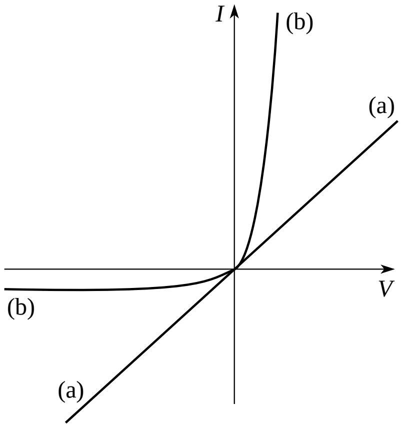


Figure 6 I–V characteristic graphs for (a) an ohmic metal and (b) a semiconductor diode. The diode is highly non-linear.

Ohm's law is not a fundamental law of physics. It is an empirical law based on observations of certain metals in certain circumstances and, even for these metals, applies only if the temperature of the metal is kept constant. For an *ohmic metal* (i.e. one described by Equation 4),

$$I = \frac{V}{R} \qquad \textit{Ohm's law} \qquad \text{(Eqn 4)}$$

a graph of current against voltage (Figure 6a) is a straight line, the gradient of which is the reciprocal of the resistance, $1/R$. Such a graph for an electrical component is known as an ***I–V* characteristic** graph.

If a particular electrical component has a linear (i.e. straight line) *I–V* graph, it is called a **linear component**. (This property is of importance when we come to consider *superposition* in Subsection 3.1.) Not all electronic components are linear. For example, Figure 6b shows the *I–V* characteristic of a component known as a *semiconductor* diode, which is highly non-linear. Such a component does not have a single value of resistance: its effective resistance will depend on the voltage at which it is being operated.

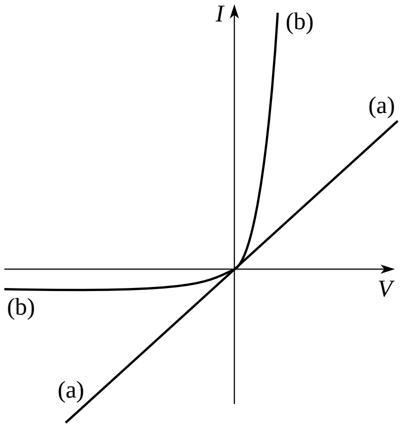


Figure 6 *I–V* characteristic graphs for (a) an ohmic metal and (b) a semiconductor diode. The diode is highly non-linear.

2.4 Resistivity and conductivity

To appreciate how the dimensions of a wire influence the flow of charge we can again appeal to our intuitive understanding of water flow in pipes — short fat pipes allow water to flow more easily than long thin pipes. When we examine the factors determining the current in a wire we find that for a given applied voltage difference between the ends of the wire:

- Increasing the length of wire decreases the current: $I \propto 1/l$, i.e. $R \propto l$.
- Increasing the cross-sectional area of the wire increases the current: $I \propto A$, i.e. $R \propto 1/A$.

These two proportionalities may be combined in the approximate equation

$$R = \rho \frac{l}{A} \tag{5}$$ ☞

where the constant of proportionality ρ (Greek letter rho) is called the **resistivity** and has the units of Ω m. ☞

The reciprocal of resistivity is **conductivity** which is usually given the symbol σ (Greek letter sigma) and the units are either $(\Omega\,\mathrm{m})^{-1}$ or, equivalently, **siemens** per metre $(\mathrm{S\,m^{-1}})$.☞

$$\sigma = \frac{1}{\rho} \tag{6}$$

The principal contribution to the resistance of a metal usually comes from electron collisions with the lattice ions. The ions vibrate with an amplitude which increases with temperature, and it is these lattice vibrations which impede the flow of current. Interestingly a sophisticated analysis of solids indicates that a perfect lattice of *stationary* ions would offer no resistance at all to electrons moving through it. As you will see in Subsection 2.5, some materials (*superconductors*) do show exceptionally low resistance at very low temperature but this is not, so far, a phenomenon that can be produced at room temperature.

The size of the current in a material when a voltage is applied across it will depend not only on the impedance to charge flow offered by the lattice vibrations but also on the number of mobile charged particles in the material. This latter factor, the number of mobile **charge carriers**, shows enormous variations from one type of material to another and is responsible for giving resistivity the widest variation of any known physical property. Table 1 lists some typical resistivities and also shows how materials are classified according to their resistivity.

A **conductor** is a material with a large number of mobile charge carriers (usually free electrons) and hence a very low resistivity. All metals are conductors.

An **insulator** has very few mobile charge carriers and its resistivity is extremely high.

A **semiconductor** has a moderate number of mobile charge carriers and intermediate resistivity: the number of charge carriers, and hence the resistivity, is strongly dependent on temperature and on the presence of impurities. Raising the temperature of a semiconductor usually greatly increases the number of mobile charge carriers and thus reduces the resistivity.

Table 1 Resistivities of some conductors, semiconductors and insulators.

	Material	Resistivity $/\Omega\,\mathrm{m}$
conductor	silver	1.65×10^{-8}
	copper	1.67×10^{-8}
	gold	2.35×10^{-8}
	aluminium	2.63×10^{-8}
	tungsten	5.51×10^{-8}
	nickel	6.84×10^{-8}
	iron	9.71×10^{-8}
semi-conductor	germanium	$\sim 10^{-1}$–10
	silicon	$\sim 10^{2}$–10^{5}
	carbon	$\sim 4 \times 10^{-3}$
insulator	glass	10^{10}–10^{14}
	PVC	$> 10^{10}$
	mica	$> 10^{12}$
	PTFE	$> 10^{15}$
	fused quartz	$> 10^{15}$

Question T4

A piece of nickel wire is 1.20 m long and has a cross-sectional area of $2.50 \times 10^{-8}\,\mathrm{m}^2$.

Calculate (a) the resistance of the wire (b) the current in the wire when there is a p.d. of 6.40 V between its ends. ❏

Question T5

A certain material has a conductivity of $4.60 \times 10^7\,\mathrm{S\,m^{-1}}$. Would you classify it as a conductor, semiconductor or insulator? ❏

2.5 Resistance and temperature

In Subsection 2.4 we explained that as the temperature of a metal is raised, the amplitude of lattice ion oscillations increases, causing the resistivity, and hence the resistance of a given specimen, to increase. The relationship between resistance R and temperature T is approximately linear over a temperature range $-50\,°\text{C} < T < +150\,°\text{C}$ and may thus be represented by the approximate equation

$$\boxed{R_T \approx R_0\,(1 + \alpha T) \qquad (7)}$$

where R_T is the resistance at temperature T, R_0 is the resistance at 0 °C, and α, the **temperature coefficient of resistance**, is defined as the mean fractional change of resistance per °C over the temperature range $T = 0\,°\text{C}$ to $T = 100\,°\text{C}$,

i.e. $$\alpha = \frac{R_{100} - R_0}{100R_0}\,(°\text{C})^{-1} \qquad (8)$$

Typical metals have α between about $3.5 \times 10^{-3}\,(°\text{C})^{-1}$ and $6.5 \times 10^{-3}\,(°\text{C})^{-1}$. Such values may seem small but the effect is quite large and is important in many applications.

✦ The temperature coefficient of resistance of copper is $4.26 \times 10^{-3}\,(°\text{C})^{-1}$. The resistance of a certain piece of copper at 0 °C is 100 Ω. What will be its resistance at 100 °C?

$$R_T \approx R_0\,(1 + \alpha T) \qquad \text{(Eqn 7)}$$

Equation 7 is only an approximation and, for most materials, its use for temperature changes of more than a hundred or so degrees will lead to large inaccuracies. However, the reproducibility of the temperature variation of resistance, particularly in samples of platinum, has resulted in platinum resistance thermometers being adopted as the international practical method of measuring temperature over the range from about 14 K (–259 °C) to 904 K (1177 °C).

If we extend our discussion beyond pure metals, there are three commercially important materials with unusual temperature coefficients which should be mentioned.

- One is carbon, often deposited as a thin film on the surface of an insulator to make resistors for electronic circuitry. These carbon film resistors have a very small *negative* temperature coefficient (NTC) ($\alpha \approx -5 \times 10^{-4}\,(°\mathrm{C})^{-1}$) so the resistance decreases as the temperature is raised.
- Second, the resistors known as **thermistors** (from *therm*ally sensitive res*istor*) are made from semiconductors with large NTCs, and are widely used in temperature measurement and control circuitry.
- Third, there is an alloy of copper and nickel (UK trade name Constantan) with a very small temperature coefficient. Wire-wound resistors are frequently made from Constantan.

A graph of resistance against temperature for a typical sample of metal (Figure 7a) is roughly linear down to quite a low temperature, where it flattens out. In this region the temperature is such that the lattice vibrations have become very small and the main contribution to resistivity comes from impurities and imperfections in the lattice.

