Module P5.5 The mathematics of oscillations

Exit module

1 Opening items

1.1 Module introduction

Oscillations occur in many branches of physics, but in this module we will examine just two: mechanical systems and electrical circuits. At first sight a mass oscillating on a spring and the tuning circuit of a radio appear to have little in common; but the mathematics that models them is almost indistinguishable, and both can be described in terms of a *second-order differential equation with constant coefficients*. In Section 2 of this module we examine the way in which such equations arise and consider some of the oscillatory phenomena their solutions represent. In particular we look at *simple*, *damped* and *driven* oscillations, and we pay particular attention to the way in which solutions of the latter kind typically consist of a *steady state* term that oscillates and gradually becomes dominant, and a *transient* term that may be important initially but gradually dies away and eventually becomes insignificant. In some circumstances the amplitude of these dominant driven oscillations can become very large; this is the phenomenon of *resonance* which we consider in Subsection 2.7.

Very often it is only the steady state behaviour of the system that is of interest, and we may then assume that the transient term is zero. In such cases we are able to abandon the differential equation approach, and use a much simpler method based on *complex numbers*. This technique is particularly relevant to the analysis of *alternating currents* in electrical circuits, and in Section 3 we use it to develop a complex version of Ohm's law. We will see that the current and the applied voltage oscillate at the same rate, but they do not necessarily do so in phase due to the *complex impedance* of the circuit concerned. In Subsection 3.3 we use complex methods to calculate the power dissipated in an electrical circuit, and in the final section we examine how complex numbers may be used to combine simple harmonic motions.

Study comment Having read the introduction you may feel that you are already familiar with the material covered by this module and that you do not need to study it. If so, try the *Fast track questions* given in Subsection 1.2. If not, proceed directly to *Ready to study?* in Subsection 1.3.

1.2 Fast track questions

Study comment Can you answer the following *Fast track questions*?. If you answer the questions successfully you need only glance through the module before looking at the *Module summary* (Subsection 5.1) and the *Achievements* listed in Subsection 5.2. If you are sure that you can meet each of these achievements, try the *Exit test* in Subsection 5.3. If you have difficulty with only one or two of the questions you should follow the guidance given in the answers and read the relevant parts of the module. However, *if you have difficulty with more than two of the Exit questions you are strongly advised to study the whole module*.

Question F1

Write down the differential equation for the current $I(t)$ in a circuit containing a resistance R, a capacitance C and an inductance L in series, that is driven by an applied voltage $V_0 \sin(\Omega t)$. What is the general form of the steady state solution to this equation?

Question F2

If $\mathcal{Z} = R + i\omega L + 1/(i\omega C)$ (where R, ω, L and C are all real and positive) find expressions for $\mathrm{Re}(\mathcal{Z})$, $\mathrm{Im}(\mathcal{Z})$ and $|\mathcal{Z}|$. For what value of ω does $|\mathcal{Z}|$ take its least value?

Write down the principal values of the arguments of R, $\mathcal{Z}_L$ and $\mathcal{Z}_C$, where

$$\mathcal{Z}_L = i\omega L \quad \text{and} \quad \mathcal{Z}_C = 1/(i\omega C)$$

and illustrate these complex numbers on an Argand diagram.

Question F3

An inductance of 3.00 H and a capacitance 0.10 F are connected in parallel, and this combination is then connected in series with a resistance of 5.00 Ω. Find the current that passes through the resistor when a voltage $V(t) = a\cos(\Omega t)$, where $a = 4.00$ V and $\Omega = 3.00$ Hz, is applied to the circuit.

Study comment Having seen the *Fast track questions* you may feel that it would be wiser to follow the normal route through the module and to proceed directly to *Ready to study?* in Subsection 1.3.

Alternatively, you may still be sufficiently comfortable with the material covered by the module to proceed directly to the *Closing items*.

1.3 Ready to study?

Study comment This module is intended to form a link between the maths and physics strands of *FLAP*. It therefore makes much heavier mathematical demands than most modules, though it assumes less knowledge of physics. To begin the study of this module you will need to be familiar with the solution of *second-order differential equations* with constant coefficients (though we provide a brief review of this topic in Subsection 2.3), and you should also know how such equations arise from *Newton's second law of motion*. You should be able to manipulate *trigonometric identities*. You should also be familiar with the *Cartesian coordinate system*, *complex numbers*, including their *exponential representation* ($z = re^{i\theta}$), the *Argand diagram* and the *real part*, *imaginary part*, *modulus argument* and *complex conjugate* of a complex number (although we provide you with a short summary of the subject in Subsection 3.1). You should also be able to *differentiate* and *integrate* standard functions such as $\sin(x)$, $\cos(x)$ and $\exp(x)$. A familiarity with geometric progressions would also be useful, although not essential. It would be helpful if you have seen how *oscillatory systems* arise in physics, but we assume no prior knowledge in this area. If you are unfamiliar with any of these topics you can review them by referring to the *Glossary*, which will indicate where in *FLAP* they are developed. The following *Ready to study questions* will help you to establish whether you need to review some of the above topics before embarking on this module.

Question R1

Sketch the graph of $y = 3\sin(\omega t + \delta)$ for

(a) $\omega = 2\,\mathrm{s}^{-1}$ and $\delta = 0$, (b) $\omega = 2\,\mathrm{s}^{-1}$ and $\delta = -\pi/2$.

How would the first graph you drew change if

(c) $\omega = 2\,\mathrm{s}^{-1}$ and $\delta = -4$, (d) $\omega = 2\,\mathrm{s}^{-1}$ and $\delta = 4$.

(e) Describe (without drawing a diagram) the graph of $y = \sin(\omega t + \pi/2)$.

Question R2

Calculate the value of ϕ given that

$$\cos\phi = \frac{2}{\sqrt{2^2 + 3^2}} \text{ and } \sin\phi = \frac{-3}{\sqrt{2^2 + 3^2}}$$

☞

Use the trigonometric identity $\cos(A + B) = \cos A\cos B - \sin A\sin B$ to express $2\cos(\omega t) + 3\sin(\omega t)$ in the form $R\cos(\omega t + \phi)$.

Question R3

Solve the quadratic equation $h^2 + 5h + 6 = 0$.