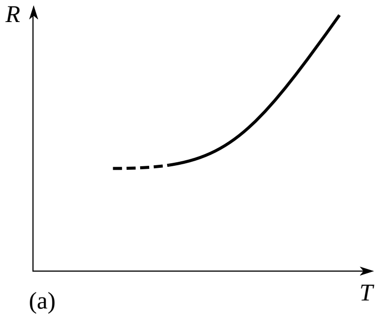


Figure 7a The resistance of (a) a metal at low temperature

Experimental observations of resistance at *very* low temperatures were not made until 1911 when H. Kammerlingh Onnes (1853–1926), who had previously discovered how to liquefy helium (boiling point 4.2 K), made the remarkable observation that the resistance of a specimen of mercury dropped abruptly to zero once a critical **transition temperature** T_c of about 4.1 K was reached (see, for example, Figure 7b). Many other metals have since been found to behave similarly. Materials which show this effect are called **superconductors**. The resistivity of a superconductor does appear to be truly zero: it is at least 10^{17} times smaller than that of metals at normal temperatures. A current flowing around a superconducting ring will continue to circulate for years — no voltage source is required to keep it going.

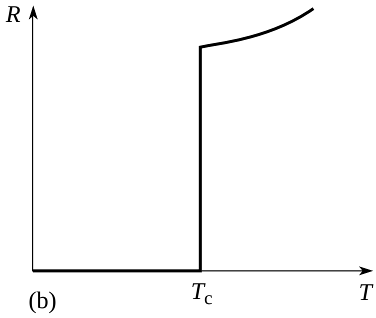


Figure 7b The resistance of (b) a superconductor above and below its transition temperature.

Until recently, no material had been found with a superconducting transition temperature (T_c) higher than about 23 K. Reaching such low temperatures needs liquid helium, which is very expensive, so uses of superconductors were few and mainly confined to the production of very large magnetic fields from large currents in superconducting rings. ☞

In 1986 Karl Muller and Johannes Bednorz found a ceramic material with a T_c of about 40 K. This was followed by intense experimentation in research laboratories around the world, resulting in ceramic materials with T_c well above 77 K, the boiling point of nitrogen — and liquid nitrogen is much cheaper and more widely available than liquid helium. Devices requiring large currents, employing high magnetic fields, or in situations (such as power transmission lines) where resistive power loss (see Subsection 2.6) is a limiting factor, may be changed beyond recognition once the technology of manufacturing these ceramics is established. If materials with room-temperature T_c are ever developed, then even further advances will become possible.

2.6 Energy and power

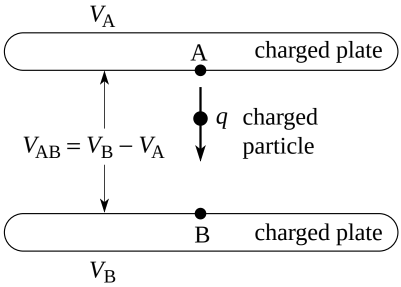


Figure 8 A charged particle moving through a potential difference.

This section is mainly concerned with the conversion of electrical energy in resistors, but we will first consider the system shown in Figure 8 (which is similar to that shown in Figure 4). Here, a particle with a charge q starts from rest at a point A and, under the influence of an electric field, accelerates in free space (a vacuum) to a point B, through a potential difference $V_{AB} = V_B - V_A$. The resulting change in the particle's electric potential energy will be qV_{AB} (from Equation 3).

$$\Delta E_{el} = q\Delta V \qquad \text{(Eqn 3)}$$

This will be a *negative* quantity since a spontaneous motion will reduce the electric potential energy and hence q and V_{AB} must have opposite signs. Now suppose that N such charges cross from A to B. The total change in electric potential energy will be $\Delta E_{el} = NqV_{AB}$ and this too will be negative since the movement of the charges will have *reduced* the electric potential energy. If the time required for the transfer of these charges is Δt, the average rate of change of electric potential energy will be the negative quantity

$$\frac{\Delta E_{el}}{\Delta t} = \frac{NqV_{AB}}{\Delta t}$$

Now, $Nq/\Delta t$ is just the average charge transferred per second, i.e. the average current $\langle I \rangle$ (Equation 1),

$$\langle I \rangle = q/t \qquad \text{(Eqn 1)}$$

so

$$\frac{\Delta E_{\rm el}}{\Delta t} = \langle I \rangle V_{\rm AB}$$

The *conservation of energy* demands that as the particles lose electric potential energy they must gain kinetic energy at the same rate. Clearly, as the particles come to rest at B they must shed this kinetic energy. If the particles are electrons, then almost all of this energy goes into increasing the thermal energy of the plates at B. So, if the plate being bombarded by particles is to remain at a constant temperature it must dissipate energy at the *positive* rate ☞

$$P = -\langle I \rangle V_{\rm AB}$$ ☞

The quantity P in this equation describes the rate at which energy is released (dissipated) at B and is called the **power**. Power is measured in **watts** (unit symbol W), where $1\,{\rm W} = 1\,{\rm J\,s^{-1}}$.

What is different when A and B are joined by a wire of finite resistance instead of by empty space? If there is still a voltage difference V between the ends of the wire, there will still be an electric field which will accelerate the free electrons. They will not accelerate smoothly, however, because they will be impeded by lattice vibrations. The electrons will collide with the lattice ions and transfer their kinetic energy (acquired from the electric field) to the ions along their path, increasing the thermal energy of the wire and tending to raise its temperature. This process is called **resistive heating** or **Joule heating**. Apart from the details of the heating, the same basic principles still apply and the rate at which energy is released will still be given by the product of a current and a voltage difference.

In general, if we wish to calculate the energy dissipated in a resistive component (i.e. the power) of a d.c. circuit we simply use the formula

$$\boxed{\text{electric power } P = IV \qquad (9)}$$

where I is the steady current *in that component*, and V is the voltage *across that component*. Similarly, the power drawn from a voltage source is the product of the output voltage of that source and the current drawn from that source. Note that by using I and V in this sense we remove the need to worry about directions and signs.

Since the resistance of a component is defined by $R = V/I$ we can rewrite Equation 9

$$\text{electric power } P = IV \qquad \text{(Eqn 9)}$$

in the following equivalent forms:

$$P = V^2/R \qquad (10)$$

and $$P = I^2R \qquad (11)$$

The powers of domestic electrical appliances are usually quoted in kilowatts (kW), e.g. a one-bar electric heater is usually 1 kW, and a typical kettle is 2.2 kW. If Equation 9 is rearranged to give $I = P/V$, we can calculate the current in an appliance and hence decide the value of the fuse to be fitted where the (UK) mains supply voltage is 240 V. (A fuse is an electrical safety device designed to shut off current flow if that current exceeds a certain fixed value.) The 1 kW heater takes a current of about 4 A ☞ In an appliance such as a heater, the quoted power (and hence the calculated current) are for the hot resistance wire. When the heater is first switched on it will be cold and the resistance will be much smaller than when fully heated, so the fuse has to cope with an initial surge of current which is greater than the value calculated. The practice is therefore to use a standard 13 A (rather than a 5 A) fuse for such cases, otherwise the fuse would blow each time the heater was switched on.

Electric supply companies bill customers for energy. We can, for example, calculate the contribution that using the kettle for five minutes will make to our next bill. The rate at which the kettle transfers energy is 2.2 kW or $2.2 \times 10^3\,\mathrm{J\,s^{-1}}$. In five minutes the total energy transferred by heating is $2.2 \times 10^3\,\mathrm{J\,s^{-1}} \times 5 \times 60\,\mathrm{s}$, i.e. $6.6 \times 10^5\,\mathrm{J}$. In order to avoid dealing with such large numbers, the much larger commercial ‘unit’ of electricity is used. This unit is the **kilowatthour** (kW h). $1\,\mathrm{kW\,h} = 1\,\mathrm{kW} \times 1\,\mathrm{h} = 10^3\,\mathrm{J\,s^{-1}} \times 3.6 \times 10^3\,\mathrm{s} = 3.6 \times 10^6\,\mathrm{J}$. The price is quoted as so much per unit. Repeating the above calculation gives us the energy used by the kettle as $2.2\,\mathrm{kW} \times (1/12)\,\mathrm{h} = 0.18\,\mathrm{kW\,h}$, i.e. 0.18 units.

If, as in our example, energy is being transferred at a constant rate P, then the total energy transferred in a time t is simply the product Pt.

If the power varies with time, however, then the calculation will need to take this into account and this will generally require calculus. ☞

Question T6

It takes 2.5 s to start a car with an electric motor that draws 90 A from its 12 V battery.

(a) How much energy is drawn from the battery?

(b) The battery is recharged from the car's 5 W generator. How long will it take to recharge the battery? ❏

Question T7

At a particular time, a city consumes 230 MW of electrical power at a voltage of 160 kV. (It is transformed to a much lower voltage before being distributed around the city!) The power lines connecting the power station to the city have a total resistance of 7.6 Ω.

(a) Find the power dissipated in the lines and express this as a percentage of the city's power.

(b) How would the power dissipated in the lines change if the supply voltage to the city were doubled while supplying the same power as before? ❏

3 Circuit laws

A note on circuit conventions The circuit components discussed in this module are mainly limited to d.c. voltage generators (batteries, etc.), resistors and measuring instruments, joined by wires. We normally assume that we can ignore any potential difference between the ends of a connecting wire. (In other words we make the resistance of connections negligible compared with that of the components.) This is readily achieved in laboratory experiments where typical voltage sources produce a few volts and components have resistances of tens, hundreds or thousands of ohms. Copper wire of 1 mm diameter has a resistance of about 2.2 Ω per hundred metres so the few centimetres we use to join our components together on the bench have very little resistance. ❏

Figure 9 illustrates some circuit diagram conventions that we will adopt for the remainder of this module. The direction of a voltage is shown by an arrow alongside the component with the arrowhead at the higher voltage end. The direction of conventional (positive) current is shown by the direction of arrows on connecting wires. The rectangular boxes represent **ohmic resistors** (i.e. resistors that obey Ohm's law). ☞ The open circle represents an **ideal voltage generator** (i.e. a device that produces a constant voltage between its terminals regardless of what is connected to it — see Subsection 4.3).

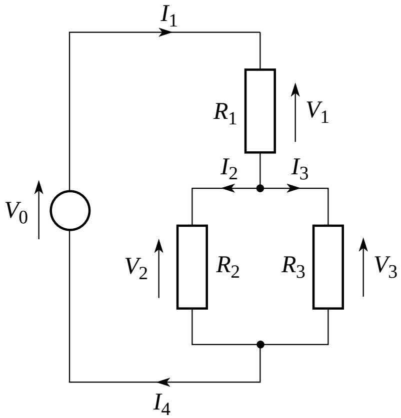
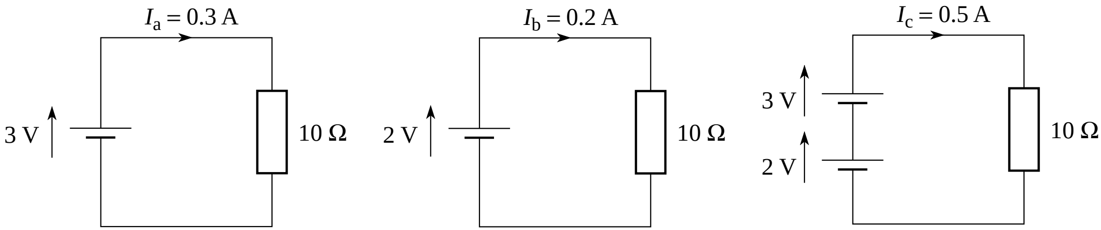


Figure 9 A circuit diagram illustrating some conventional symbols.

When assigning directions to the currents (I_1, I_2, I_3, etc.) and the voltage differences (V_1, V_2, V_3, etc.) in Figure 9, or any similar figure, it is often necessary to make the assignments on an arbitrary basis, and it is quite possible that some of those assignments will be wrong. Fortunately, this is not a major problem, since if such 'mistakes' are made it will subsequently emerge that the currents or voltages involved will be negative rather than positive quantities. Despite this, it's still a good idea to make the assignments as realistically as possible at the outset. Starting at a voltage generator and remembering that (conventional) currents flow from high voltage to low voltage will often give you a reasonable idea of the appropriate directions to choose.

3.1 Linearity and superposition

The circuits of Figure 10 ☞ show batteries connected to an ohmic resistor. Each battery is represented symbolically by a pair of parallel lines, the longer of which corresponds to the higher voltage terminal. If both batteries are connected simultaneously as in (c) then the net voltage applied to the resistor is 5 V and the net current of 0.5 A is the sum of the 0.3 A and 0.2 A of (a) and (b).



(a) (b) (c)

Figure 10 (a) A 3 V battery connected to a 10 Ω resistor. (b) A 2 V battery connected to the resistor. (c) Both batteries connected simultaneously across the resistor.

This result is also shown on the I–V characteristic of Figure 11a. It is only because the resistor is ohmic (i.e. behaves linearly) that the currents generated by different sources are additive (taking account of directions). If the I–V graph were not linear, then as Figure 11b indicates, the current when both batteries are included would *not* generally be the sum of the separate currents.

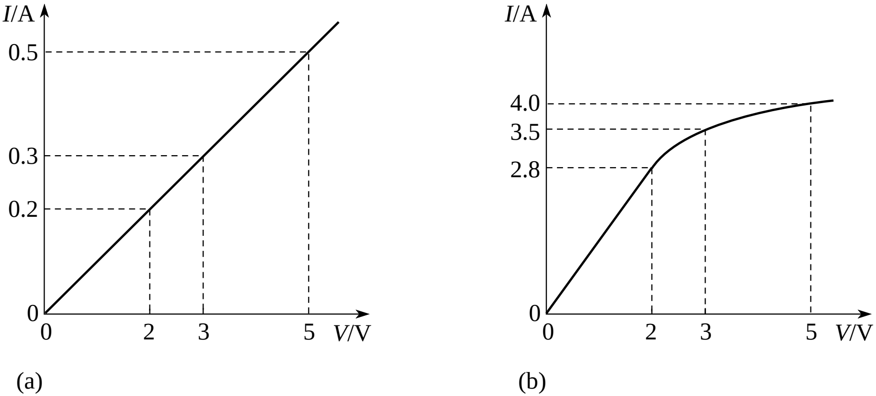


Figure 11 (a) For a linear (i.e. ohmic) device, the current when both batteries are present is the sum of the currents when each battery is present on its own. (b) This result would no longer apply if the resistor were replaced with a non-linear device.

The above example is generalized in the **principle of superposition**:

> In a circuit made up of linear components and containing several voltage generators, the resultant current in, or voltage across, any component will be the algebraic sum of those currents or voltages in or across that component when each of the voltage generators is taken in turn, with all other voltage generators replaced by *short circuits*.

The word *algebraic* is used here to mean giving the currents or voltages + or – signs to correspond to their directions. Note, too, the use of the term **short circuit**, meaning a path of (effectively) zero resistance.

3.2 Kirchhoff 's laws

When analysing a circuit the usual aim is to obtain expressions for the current and/or voltage and/or power in any component of that circuit. Kirchhoff's current and voltage laws of circuit behaviour are fundamental in such analysis, although we do not always use them explicitly we always rely on rules derived from them. In Subsection 3.3 and in Section 4 we will use Kirchhoff's laws to obtain some useful results about circuits. In this subsection each of the laws will be stated and the statement will be followed by some explanatory comments. Here is the first law:

Kirchhoff's current law

The algebraic sum of the currents at a node is zero.

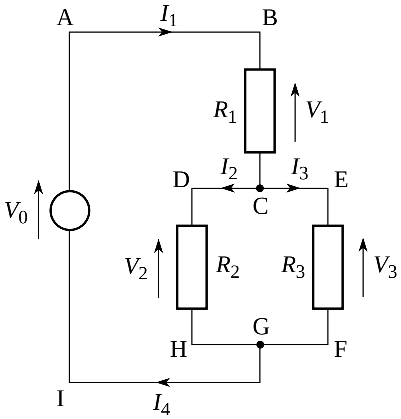


Figure 12 A circuit for analysis by Kirchhoff's laws.

A **node** is a junction of connections to two or more components such as the points A, B, C, D, etc. on Figure 12. The phrase *algebraic sum* means 'taking account of directions'; this is generally done by associating a positive or negative sign with each current (I_1, I_2, etc.) according to its assumed direction.

In our case, we shall regard any current directed *towards* a particular node as making a positive contribution at that node, and any current directed away from the node as making a negative one. Thus, for the three currents I_1, I_2 and I_3 associated with node C in Figure 12 we associate a + sign with I_1 and – signs with I_2 and I_3, so Kirchhoff's current law takes the form

$$I_1 - I_2 - I_3 = 0$$

The equivalent statement $I_1 = I_2 + I_3$ gives an alternative form of the law: *The sum of the currents entering a node is equal to the sum of the currents leaving the node*.

Kirchhoff's current law thus implies that, for a continuous circuit in the steady state, there is no net build-up or disappearance of charge at any point in the circuit. We can also apply the law to node G, where we get $I_2 + I_3 = I_4$, but we have already seen that $I_2 + I_3 = I_1$, so we can conclude that $I_1 = I_4$, thus confirming the continuity of current through the voltage generator.