Question R4

If z is defined by $z = 4e^{5i\pi/4}$, what are the principal values of arg(z), $|z|$, Re(z) and Im(z)?

If you are unsure about any of these terms consult *complex numbers* in the *Glossary*.

Question R5

(Optional) What is the sum of the geometric series

$$1 + r + r^2 + r^3 + \dots + r^{n-1}$$

What is the sum for the particular case of $r = i$ (where $i^2 = -1$) and $n = 9$?

2 Oscillations and differential equations

2.1 Introducing mechanical and electrical oscillations

Figure 1 shows a small body of mass m held between two stretched springs on a smooth horizontal table. Under the influence of the springs, the body is able to move to and fro along a line that we will take to be the x-axis of a system of Cartesian coordinates.

The equilibrium position of the body will be taken to be the point $x = 0$, so the position coordinate of the body at any time t determines its displacement from equilibrium at that time.

If the body is released from rest at a point slightly to the right of its equilibrium position at some initial time $t = 0$ (diagram A), it will subsequently oscillate back and forth about its equilibrium position, as indicated in diagrams B, C, D and E.

As a result, the instantaneous position of the body will be a *function* of time and can be denoted by $x(t)$. The dashed line in Figure 1 is a graphical representation of this function, although it would be more natural for us to draw the graph of $x(t)$ with the t-axis horizontal and the x-axis vertical,

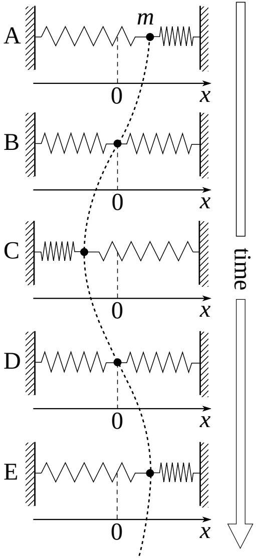


Figure 1 The vibrations of a mechanical system.

as in Figure 2a (with a suitably scaled t-axis to make the figure a manageable size). In the absence of *dissipative effects*, such as *friction* or *air resistance*, the total energy (kinetic plus potential) of the oscillator will be constant, and the displacement–time graph of Figure 2a will be *sinusoidal* (i.e. of the same general shape as the graph of a sine or cosine function). A displacement–time graph of this kind is characteristic of the particular kind of motion known as **simple harmonic motion** (SHM) which occurs in many branches of physics and engineering. The function whose graph is shown in Figure 2a may be represented algebraically by an expression of the form

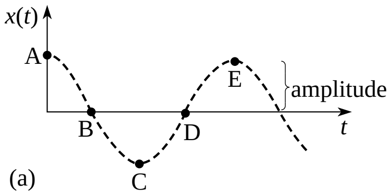


Figure 2a Time–displacement graph for the system shown in Figure 1.

$$x(t) = A_0 \sin(\omega_0 t + \pi/2)$$

and is a special case of the class of functions that provide the most general mathematical description of simple harmonic motion

$$x(t) = A_0 \sin(\omega_0 t + \phi) \qquad (1)$$

where A_0, ω_0 and ϕ are constants that characterize the motion and are, respectively, referred to as the **amplitude**, the **angular frequency** and the **phase constant** (or **initial phase**) of the oscillation. The amplitude is equal to the magnitude of the maximum displacement from equilibrium that occurs during each cycle of oscillation.

The angular frequency is related to the **period** T (i.e. the time required for one complete oscillation such as that from A to E in Figure 1) and to the **frequency** f by the relation

$$\omega_0 = 2\pi/T = 2\pi f$$

Thus the frequency ($f = 1/T$) is the number of oscillations per second, and the angular frequency is just 2π times that value. The phase constant determines the value of x at $t = 0$, since $x(0) = A_0 \sin(\phi)$. Note that the alternative name for ϕ, the *initial phase*, arises because the quantity $(\omega_0 t + \phi)$ which determines the stage that the oscillator has reached in its cycle at any time t is called the **phase**, and ϕ is simply the value of the phase at $t = 0$.

Figure 2b shows the effect of changing the initial phase of an oscillator. The equation describing the dashed line was given above as $x(t) = A_0 \sin(\omega_0 t + \pi/2)$, so it corresponds to a phase constant $\phi = \pi/2$. This may be contrasted with the equation describing the solid curve, which may be written

$$x(t) = A \sin(\omega t + \pi/2 - \omega_0 t_0)$$

and which corresponds to a phase constant $\phi = \pi/2 - \omega_0 t_0$. The quantity t_0 indicates the extent to which the behaviour of the oscillator represented by the solid curve lags behind that represented by the dashed curve. We can therefore say that there is a phase difference between the two oscillators, and that the former (represented by the solid curve) lags the latter (represented by the dashed curve) by $\omega_0 t_0$.

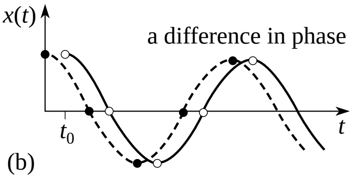


Figure 2b Effect of varying the iniyial phase on the time–displacement graph for the system shown in Figure 1.

In practice, an oscillating system of the kind shown in Figure 1 would be subject to friction and other dissipative effects and would lose energy to its environment. As a result of this energy transfer, the maximum displacement attained during each oscillation generally tends to decrease with time, resulting in the sort of **damped oscillations** indicated in Figure 2c. Provided the damping is sufficiently light it is possible to describe this kind of oscillation in a similar way to the simple harmonic oscillation described above. Of course, the description is not exactly the same; the damping generally tends to reduce the angular frequency from ω_0 to some lower value ω, and causes the amplitude to become a (decreasing) function of time $A(t)$, but apart from these changes we can often describe the damped oscillations by a function of the form

$$x(t) = A(t)\sin(\omega t + \phi) \qquad (2)$$

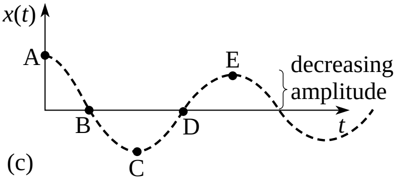
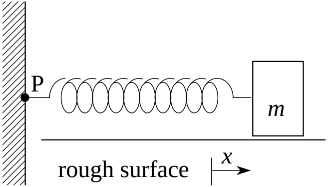


Figure 2c Effect of damping on the time–displacement graph for the system shown in Figure 1.

Now consider the system shown in Figure 3a, in which a body of mass m is attached to one end of a horizontal spring, the other end of which is attached to a fixed point P. The body can slide back and forth along a straight line, which we will again take to be the x-axis, but this time it is subject to an externally imposed force (acting along the x-axis) in addition to the force due to the spring and any dissipative force that may act. In this situation the externally imposed force is called a **driving force** and the oscillations that it helps to produce and sustain in the oscillator are called **forced** or **driven oscillations**. If the driving force varies sinusoidally with time, at angular frequency Ω, ☞ so that it may be described by an expression of the form $F_0 \sin(\Omega t)$, we will *eventually* find that the motion of the oscillating body under the influence of the driving force is described by



(a)

Figure 3a A mass subject to restoring, damping and driving forces.

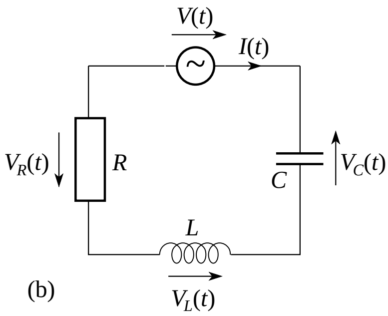
$$x(t) = A \sin(\Omega t - \delta) \quad (3)$$ ☞

where A and δ are constants whose values depend on the angular frequency of the driving force, Ω, and the characteristics of the oscillator, but are independent of time. The steady nature of the eventual motion shows that in this case work done by the driving force is somehow compensating the oscillator for the energy it loses due to dissipative effects.

❖ What physical interpretation can you give to the parameters A and δ that appear in Equation 3?

Oscillations, whether simple, damped or driven are not confined to mechanical systems. All sorts of physical systems exhibit oscillations. Temperatures may oscillate from day to day or season to season; concentrations of different chemicals may rise and fall in oscillating chemical reactions; electric charges may oscillate back and forth in appropriately constructed electrical circuits; and so on. The properties of electrical oscillations are particularly important and provide interesting analogies with mechanical oscillations. We will now briefly describe some of the situations in which electrical oscillations arise, and then investigate the reasons why such apparently different systems should exhibit such closely similar behaviour.

An electric **circuit** is a closed path around which electric charge may flow. A typical circuit, such as that of Figure 3b, contains a number of electrical components that assist or retard the flow of charge and thereby give the circuit its particular characteristics. In order for the flow of charge to occur at all there must generally be a *potential difference* between one part of the circuit and another; this is measured in *volts* (V, where $1\,\mathrm{V} = 1\,\mathrm{J\,C^{-1}}$) and is often referred to as a **voltage**. It might be supplied by a battery, but in the case of Figure 3b there is a **voltage generator**, shown by the symbol at the top of the diagram, which produces a time dependent potential difference $V(t)$ between its terminals. The instantaneous rate of flow of charge at any point in the circuit constitutes the instantaneous **current** $I(t)$ at that point, and may be measured in *amperes* (A, where $1\,\mathrm{A} = 1\,\mathrm{C\,s^{-1}}$). The conventional current direction is taken to be that of positive charge flow. The rest of the circuit shown in Figure 3b consists of a *resistor* (shown as the rectangle), a *capacitor* (shown as parallel bars) and an *inductor* (shown as the coil), connected in **series**, so that the same current flows through each component. Such circuits are called series LCR circuits.



(b)

Figure 3b A simple LCR circuit containing a resistor, a capacitor and an inductor connected in series. At the instant shown the current is increasing in the direction shown and the directions (polarity) of the voltages are shown by arrows.

A **resistor** is a component that dissipates energy. When there is a potential difference V_R across a resistor, a current will flow through it. The current and the voltage will be related by **Ohm's law**

$$V_R(t) = I(t)R \tag{4}$$

where R is a constant, called the **resistance** of the resistor, which may be measured in *ohms* (Ω, where 1 Ω = 1 V A^{-1}). ☞ The energy dissipated per second (i.e. the power dissipated) when a current I flows through a resistor of resistance R is

$$P = I^2R = V_R^2/R = IV_R \tag{5}$$

A **capacitor** is a device for storing electrical charge (and thereby storing energy in the associated *electric field*). The simplest such device consists of a pair of parallel metal plates placed a short distance from one another and the symbol used to represent the capacitor reflects this. When charged, these plates carry charges of equal magnitude but opposite sign, $+q$ and $-q$. The charge q is measured in *coulomb* (C, where 1 C = 1 A s), and is related to the voltage V_C across a capacitor by the relation

$$V_C(t) = \frac{q(t)}{C} \tag{6}$$

where the constant C that characterizes the capacitor is called its **capacitance** and is measured in *farad* (F, where 1 F = 1 A s V^{-1}).

An **inductor** is another device that can be used to store energy, in the *magnetic field* produced when a current flows through it. In the case of an inductor, the instantaneous voltage across the inductor is proportional to the *rate of change* of the instantaneous current through the inductor, hence

$$V_L(t) = L\frac{dI(t)}{dt} \tag{7}$$

where the constant L that characterizes the inductor is called its **inductance** and is measured in *henry* (H, where $1\,\mathrm{H} = 1\,\mathrm{V\,s\,A^{-1}}$). The direction or *polarity* of the induced voltage is always such as to oppose the change which causes it — this is known as **Lenz's law**.

The quantities V_R, V_C, V_L that have been introduced above have polarities and so may each be positive or negative. We must ensure that we understand the significance of the signs of these quantities. Figure 3b shows the situation at the instant where the capacitor charge is increasing on the upper plate and V_C is growing with the polarity shown. We are also illustrating the case where the current is *increasing* in the direction shown, so Lenz's law gives the polarity of V_L as being opposite to the increasing current direction.

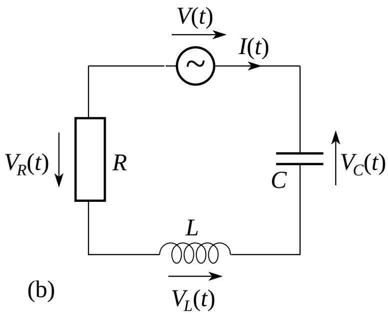


Figure 3b A simple LCR circuit containing a resistor, a capacitor and an inductor connected in series. At the instant shown the current is increasing in the direction shown and the directions (polarity) of the voltages are shown by arrows.