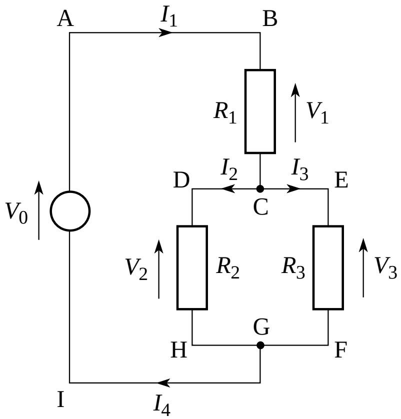


Figure 12 A circuit for analysis by Kirchhoff's laws.

Kirchhoff's voltage law

The algebraic sum of the voltages across all components in a closed loop of a circuit is zero.

Here we have taken *component* to mean any part of an electrical circuit, e.g. resistor or voltage generator. A closed loop (sometimes called a **mesh**) refers to any path in a circuit which may be followed continuously around to its starting point.

The circuit of Figure 12 has three closed loops (meshes): ABCDHGI, ABCEFGI and DEFH. We have labelled the voltages across the resistors in Figure 12 and added direction arrows to these voltages. Once again, the law demands an *algebraic sum*, so we must again associate signs with each voltage in any particular loop. To do this we arbitrarily assign a positive direction for voltages to each loop, a + sign is then associated with voltages that point in this direction and a – sign with those that point in the opposite direction. ☞ For example, in the loop ABCDHGI we may decide (arbitrarily) to call clockwise-directed voltages positive, and anticlockwise ones negative.

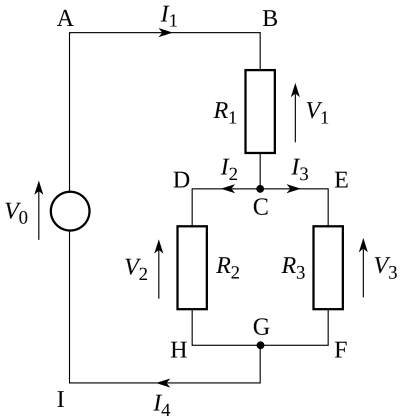


Figure 12 A circuit for analysis by Kirchhoff's laws.

Then, proceeding from A we have: $-V_1 - V_2 + V_0 = 0$, i.e. $V_1 + V_2 = V_0$.

For the loop ABCEFGI, using the same convention for assigning signs: $-V_1 - V_3 + V_0 = 0$, i.e. $V_1 + V_3 = V_0$.

For loop DEFH: $-V_3 + V_2 = 0$, i.e. $V_3 = V_2$.

These equations lead to an alternative statement of Kirchoff's voltage law: *The potential difference between two points joined by more than one continuous path is independent of the particular path considered.*

This is equivalent to saying that if we connect a voltmeter between two points of a circuit then we will only get one reading of the voltage, regardless of which particular path we are *thinking of* at the time — which is indeed what happens!

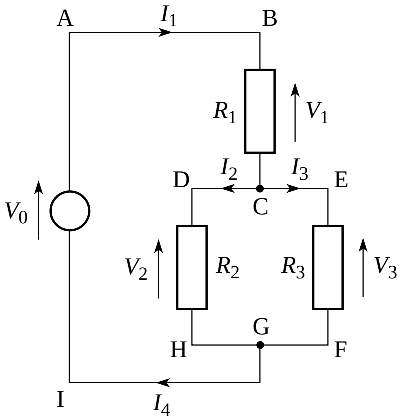


Figure 12 A circuit for analysis by Kirchhoff's laws.

Question T8

Use Kirchhoff's laws to find the currents in the circuit shown in Figure 13. ❏

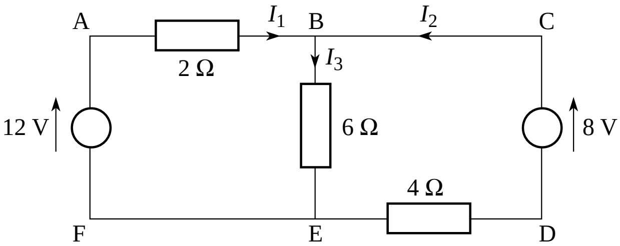


Figure 13 See Question T8.

In Figure 12, Kirchhoff’s laws can also be used to show how the current is divided between the resistors R_2 and R_3. First, Ohm’s law gives us $V_2 = I_2R_2$ and $V_3 = I_3R_3$ which, together with $V_2 = V_3$ from the voltage law, gives $I_2R_2 = I_3R_3$. If this expression is rearranged, we get $I_2/I_3 = R_3/R_2$, i.e. the ratio of currents is the reciprocal of the corresponding ratio of resistances. Going a step further, if we use $I_1 = I_2 + I_3$ from the current law, and substitute for I_3 since ($I_3 = I_2R_2/R_3$) we obtain:

$$I_1 = I_2 + \frac{I_2R_2}{R_3} = I_2\left(\frac{R_2 + R_3}{R_3}\right)$$

A similar expression can be found for I_3 in terms of I_1, R_2 and R_3.

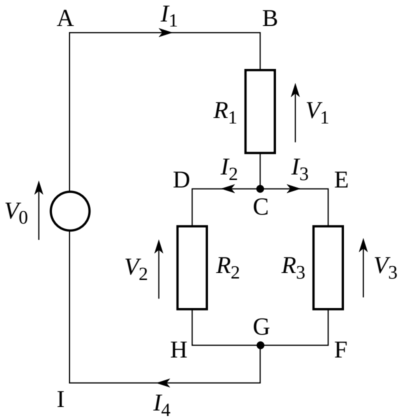


Figure 12 A circuit for analysis by Kirchhoff’s laws.

If both of these are rearranged we get the so called current divider equations.

The **current divider equations**:

$$\frac{I_2}{I_1} = \frac{R_3}{R_2 + R_3} \tag{12a}$$

$$\frac{I_3}{I_1} = \frac{R_2}{R_2 + R_3} \tag{12b}$$

3.3 Applying circuit laws: the Wheatstone bridge

In the 1840s, Charles Wheatstone ☞ described the **bridge circuit**, composed of a voltage source and four resistors, which carries his name. It is usually drawn in a diamond shape (Figure 14a) but we will use a rectangular format (Figure 14b) which is easier to draw and to appreciate. The Wheatstone bridge has been used since the days of its invention for determining the value of an unknown resistance.

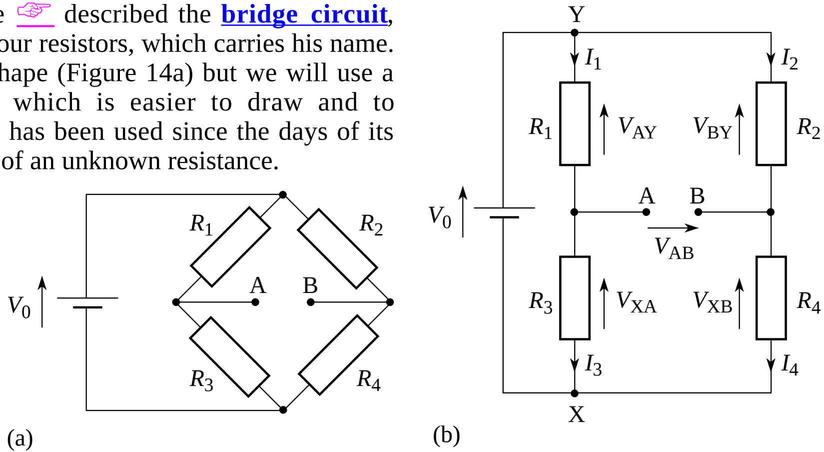
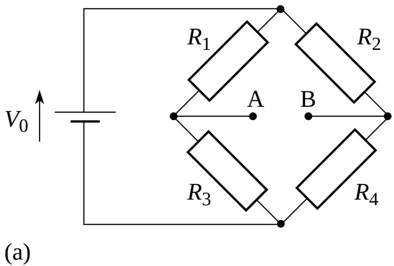


Figure 14 The Wheatstone bridge circuit drawn in (a) the traditional diamond pattern and (b) a rectangular pattern.

The usual procedure is to use a **balanced bridge**, i.e. to adjust the values of the resistances until there is a zero reading on a sensitive current meter connected between A and B. Examining the conditions that bring this about shows us how the value of an unknown resistance can be measured. Since there is no current between A and B ($I_{\rm meter} = 0$), the current law tells us that $I_1 = I_3$, and $I_2 = I_4$. Also, since there is no current there must be no potential difference between A and B: so $V_{\rm AB} = 0$. It then follows from the voltage law that $V_{\rm XA} = V_{\rm XB}$ and $V_{\rm AY} = V_{\rm BY}$.



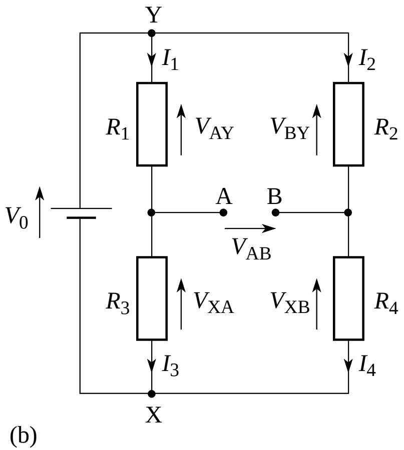


Figure 14 The Wheatstone bridge circuit drawn in (a) the traditional diamond pattern and (b) a rectangular pattern.

❖ Use Ohm's law to write down expressions for V_{AY} and V_{BY} and for V_{XA} and V_{XB} in terms of I_1, I_2, R_1, R_2, R_3 and R_4. Hence obtain, and equate, two different expressions for I_2/I_1.

Now suppose R_4 is an unknown resistance, R_1 and R_2 have known values and R_3 is a finely-adjustable known resistance. By alterations to R_3, the current in the meter can be reduced to zero and R_4 found from Equation 13.

$$\frac{R_1}{R_2} = \frac{R_3}{R_4} \qquad \text{(Eqn 13)}$$

Question T9

(a) Rearrange Equation 13 to make R_4 the subject. (b) Calculate the value of R_4 that would balance a bridge circuit in which $R_1 = 500\,\Omega$, $R_2 = 470\,\Omega$ and $R_3 = 327\,\Omega$. ❏

Nowadays the bridge is widely used to monitor changes in the resistance of sensors such as strain gauges and resistance thermometers. This requires an *unbalanced bridge*, which we will discuss in Subsection 4.4 after some further techniques of circuit analysis have been introduced.

4 Equivalent circuit techniques

For any other than the simplest of circuits, the application of Kirchhoff's laws may require the solution of a large number of *simultaneous equations*. This is precisely the sort of task for which computers are eminently suitable. However, for analysis 'by hand', there are other methods that provide useful short cuts, such as replacing a complicated circuit by a simpler **equivalent circuit** which has similar properties. One important technique is to replace several resistors by a single resistor, applying rules derived using Kirchhoff's laws.

4.1 Resistors in series and parallel

The circuit in Figure 15a shows an ideal voltage source and three resistors connected in **series**, i.e. joined sequentially like the links of a chain. We now seek the value of a single equivalent resistance which, when connected across the same voltage source, will draw the same current from the source. It is clear from Kirchoff's current law that the current in each resistor is the same. In addition we can apply the voltage law to get $V_0 = V_1 + V_2 + V_3$. Using Ohm's law this becomes:

$$V_0 = IR_1 + IR_2 + IR_3 = I(R_1 + R_2 + R_3)$$

If we had a circuit consisting of the battery and a single resistor R_{series} then the same current would be drawn from the battery if we made

$$R_{\text{series}} = R_1 + R_2 + R_3$$

This R_{series} is equivalent (as far as the rest of the circuit is concerned) to the three resistors R_1, R_2 and R_3.

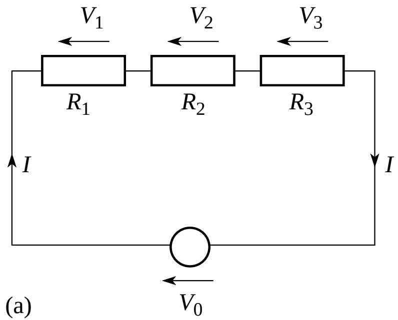


Figure 15a Three resistors connected in series.

Clearly this argument can be extended for any number of resistors and we could write the equivalent of N resistors in series as

$$R_{\text{series}} = \sum_{j=1}^{N} R_j \qquad \textit{resistors in series} \qquad (14)$$

☞

Figure 15b shows three resistors connected in **parallel**, i.e. forming three different paths between the nodes A and B. If the current law is applied at node A we have $I = I_1 + I_2 + I_3$, while the voltage law gives $V_0 = V_1 = V_2 = V_3$. If Ohm's law is used and these two expressions are combined we obtain

$$I = \frac{V_1}{R_1} + \frac{V_2}{R_2} + \frac{V_3}{R_3} = V_0\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right)$$

If we had a single resistor which drew the same current and had a value $R_{\rm parallel}$ then

$$I = V_0\left(\frac{1}{R_{\rm parallel}}\right)$$

If these equations are compared we find

$$\frac{1}{R_{\rm parallel}} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$$

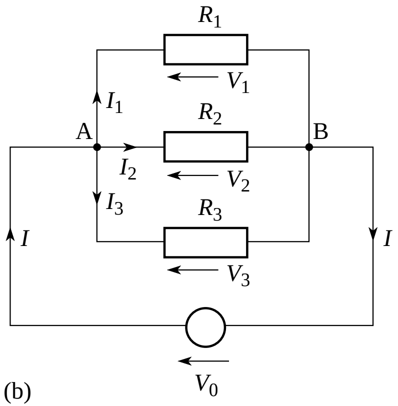


Figure 15b Three resistors connected in parallel.

The above argument can be extended to N parallel resistors, for which

$$\frac{1}{R_{\text{parallel}}} = \sum_{j=1}^{N} \frac{1}{R_j} \qquad \textit{resistors in parallel} \tag{15}$$

A useful alternative expression for *two* resistors in parallel is:

$$\frac{1}{R_{\text{parallel}}} = \frac{1}{R_1} + \frac{1}{R_2} = \frac{R_1 + R_2}{R_1 R_2}$$

Thus, for *two* resistors in parallel

$$R_{\text{parallel}} = \frac{R_1 R_2}{R_1 + R_2} \tag{16}$$

Note that Equation 16 *cannot* be extended to three or more resistors. Also note that the equivalent resistance of a pair of resistors in parallel is always *less* than the resistance of either individual resistor.

Question T10

Find the resistance equivalent to (a) $10\,\Omega$ in parallel with $20\,\Omega$ (b) three $5\,\Omega$ resistors in parallel. ❏

The equivalent circuits for resistors are often helpful when we want to find the current (or voltage or power) elsewhere in a circuit.

✥ Use the above techniques to find an expression for the current I_1 in Figure 12 in terms of R_1, R_2, R_3 and V_0.

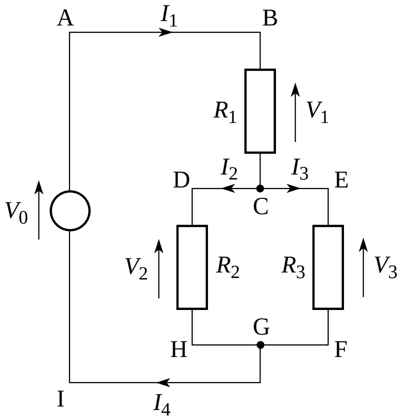


Figure 12 A circuit for analysis by Kirchhoff's laws.

Suppose we now want to extend the analysis of this circuit (Figure 12) to find the currents I_2 and I_3 and the voltages V_1, V_2, V_3. One way of proceeding is as follows: if we know I_1 we can use Ohm's law to find V_1; if we know V_0 and V_1 we can then find V_2 and V_3 since Kirchhoff's voltage law tells us $V_2 = V_3 = V_0 - V_1$. Finally, if we know V_2 and V_3 we can use Ohm's law twice to find I_2 and I_3.

Question T11

Find the total resistance between each of the following pairs of terminals A and B, B and C, A and C of Figure 16. ❏

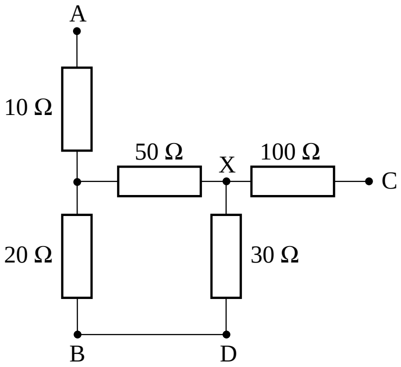


Figure 16 See Question T11.

4.2 Adapting ammeters and voltmeters

Ammeters and **voltmeters** are devices for measuring currents and potential differences, respectively. The techniques developed in the preceding subsections can now be applied to ammeters and voltmeters to see how a general meter may be adapted for different purposes. Many meters nowadays are digital instruments, but analogue meters with needles and dials, usually based on a **moving-coil galvanometer** (MCG), ☞ are also used. A so-called **multimeter** can act either as an ammeter or a voltmeter, and may also include a means of measuring resistance.

Ideally, adding a meter to a circuit should not alter the currents and voltages in that circuit. To measure a current in a loop, the loop must be broken and an ammeter connected in series with it so that all the current flows through it. For the ammeter not to change the current significantly, its resistance must be *negligible* compared with that of the rest of the loop. An ideal ammeter therefore has zero resistance.