The voltage V_R has a polarity opposite to the direction of the current. Each of these instantaneous voltages and that of the generator are conveniently shown by arrows, with the arrows pointing in the direction of increasing positive voltage as shown. A positive value for the voltage across the capacitor implies that the upper plate is at a higher voltage than the lower plate and a positive value for V_L and V_R implies that in each case the end nearer to the generator is positive.

If we let q represent the instantaneous charge on the upper plate of the capacitor, then q will be positive when the capacitor voltage is positive. Moreover, if the instantaneous current is positive, then positive charge will be flowing onto the upper plate of the capacitor and q will be increasing, this tells us nothing about the sign of q but it does ensure that

$$I = \frac{dq}{dt} \tag{8}$$

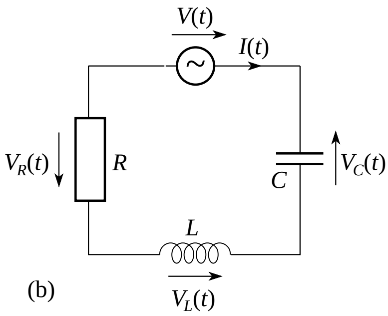


Figure 3b A simple LCR circuit containing a resistor, a capacitor and an inductor connected in series. At the instant shown the current is increasing in the direction shown and the directions (polarity) of the voltages are shown by arrows.

As you can see from the above discussion, the circuit shown in Figure 3b will be characterized by the relevant values of R, C and L. Given these three values and the specific form of the externally supplied voltage $V(t)$, it is possible to determine the current $I(t)$ that flows through the circuit, and the associated charge $q(t)$ on the upper plate of the capacitor. Interestingly the circuit turns out to be an electrical analogue of the driven mechanical oscillator shown in Figure 3a. In particular, if the external voltage is of the form $V(t) = V_0 \sin(\Omega t)$, then eventually, after any transient currents have died away:

$$q(t) = A \sin(\Omega t - \delta) \tag{9}$$

and, consequently $$I(t) = \frac{dq}{dt} = A\Omega \cos(\Omega t - \delta) \tag{10}$$

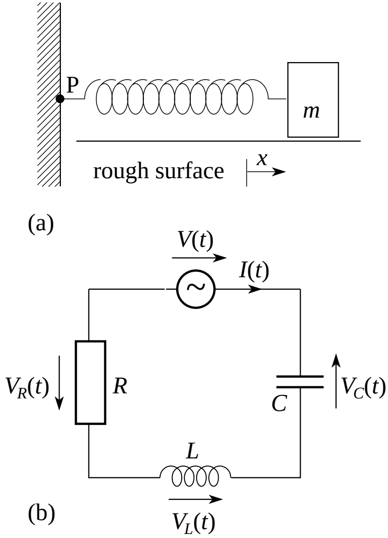


Figure 3 (a) A mass subject to restoring, damping and driving forces. (b) A simple LCR circuit containing a resistor, a capacitor and an inductor connected in series. At the instant shown the current is increasing in the direction shown and the directions (polarity) of the voltages are shown by arrows.

If you compare Equation 9 with Equation 3

$$q(t) = A\sin(\Omega t - \delta) \qquad \text{(Eqn 9)}$$

$$x(t) = A\sin(\Omega t - \delta) \qquad \text{(Eqn 3)}$$

you will see that they are both of the same form. It is in this sense that the charge oscillations in the series LCR circuit driven by an externally supplied sinusoidal voltage may be said to be analogous to the displacement oscillations of the mechanical oscillator driven by an externally supplied sinusoidal force.

In the next subsection you will see why these two very different physical systems give rise to essentially identical oscillatory phenomena. The essential point is that the underlying physics of both systems is described by very similar equations.

2.2 Mathematical models of mechanical and electrical oscillators

A driven mechanical oscillator

The oscillating body in Figure 3a is subject to three forces:

1 A *restoring force* F_{1x} due to the spring that tends to return the body to its equilibrium position. This will be taken to be

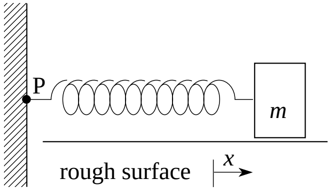
$$F_{1x}(t) = -kx(t) \qquad (11a)$$

where k is the positive *spring constant* that characterizes the spring.

2 A *damping force* F_{2x}, due to friction and air resistance, that opposes the motion of the body. We will assume that the magnitude of this force is proportional to the instantaneous velocity of the sliding body, so that

$$F_{2x}(t) = -b\frac{dx(t)}{dt} \qquad (11b)$$

where b is a positive constant that characterizes the dissipative forces.



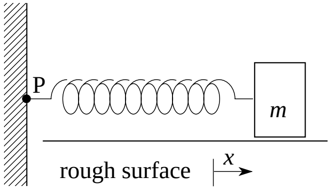
(a)

Figure 3a A mass subject to restoring, damping and driving forces.

3 A *driving force* F_{3x} provided by an external agency. We will assume that this force varies with time in a periodic way, and has the relatively simple form

$$F_{3x}(t) = F_0 \sin(\Omega t) \tag{11c}$$

where F_0 is the maximum magnitude that the driving force attains, and Ω is the angular frequency of the driving force. Note that the angular frequency Ω is externally imposed and is not necessarily related in any way to the natural frequency of the system in Figure 3a.



(a)

Figure 3a A mass subject to restoring, damping and driving forces.

Using *Newton's second law of motion* we can therefore say that the sliding body must obey an *equation of motion* of the form

$$m\frac{d^2x(t)}{dt^2} = F_{1x}(t) + F_{2x}(t) + F_{3x}(t)$$

so, in this case

$$m\frac{d^2x(t)}{dt^2} = -kx(t) - b\frac{dx(t)}{dt} + F_0 \sin(\Omega t) \tag{12}$$

which can be rearranged to isolate the time-dependent driving term as follows:

$$m\frac{d^2x(t)}{dt^2} + b\frac{dx(t)}{dt} + kx(t) = F_0\sin(\Omega t) \qquad (13)$$

This is known as the equation of motion of a **harmonically driven linearly damped oscillator**. The function $x(t) = A\sin(\Omega t - \delta)$ that we introduced in Subsection 2.1 (Equation 3) to describe the steady state behaviour of the driven oscillator is a solution of this equation, provided we choose A and δ appropriately, as we will demonstrate in the next subsection.