To measure the voltage difference between two points in a circuit, a voltmeter is connected to those two points so that it is in parallel with the circuit component(s). For a voltmeter not to affect the voltage in the circuit, its resistance must be *very large* compared to that of the other component(s), so that no current flows through it. An ideal voltmeter therefore has infinite resistance, but most voltmeters have to draw some current in order to operate and therefore have a large but finite resistance. A digital voltmeter generally has a higher resistance than a moving-coil instrument, typically 10 MΩ.

Now let us examine how to convert an MCG into an ammeter. Suppose our MCG has a resistance of 180 Ω and produces a full-scale deflection in response to a current through it of 100 μA. Provided the resistance of the rest of the circuit is much larger than 180 Ω, the meter can be used to measure currents up to 100 μA ☞. A current larger than 100 μA would damage the meter. However, we can use the MCG in circuits with larger currents if we connect a bypass or **shunt resistor** ($R_{\rm sh}$) in parallel with it, this is shown in Figure 17a where we represent the real MCG ☞ (of resistance 180 Ω) by an ideal meter (of zero resistance) in series with a resistor $R_{\rm M}$. Suppose we want a full-scale deflection on the meter when a current of $I = 1.00$ A flows into the combination of MCG and shunt resistor.

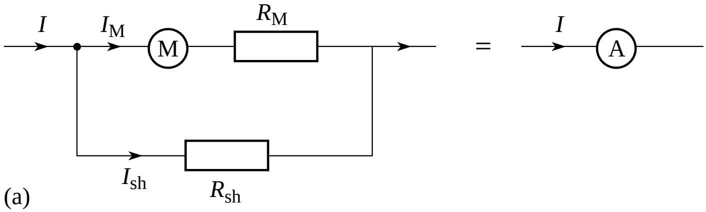


Figure 17a A moving-coil galvanometer used with a shunt resistor to make an ammeter.

We know that we need a current $I_M = 100\,\mu A$ to flow through the meter, so we use the current divider equation (Equation 12)

$$\frac{I_2}{I_1} = \frac{R_3}{R_2 + R_3} \qquad (12a)$$

$$\frac{I_3}{I_1} = \frac{R_2}{R_2 + R_3} \qquad (12b)$$

to find R_{sh}

$$I_M = \frac{R_{sh}}{R_M + R_{sh}}\,I \qquad (17)$$

If we substitute $I = 1.00\,A$, $I_M = 1.00 \times 10^{-4}\,A$ and $R_M = 180\,\Omega$ into Equation 17 we find $R_{sh} = 0.018\,\Omega$.

(Note that, since $R_{sh} << R_M$, ☞ Equation 17 simplifies to $R_{sh} \approx R_M I_M / I$.)

The same MCG can be converted into a voltmeter, but now we have to add a series resistor (R_s) to the MCG (Figure 17b). Suppose we want to produce a full-scale deflection for a potential difference of 10 V across the combination of meter and resistor: we need that potential difference to produce a current of 100 μA. If Ohm's law is applied we obtain

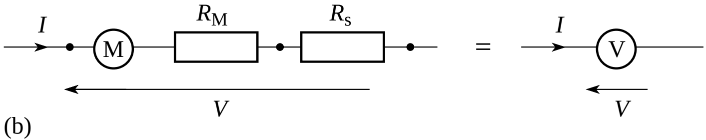


Figure 17b A moving-coil galvanometer used with a series resistor to make a voltmeter.

$$V = I(R_s + R_M) \qquad (18)$$

which for $V = 10$ V and $R_M = 180\,\Omega$, $I = 1.00 \times 10^{-4}$ A gives $R_s = 9.98 \times 10^4\,\Omega$.

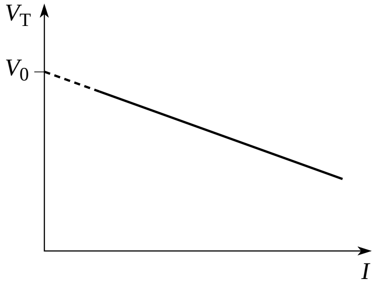
Question T12

An MCG has a resistance of $2500\,\Omega$, and a current of 1 mA pro-duces a full-scale deflection. Find the values of the shunt and series resistors, respectively, to convert it into (a) an ammeter to measure currents up to 10 A and (b) a voltmeter to measure voltages up to 30 V. ❏

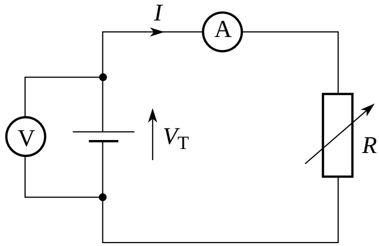
4.3 Voltage generators

As the current drawn from a realistic voltage generator, such as a battery is increased, the **terminal p.d.** (the voltage V_T between the terminals of the generator) decreases linearly, as shown in Figure 18. ☞ This behaviour is described empirically by the equation $V_T = V_0 - kI$. The voltage V_0, the value of V_T when $I = 0$, is called the **open circuit voltage** (the voltage when there is an open circuit, i.e. no connection — effectively an infinite resistance — between the terminals). Sometimes this is called the **electromotive force** (**e.m.f.**) of the generator ☞. The constant k is the gradient of the straight-line graph and, as all three terms of the equation must have the units of voltage, k must have the units of resistance. We emphasize this by rewriting k as R_0, so the equation becomes

$$V_T = V_0 - IR_0 \tag{19}$$



(a)



(b)

Figure 18 (a) The output voltage of a non-ideal voltage generator decreases as the current increases. (b) The circuit used to obtain the results plotted in (a). (A variable resistor is represented in a circuit diagram by a box with a diagonal arrow through it.)

Now consider the behaviour of the circuit in Figure 19 which includes an ideal voltage generator ☞ and a resistance R_0 as well as a variable resistance R. If the current I in this circuit is changed by varying R, then the voltage V recorded by the (ideal) voltmeter is

$$V = V_0 - IR_0 \qquad (20)$$ ☞

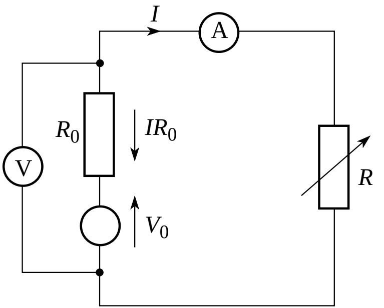


Figure 19 The equivalent circuit of a voltage generator which has an *internal* (or *output*) *resistance*, R_0.

The right-hand sides of Equations 19 and 20 are identical, so the circuit of Figure 19 behaves just like that of Figure 18b and we can therefore say it is an *equivalent circuit*. The behaviour of our real generator is identical to that of an ideal generator in series with a resistance given that:

1 The voltage of the equivalent ideal generator is equal to the open-circuit voltage, V_0 of the real generator.

2 The resistance R_0 in series with the ideal generator is determined from the V–I relationship of the real generator.

R_0 is called the **internal resistance**, or **output resistance**, of the voltage generator.

For the purposes of circuit analysis, any real voltage generator may be replaced by an ideal voltage generator in series with an output resistance. The output resistance of a generator may be ignored if it is negligible in comparison with the resistance of the circuit to which it is connected, since $V_{\rm T}$ is then little different from V_0.

Question T13

Find the open circuit voltage and internal resistance of a battery the terminal p.d. of which is 9 V when it supplies 1 A, and 6 V when it supplies 4 A. ❏

4.4 Thévenin's theorem

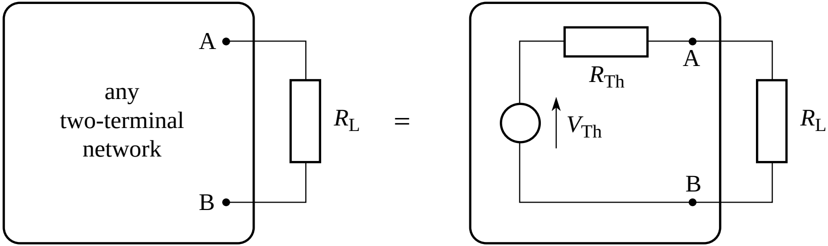


Figure 20 Thévenin's theorem gives a simple equivalent circuit for any circuit composed of resistors and voltage generators, between two terminals.

You have seen how to replace combinations of resistors with a single resistor, and a real voltage generator by an ideal generator in series with a resistor. These are just two examples of a much more general technique that can be applied to any circuit consisting of resistors and voltage generators that supplies current to an external resistor (usually called the **load resistor**). This technique, illustrated in Figure 20, is based on *Thévenin's theorem* which may be expressed thus:

> **Thévenin's theorem**: For the purpose of calculating the current and voltage in a load resistor R_L, *any* two-terminal network of voltage generators and resistors can be replaced by an equivalent circuit consisting of a single ideal voltage generator in series with a single resistor.

The properties of the two parts of the equivalent circuit are found by applying two rules:

1. The voltage of the ideal voltage generator (the **Thévenin voltage**, $V_{\rm Th}$) is the open-circuit voltage (i.e. when $R_{\rm L} = \infty$) between the two terminals.
2. The value of the series resistance (the **Thévenin resistance**, $R_{\rm Th}$) is the resistance between the two terminals when all voltage generators in the circuit are replaced by their output resistances (or by short circuits if the output resistances are negligible).

As an example, we will apply Thévenin's theorem to a **voltage divider** circuit (Figure 21). This important circuit is used when we have a voltage supply V_0 which is too large for the purpose we have in mind, so we wish to reduce it. This is accomplished by connecting two resistors R_1 and R_2 in series with the supply (assumed here to have negligible internal resistance) and then using the reduced voltage across one of the resistors (R_2). Using Rule 1, V_{Th} is obtained by removing R_{L}, finding the current through R_1 and R_2 (these are in series so this current is $V_0/(R_1 + R_2)$) and then using Ohm's law to find the voltage across R_2.

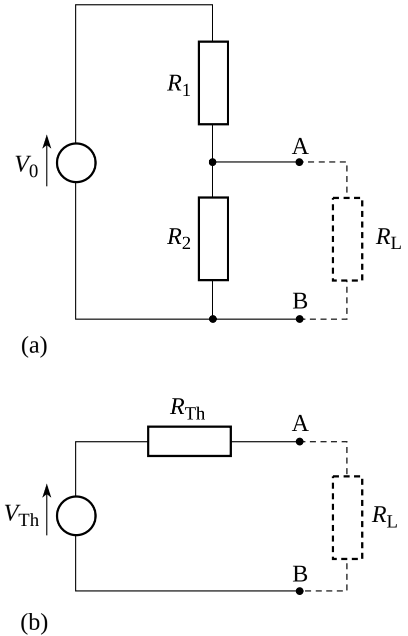


Figure 21 Thévenin's theorem applied to a voltage divider circuit.

Thus, the open circuit voltage between A and B from a voltage divider circuit is given by:

The **voltage divider equation**:

$$V_{\mathrm{Th}} = \frac{R_2}{R_1 + R_2} V_0 \tag{21}$$

From Rule 2,

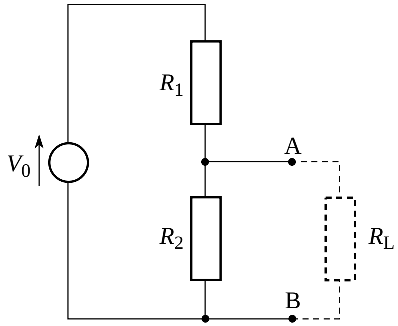
R_{Th} is the resistance between terminals A and B when the voltage generator is replaced by a short circuit. But be careful! R_1 and R_2 are connected in *series* with the voltage generator, but when that generator is replaced by a short circuit each resistor is connected directly between A and B so they are actually in *parallel* with A and B, ☞ thus,

$$R_{\mathrm{Th}} = \frac{R_1 R_2}{R_1 + R_2}$$

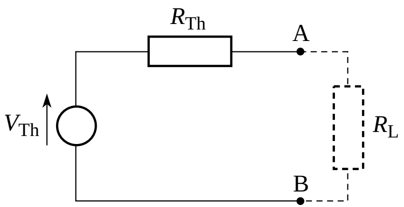
When a load resistance R_{L} is connected between terminals A and B of the voltage divider circuit the voltage between A and B will change, as you can confirm by answering the next question.

Question T14

Find the voltage across R_L for the circuit of Figure 21. Express your answer in terms of V_0, R_1, R_2 and R_L. ❏



(a)



(b)

Figure 21 Thévenin's theorem applied to a voltage divider circuit.

For a further example of Thévenin's theorem we will return to the (Wheatstone) bridge circuit of Subsection 3.3. As stated in the introduction to this module, many physical quantities are nowadays measured electrically using sensors. For example, *strain gauges* ☞ exploit the change in resistance with length (Subsection 2.4) to measure the strains in engineering structures, and resistance thermometers use the dependence of resistance on temperature (Subsection 2.5) to measure temperature changes. The Wheatstone bridge is used to monitor such changes in resistance.

If one of the resistors in the bridge (Figure 14b) changes with time, then V_{AB} (which we will call the output voltage) will also change and we will have an **unbalanced bridge** — there will be a p.d., and hence a current, between A and B. Now, V_{AB} is just the difference between the potentials at A and B, both of which can be measured with respect to X, so $V_{AB} = V_{XB} - V_{XA}$.

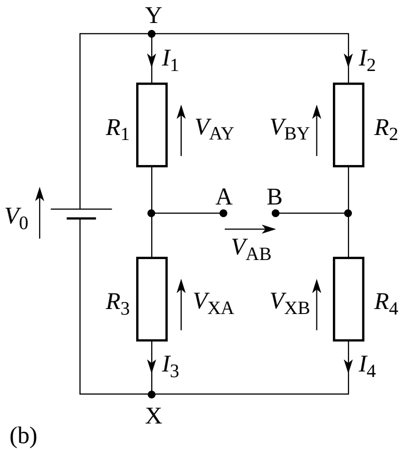


Figure 14b The Wheatstone bridge circuit drawn in a rectangular pattern.

By comparing the voltage divider circuit of Figure 21 with the bridge circuit of Figure 14 you should be able to convince yourself that

$$V_{\rm XA} = \frac{V_0 R_3}{R_1 + R_3}$$

and similarly,

$$V_{\rm XB} = \frac{V_0 R_4}{R_2 + R_4}$$

Thus $V_{\rm AB} = V_{\rm XB} - V_{\rm XA}$

$$= \left(\frac{R_4}{R_2 + R_4} - \frac{R_3}{R_1 + R_3} \right) V_0 \qquad (22)$$

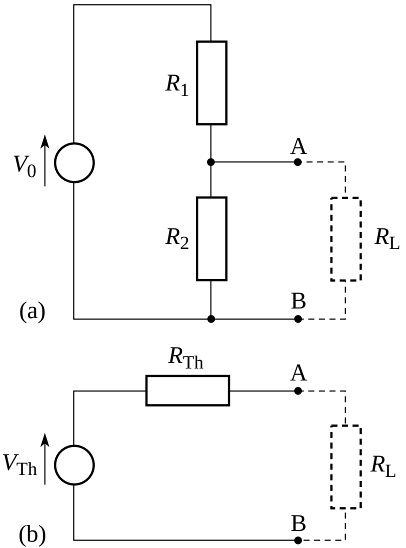
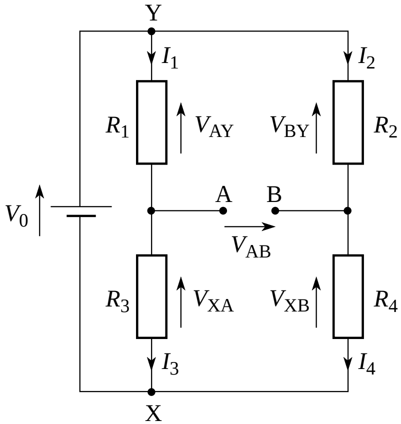


Figure 21 Thévenin’s theorem applied to a voltage divider circuit.



(b)

Figure 14b The Wheatstone bridge circuit drawn in a rectangular pattern.

The situation becomes slightly simpler if we consider the circuit of Figure 22 where all resistors have the same value and the bridge is initially balanced. If one of those resistors (the one at the bottom right, say) increases its value by a small amount ΔR the bridge will become unbalanced, and there will be a non-zero output voltage:

$$V_{\rm AB} = \left(\frac{R+\Delta R}{R+R+\Delta R} - \frac{R}{R+R}\right)V_0 = \left(\frac{R+\Delta R}{2R+\Delta R} - \frac{1}{2}\right)V_0$$

i.e. $$V_{\rm AB} = \frac{\Delta R}{2(2R+\Delta R)}V_0 \qquad (23)$$

If the change ΔR is much less than $2R$ then Equation 23 gives

$$V_{\rm AB} \approx \frac{\Delta R}{4R}V_0\text{, i.e. } V_{\rm AB} \propto \Delta R$$

If the output voltage $V_{\rm AB}$ is connected to a chart recorder or to a computer, then we can obtain an automated record of the changes in resistance and hence of whatever causes them.

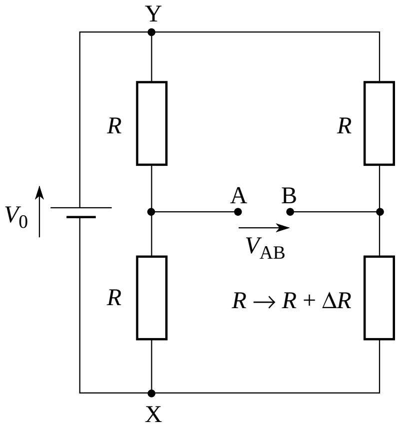


Figure 22 An unbalanced Wheatstone bridge.