A series LCR circuit

In order to determine the differential equation that describes the behaviour of the series LCR circuit of Figure 3b we need to introduce two basic principles of circuit analysis (based on *Kirchhoff's laws*):

1. In a series circuit the instantaneous current $I(t)$ through each component is the same. The physical basis for this is the principle of the *conservation of electric charge*.
2. In a series circuit the sum of the instantaneous voltages across each passive component ☞ is equal to the externally supplied voltage $V(t)$. The physical basis for this is the principle of the *conservation of energy.*

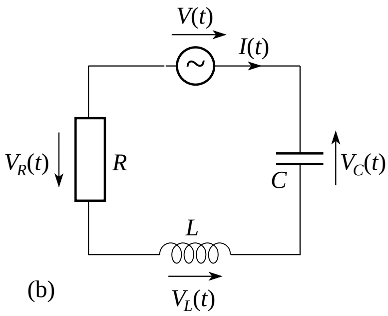


Figure 3b A simple LCR circuit containing a resistor, a capacitor and an inductor connected in series. At the instant shown the current is increasing in the direction shown and the directions (polarity) of the voltages are shown by arrows.

Now, we already know from Equations 4, 6 and 7 that

$$V_R(t) = I(t)R \qquad \text{(Eqn 4)}$$

$$V_C(t) = \frac{q(t)}{C} \qquad \text{(Eqn 6)}$$

$$V_L(t) = L\frac{dI(t)}{dt} \qquad \text{(Eqn 7)}$$

So we can use the second of the two principles given above (*conservation of energy.*) to write

$$V(t) = V_R(t) + V_C(t) + V_L(t)$$

Using the first principle (*conservation of electric charge*.), together with Equations 4, 6 and 7, this gives us

$$V(t) = RI(t) + \frac{q(t)}{C} + L\frac{dI(t)}{dt} \qquad (14)$$

However, we also know that the current in the circuit is given by the rate of change of the charge q on the upper plate of the capacitor, so we can write

$$I(t) = \frac{dq(t)}{dt} \qquad \text{(Eqn 8)}$$

and hence $$\frac{dI(t)}{dt} = \frac{d^2q(t)}{dt^2} \qquad (15)$$

Substituting Equations 8 and 15 into Equation 14

$$V(t) = RI(t) + \frac{q(t)}{C} + L\frac{dI(t)}{dt} \qquad \text{(Eqn 14)}$$

we see that

$$V(t) = R\frac{dq(t)}{dt} + \frac{1}{C}q(t) + L\frac{d^2q(t)}{dt^2}$$

which may be rearranged to give

$$L\frac{d^2q(t)}{dt^2} + R\frac{dq(t)}{dt} + \frac{1}{C}q(t) = V(t) \tag{16}$$

Finally, substituting the relevant expression for $V(t)$ we obtain

$$L\frac{d^2q(t)}{dt^2} + R\frac{dq(t)}{dt} + \frac{1}{C}q(t) = V_0\sin(\Omega t) \tag{17}$$

Now, if you compare Equations 13 and 17

$$m\frac{d^2x(t)}{dt^2} + b\frac{dx(t)}{dt} + kx(t) = F_0\sin(\Omega t) \quad \text{(Eqn 13)}$$

you will see that they have the same form. One may be obtained from the other by making the following substitutions:

q	$\Leftrightarrow$	x	L	$\Leftrightarrow$	m	☞		
R	$\Leftrightarrow$	b	$1/C$	$\Leftrightarrow$	k	V_0	$\Leftrightarrow$	F_0

Note that we are not claiming some sort of mystical link between charge and displacement, or inductance and mass, but simply drawing attention to the fact that the two very different physical systems can both be described by similar equations. It is the mathematical model that is the same in both cases, not the system it is representing.

Equations 13 and 17

$$m\frac{d^2x(t)}{dt^2} + b\frac{dx(t)}{dt} + kx(t) = F_0 \sin(\Omega t) \qquad \text{(Eqn 13)}$$

$$L\frac{d^2q(t)}{dt^2} + R\frac{dq(t)}{dt} + \frac{1}{C}q(t) = V_0 \sin(\Omega t) \qquad \text{(Eqn 17)}$$

are examples (essentially the same example) of *second-order differential equations* *with constant coefficients*. (From a mathematical point of view they are also *linear* and *inhomogeneous*). Solving such equations is an inherently mathematical process, but it is of great interest to physicists since the possible solutions include the various forms of harmonic motion that were described in Subsection 2.1.

❖ By differentiating Equation 16,

$$L\frac{d^2q(t)}{dt^2} + R\frac{dq(t)}{dt} + \frac{1}{C}q(t) = V(t) \qquad \text{(Eqn 16)}$$

with respect to time, show that the instantaneous current in a series LCR circuit obeys a differential equation similar to that satisfied by the charge $q(t)$.

2.3 Second-order differential equations — a brief review

This subsection describes the mathematical principles involved in solving *second-order differential equations with constant coefficients*. This topic is discussed from the same point of view, but in greater detail, in the maths strand of *FLAP*.

The general linear second-order differential equation with constant coefficients is of the form

$$a\frac{d^2x(t)}{dt^2} + b\frac{dx(t)}{dt} + cx(t) = f(t) \qquad (20)$$

☞

where a, b and c are constants and $f(t)$ is independent of x.

If $f(t) = 0$ for all values of t the equation is said to be **homogeneous**, otherwise the equation is said to be **inhomogeneous**. The equation is said to be **linear** because the dependent variable x only appears once, and only to the first power, in each of the terms that involves it at all — there are no terms involving x^2 or $(dx/dt)^2$ or $x(dx/dt)$ or anything else of that kind. The equation is **second order** because it involves no derivative of x higher than the second derivative.

As you can see, Equations 13 and 17

$$m\frac{d^2x(t)}{dt^2} + b\frac{dx(t)}{dt} + kx(t) = F_0\sin(\Omega t) \qquad \text{(Eqn 13)}$$

$$L\frac{d^2q(t)}{dt^2} + R\frac{dq(t)}{dt} + \frac{1}{C}q(t) = V_0\sin(\Omega t) \qquad \text{(Eqn 17)}$$

are both of this general form.

$$a\frac{d^2x(t)}{dt^2} + b\frac{dx(t)}{dt} + cx(t) = f(t) \qquad \text{(Eqn 20)}$$

In the cases that are of interest to us the constants a, b, and c are all positive, and the function $f(t)$ corresponds to the external driving term.

When confronted with an equation such as Equation 20 our usual aim is to find its **general solution**. For such a second-order differential equation this general solution expresses x in terms of t, the given constants that appear in the equation and two additional **arbitrary constants**. The values of the arbitrary constants cannot be determined from Equation 20 itself but must be found from supplementary conditions such as the initial values of x and its derivative, $x(0)$ and $\dfrac{dx(0)}{dt}$. These supplementary conditions are generally referred to as **boundary conditions** or **initial conditions** as appropriate.

In the case of Equation 20

$$a\frac{d^2x(t)}{dt^2} + b\frac{dx(t)}{dt} + cx(t) = f(t) \qquad \text{(Eqn 20)}$$

the general solution is the sum of two parts, a **particular solution** $x_p(t)$, which may be any solution of Equation 20 that does not contain arbitrary constants, and a **complementary function** $x_c(t)$ which does contain two arbitrary constants, and which satisfies the corresponding homogeneous equation

$$a\frac{d^2x(t)}{dt^2} + b\frac{dx(t)}{dt} + cx(t) = 0 \qquad (21)$$

Question T1

Show that $x(t) = x_c(t) + x_p(t)$ will satisfy Equation 20 and will contain two arbitrary constants, as the general solution should. ❏

We will now outline the procedure for determining the complementary function in any give case, and then comment on the determination of a particular solution.

Finding the complementary function

1 Given the values of a, b and c in Equation 21 write down the so-called **auxiliary equation**

$$ap^2 + bp + c = 0$$

2 Find the solutions p_1 and p_2 (of this quadratic equation, see Question R3).

3 (a) If $b^2 > 4ac$ the solutions will be two different real numbers and the comp-lementary function will be

$$x_c(t) = B\exp(p_1t) + D\exp(p_2t) \tag{22}$$

(b) If $b^2 < 4ac$ the solutions will be two different complex numbers which may be written

$$p_1 = -2\gamma + i\omega \quad \text{and} \quad p_2 = -2\gamma - i\omega$$

where $\gamma = b/a$ and $\omega = \sqrt{\dfrac{c}{a} - \dfrac{\gamma^2}{4}}$ (23)

and the complementary function will be

$$x_c(t) = e^{-\gamma t/2}[E\cos(\omega t) + G\sin(\omega t)] \tag{24}$$

(c) If $b^2 = 4ac$ the solutions will be two equal real numbers $p_1 = p_2 = -b/2a = -\gamma/2$, and the complementary function will be

$$x_c(t) = (H + Jt)e^{-\gamma t/2} \tag{25}$$

Finding a particular solution

Determining a particular solution is generally much more difficult, and usually comes down to educated guesswork. However, in the cases that will be of interest to us in this module the driving term $f(t)$ will usually have the general form

$$f(t) = f_0 \sin(\Omega t) \tag{26}$$

and the particular solution will have the corresponding form

$$x_{\rm p}(t) = A \sin(\Omega t - \delta) \tag{27}$$

where $$A = \frac{f_0}{\sqrt{(c/a - \Omega^2)^2 + (\gamma\Omega)^2}} \quad \text{and} \quad \delta = \arctan\left(\frac{\gamma\Omega}{c/a - \Omega^2}\right) \tag{28}$$ ☞

Note that the constants A and δ appearing in the particular solution are *not* arbitrary constants; their values are determined by the given values of a, b, c, f_0 and Ω and not by any initial or boundary conditions.

Note also that for the homogeneous equation the particular solution can always be set equal to zero since f_0 can then be set equal to zero, so $A = 0$.

Using Equations 22 to 28 it is now possible to solve a wide a range of oscillatory problems.

Example 1 Write down the complementary function, a particular solution, and finally the general solution of the differential equation

$$L\frac{d^2q(t)}{dt^2} + R\frac{dq(t)}{dt} + \frac{q(t)}{C} = V_0 \sin(\Omega t) \tag{29}$$

when $L = 1.0\,\text{H}$, $R = 5.0\,\Omega$, $C = 1/6\,\text{F}$, $V_0 = 0.6\,\text{V}$ and $\Omega = 5.0\,\text{s}^{-1}$.

Solution Comparing Equation 29 with Equation 20,

$$L\frac{d^2q(t)}{dt^2} + R\frac{dq(t)}{dt} + \frac{q(t)}{C} = V_0\sin(\Omega t) \qquad \text{(Eqn 29)}$$

$$a\frac{d^2x(t)}{dt^2} + b\frac{dx(t)}{dt} + cx(t) = f(t) \qquad \text{(Eqn 20)}$$

and identifying a with L, b with R and c with $1/C$, we see that in this case $b^2 > 4ac$. Solving the auxiliary equation leads to $p_1 = -2\,\mathrm{s}^{-1}$ and $p_2 = -3\,\mathrm{s}^{-1}$ so the complementary function takes the form of Equation 22

$$x_{\rm c}(t) = B\exp(p_1t) + D\exp(p_2t) \qquad \text{(Eqn 22)}$$

and is given by

$$q_{\rm c}(t) = B\exp[-(2\,\mathrm{s}^{-1})t] + D\exp[-(3\,\mathrm{s}^{-1})t]$$

where B and D are arbitrary constants.