5 Closing items

5.1 Module summary

1 The electric *current* $I = dq/dt$ in a wire describes the rate at which net charge is transferred across a plane perpendicular to the axis of that wire. Current in a metal is due to the flow of free electrons. The conventional direction of current is that in which positive charge would flow.

2 Free charged particles move so as to minimize the *electric potential energy* $E_{\rm el}$ of a system. The difference in electric potential energy per unit charge between two points is called the (*electric*) *potential difference*, or *voltage difference*, between those points, so $\Delta V = \Delta E_{\rm el}/q$. A point connected directly to the Earth, or to the negative terminal of a battery is usually chosen to be a point of zero electric potential energy. The *electric potential* or *voltage* at any other point is then defined as the electric potential difference between that point and the chosen point of zero electric potential.

3 *Ohmic resistors* are *linear components* that obey Ohm's law $V = IR$. Where V is a potential difference measured in volts (V), I is a current measured in amps (A) and R is a *resistance* measured in ohms (Ω).

4 The resistance of a sample length l and cross-sectional area A is given by $R = \rho l/\mathrm{A}$ where ρ is the *resistivity* of the material. *Conductors* have low resistivity, *insulators* have high resistivity and *semiconductors* have intermediate resistivity. Resistivity tends to increase with temperature, though some materials, particularly semiconductors, show the opposite behaviour.

5 When a steady current I flows between points separated by a potential difference V the rate of energy transfer is the *power* $P = IV$.

6 For *linear* components, the *principle of superposition* allows the analysis of *circuits* with more than one voltage source.

7 The usual aim of circuit analysis is the evaluation of the *current*, *voltage* and *power* in any component of a circuit. *Kirchhoff's current law* (the algebraic sum of the currents at a *node* is zero) and *Kirchhoff's voltage law* (the algebraic sum of the voltages across all components in a closed loop is zero) underlie all circuit analysis.

8 Analysis is often helped by replacing part of a circuit with a simpler *equivalent circuit* as described by *Thévenin's theorem*.

9 *Equivalent circuit* techniques include replacing resistors in *series* by a resistance $R_{\text{series}} = \sum\limits_{j=1}^{N} R_j$, replacing resistors in *parallel* by a resistance $1/R_{\text{parallel}} = \sum\limits_{j=1}^{N} 1/R_j$, replacing a voltage source by an *ideal voltage generator* plus an *output resistance*, and adding either a *shunt* or a series resistor to a galvanometer to produce a combination equivalent to an ammeter or voltmeter (respectively) with a desired range.

10 Results obtained from Kirchoff's laws and Thévenin's theorem include: the *current divider equations* for resistors in parallel (Equation 12); the *voltage divider equation* for resistors in series (Equation 21); and the condition for a Wheatstone *bridge circuit* to be *balanced* (Equation 13).

5.2 Achievements

Having completed this module, you should be able to:

A1 Define the terms that are emboldened and flagged in the margins of the module.

A2 Relate the flow-rate of charged particles to a current.

A3 Calculate the resistances of specimens of known dimensions and resistivity or conductivity.

A4 Apply Ohm's law to the currents and voltages in resistors.

A5 Make calculations relating resistance and temperature.

A6 Distinguish the resistive behaviour of metals when in the normal and superconducting states and appreciate the importance of high T_c superconducting materials.

A7 Carry out simple calculations involving current, voltage, power and the release of electric potential energy.

A8 Explain how superposition is a consequence of linearity and apply the principle of superposition in the analysis of circuits containing several voltage sources.

A9 Apply Kirchhoff's current and voltage laws to the analysis of simple circuits.

A10 Determine the conditions for zero current output from a balanced Wheatstone bridge.

A11 Find equivalent circuits for resistors in series and parallel and calculate the resistance between pairs of terminals in complicated resistive circuits.

A12 Calculate the values of shunt and series resistors to convert moving-coil galvanometers into ammeters and voltmeters with specified ranges.

A13 Use the equivalent circuit for a voltage generator with an output resistance and appreciate when it is possible to ignore the presence of such an output resistance.

A14 Use Thévenin's theorem to calculate the Thévenin equivalent voltage and resistance and find simplified equivalents of circuits.

Study comment You may now wish to take the *Exit test* for this module which tests these Achievements. If you prefer to study the module further before taking this test then return to the *Module contents* to review some of the topics.

5.3 Exit test

Study comment Having completed this module, you should be able to answer the following questions each of which tests one or more of the Achievements.

Question E1

(*A3*, *A4* and *A7*) The resistive element (a coil of wire) of an electric heater dissipates energy at the rate of 1 kW when operated at 240 V. By how much would the length of wire have to be changed to produce the same power if the supply voltage were 110 V? (Ignore the fact that the operating temperatures may be different.)

Question E2

(*A4* and *A5*) A 100 W bulb operates at 240 V and the filament reaches a temperature of 2200 °C. The filament metal has an average temperature coefficient of resistance of $4.7 \times 10^{-3}\,(°\mathrm{C})^{-1}$ over the range of 0 – 2200 °C (and you can take the variation of resistance with temperature to be linear over this range).

(a) Find the resistance of the filament at 2200 °C and at 20 °C.

(b) Hence find the current when the bulb is first switched on.

Question E3

(*A9*) Use Kirchhoff's laws to find the currents in Figure 23.

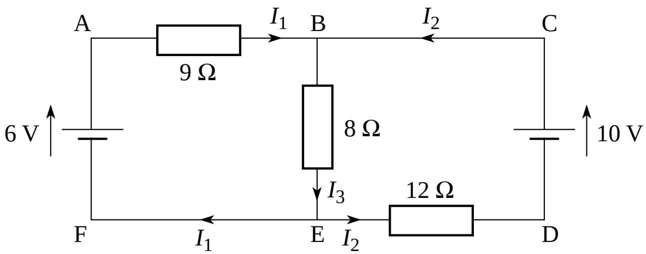


Figure 23 See Question E3.

Question E4

(*A7* and *A11*) A resistor R is connected to a supply voltage V. Find the power dissipated in R and compare it with that dissipated when R is replaced by two resistors, each of resistance R, connected (a) in series and (b) in parallel. Assume the resistances do not change with temperature.

Question E5

(*A5* and *A10*) Figure 24 shows a balanced Wheatstone bridge circuit in which $R_1 = 12.5\,\Omega$, $R_2 = 1.60\,\Omega$ and $R_3 = 9.20\,\Omega$.

(a) Find the value of R_4.

(b) R_4 is a length 2.70 m of a material sample with a cross-sectional area of $1.80 \times 10^{-7}\,\text{m}^2$. Calculate the resistivity of the material.

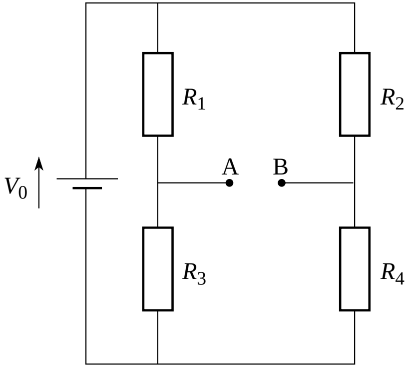


Figure 24 See Question E5.

Question E6

(*A14*) (a) Find the Thévenin equivalent of the circuit of Figure 25 between the terminals A and B.

(b) Find the values of the Thévenin equivalent voltage and resistance for $V_0 = 12\,\text{V}$, $R_1 = 200\,\Omega$, $R_2 = 200\,\Omega$, $R_3 = 1\,\text{k}\Omega$ and $R_4 = 800\,\Omega$.

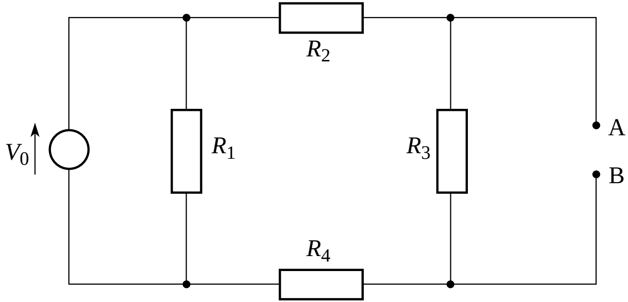


Figure 25 See Question E6.

Study comment This is the final *Exit test* question. When you have completed the *Exit test* go back to Subsection 1.2 and try the *Fast track questions* if you have not already done so.

If you have completed **both** the *Fast track questions* and the *Exit test*, then you have finished the module and may leave it here.