A particular solution is $q_p(t) = A\sin[(5\,s^{-1})t - \delta]$

where, from Equation 28,

$$A = \frac{f_0}{\sqrt{(c/a - \Omega^2)^2 + (\gamma\Omega)^2}} \text{ and } \delta = \arctan\left(\frac{\gamma\Omega}{c/a - \Omega^2}\right) \qquad \text{(Eqn 28)}$$

$A = 1.9 \times 10^{-2}\,C$ and $\delta = -0.92$. ☞

The general solution of the differential equation is therefore

$$q(t) = q_c(t) + q_p(t) = B\exp[-(2\,s^{-1})t] + D\exp[-(3\,s^{-1})t] + (1.9 \times 10^{-2}\,C)\sin[(5\,s^{-1})t + 0.92]$$

❑

Note that in this case the charge $q(t)$ and hence $V_c(t)$ lead the applied voltage $V_0 = A\sin(\Omega t)$.

The constants B and D in the above example are determined by the initial state of the system, i.e. the initial charge on the capacitor and the initial current. In practice however their values are usually unimportant, as the following question invites you to show.

✥ Suppose that in Example 1 we have $B = 0.6\,\mathrm{C}$ and $D = 0.6\,\mathrm{C}$. Calculate the values of $q(t)$, $q_c(t)$ and $q_p(t)$ when $t = 0\,\mathrm{s}$, $t = 1\,\mathrm{s}$, $t = 3\,\mathrm{s}$, $t = 5\,\mathrm{s}$ and $t = 8\,\mathrm{s}$. What do you notice about the values of $q(t)$, $q_c(t)$ and $q_p(t)$ as t increases? What part do B and D play in determining the eventual behaviour of $q(t)$?

Figure 4 shows the graphs of $q_p(t)$ (the dashed sinusoidal curve) and $q(t) = q_c(t) + q_p(t)$ (the solid curve). Initially they are quite different, but for $t > 4\,\text{s}$ they are indistinguishable. Because of this behaviour $q_p(t)$ is said to represent the **steady state** behaviour, whereas $q_c(t)$ is said to represent the **transient** behaviour.

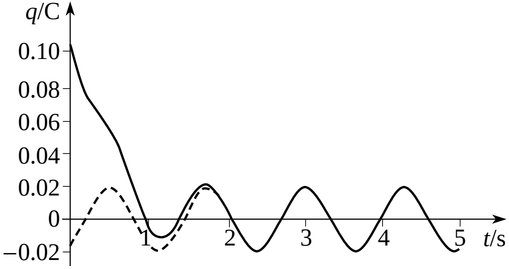


Figure 4 The graphs of $q(t) = q_c(t) + q_p(t)$ (solid curve) and $q_p(t)$ (dashed curve) in Example 1.

The important points to notice from Example 1 are:

- the transient part of the general solution is insignificant for large values of t, so that eventually the steady state term $A\sin(\Omega t - \delta)$ will dominate the solution;
- the constants B, D, E, G, H and J determine the initial state of the system, but if we are only interested in what happens for large values of t their values are irrelevant;
- the steady state term $A\sin(\Omega t - \delta)$ is a sinusoidal function with the same angular frequency as the driving term $V(t)$.

An important consequence

If we start to drive the mechanical oscillator shown in Figure 3a, or close a switch so as to complete the circuit shown in Figure 3b, there will be a short period of time when the transient behaviour is significant, but in most practical situations the behaviour rapidly moves to a steady state in which the oscillation is sinusoidal. If we are only interested in the steady state, which is usually the case, then we can totally ignore the transient term in the solution of the differential equation. Even more significantly, if we are only interested in the steady state behaviour, we can often abandon this approach entirely and avoid the difficult business of solving differential equations altogether. In Section 3 we will introduce a much simpler method of analysing harmonically driven oscillators based on the *assumption* that the oscillation is sinusoidal.

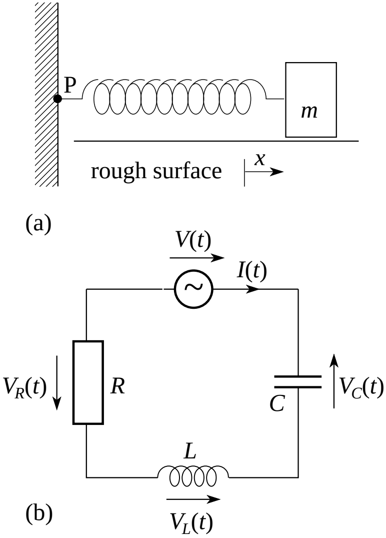


Figure 3 (a) A mass subject to restoring, damping and driving forces. (b) A simple LCR circuit containing a resistor, a capacitor and an inductor connected in series. At the instant shown the current is increasing in the direction shown and the directions (polarity) of the voltages are shown by arrows.

Question T2

A mechanical oscillator satisfies the differential equation

$$a\frac{d^2x(t)}{dt^2} + b\frac{dx(t)}{dt} + cx(t) = 0$$

where $a = 1\,\mathrm{s}^2$, $b = 4\,\mathrm{s}$ and $c = 3$. Write down the auxiliary equation and solve it. Hence find the general solution of this (homogeneous) differential equation. ❏

2.4 Harmonic oscillations: simple, damped and driven

In this subsection we consider some special cases of the second-order linear differential equation

$$a\frac{d^2x(t)}{dt^2} + b\frac{dx(t)}{dt} + cx(t) = f_0 \sin(\Omega t) \tag{30}$$

Each of the cases we consider will correspond to a particular kind of oscillatory behaviour, and may be applied to mechanical or electrical systems (or any other system that may be similarly modelled).

b = 0, f_0 = 0; the case of simple harmonic oscillation

In this case Equation 30 may be written in the form

$$a\frac{d^2x(t)}{dt^2} + cx(t) = 0 \quad \text{where } a \neq 0$$

It is conventional to rewrite this by dividing both sides by *a* and introducing

the *natural angular frequency* $\omega_0 = \sqrt{c/a}$ (31)

so that $$\frac{d^2x(t)}{dt^2} + \omega_0^2 x(t) = 0 \tag{32}$$

In this (homogeneous) case a particular solution is $x_p(t) = 0$. Moreover, $b^2 < 4ac$, so the complementary function, and hence the general solution, takes the form of Equation 24 with $\gamma = b/c = 0$, and $\omega = \sqrt{c/a} = \omega_0$

i.e. $x(t) = E\cos(\omega_0 t) + G\sin(\omega_0 t)$ (33)

✧ Show that this solution can be written in the equivalent form

$$x(t) = A_0 \sin(\omega_0 t + \phi) \qquad (34)$$

and hence confirm that it can be used to represent simple harmonic motion with amplitude A_0, angular frequency $\omega_0 = \sqrt{c/a}$ and phase constant ϕ.

Question T3

A simple series circuit consists of a capacitor connected in series with an inductor. If the charge on the capacitor at time $t = 0$ is q_0, and there is no current in the circuit at that time, determine the differential equation that describes the variation of q with time, write down its general solution, and show that the charge exhibits simple harmonic oscillations with angular frequency $\omega_0 = \sqrt{1/(LC)}$. ❏

f_0 = 0; the case of linearly damped oscillation

In this case Equation 30

$$a\frac{d^2x(t)}{dt^2}+b\frac{dx(t)}{dt}+cx(t)=f_0\sin(\Omega t) \qquad \text{(Eqn 30)}$$

may be written in the form

$$a\frac{d^2x(t)}{dt^2}+b\frac{dx(t)}{dt}+cx(t)=0 \qquad (35)$$

It is conventional to rewrite this by dividing both sides by a and introducing

the *damping constant* $\gamma = b/a$ (36a)

and the *natural angular frequency* $\omega_0=\sqrt{c/a}$ (36b)

so that $$\frac{d^2x(t)}{dt^2}+\gamma\frac{dx(t)}{dt}+\omega_0^2x(t)=0 \qquad (36c)$$

In this case a particular solution is $x_p(t) = 0$, and the complementary function may take any of the forms described by Equations 22, 24 or 25,

$$x_c(t) = B\exp(p_1t) + D\exp(p_2t) \qquad \text{(Eqn 22)}$$

$$x_c(t) = e^{-\gamma t/2}[E\cos(\omega t) + G\sin(\omega t)] \qquad \text{(Eqn 24)}$$

$$x_c(t) = (H + Jt)e^{-\gamma t/2} \qquad \text{(Eqn 25)}$$

depending on the values of γ and ω_0.

(a) If $\gamma^2 > 4\omega_0^2$ the oscillator is said to be **overdamped** and the general solution has the form

$$x(t) = B\exp(p_1 t) + C\exp(p_2 t) \tag{37}$$

where $$p_1 = \frac{-\gamma}{2} + \sqrt{\frac{\gamma^2}{4} - \omega_0^2} \quad \text{and} \quad p_2 = \frac{-\gamma}{2} - \sqrt{\frac{\gamma^2}{4} - \omega_0^2} \tag{38}$$

(b) If $\gamma^2 < 4\omega_0^2$ the oscillator is said to be **underdamped** and the general solution has the form

$$x(t) = e^{-\gamma t/2}[E\cos(\omega t) + G\sin(\omega t)] \tag{39}$$

where $$\omega = \sqrt{\omega_0^2 - \frac{\gamma^2}{4}} \tag{40}$$

(c) If $\gamma^2 = 4\omega_0^2$ the oscillator is said to be **critically damped** and the general solution has the form

$$x(t) = (H + Jt)\,e^{-\gamma t/2} \tag{41}$$

☞

Some typical examples of damped oscillatory behaviour are shown in Figure 5. Notice that only in the underdamped case is it mathematically justified to claim that there is a well defined angular frequency associated with the oscillation since only then is at least one full cycle completed. Also notice that the vibrations decrease more rapidly as the value of b (and hence γ) is increased. In practical situations there is always resistance in a circuit, or friction in a mechanical system, so we are generally justified in assuming that $b > 0$.

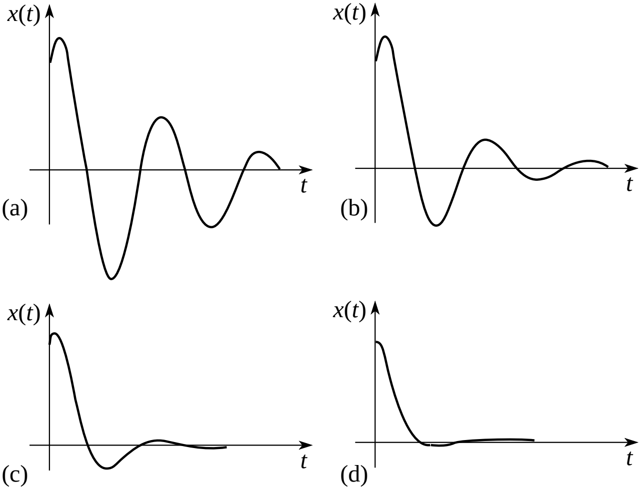


Figure 5 Oscillatory behaviour as the damping increases. from a to d

Question T4

Show that in the case of underdamped oscillations, the general solution may be written in the form

$$x(t) = e^{-\gamma t/2}[A_0 \sin(\omega t + \phi)] \qquad (42)$$ ❏

The general case of harmonically driven linearly damped oscillation

In this case we take Equation 30

$$a\frac{d^2x(t)}{dt^2} + b\frac{dx(t)}{dt} + cx(t) = f_0\sin(\Omega t) \qquad \text{(Eqn 30)}$$

and, as with the undriven case, it is conventional to rewrite this by dividing both sides by a and introducing

the *damping constant* $\gamma = b/a$ (43)

the *natural angular frequency* $\omega_0 = \sqrt{c/a}$ (44)

and $a_0 = f_0/a$ (45)

so that $$\frac{d^2x(t)}{dt^2} + \gamma\frac{dx(t)}{dt} + \omega_0^2 x(t) = a_0\sin(\Omega t) \qquad (46)$$

In this case the general solution is the sum of a transient term (a complementary function) given by the appropriate solution to the linearly damped oscillator equation, and a steady state term (particular solution) as given by Equations 27 and 28

$$x_p(t) = A\sin(\Omega t - \delta) \qquad \text{(Eqn 27)}$$

$$A = \frac{f_0}{\sqrt{(c/a - \Omega^2)^2 + (\gamma\Omega)^2}} \text{ and } \delta = \arctan\left(\frac{\gamma\Omega}{c/a - \Omega^2}\right) \qquad \text{(Eqn 28)}$$

with f_0 replaced by a_0.

Thus, in the physically important case of underdamping

$$x(t) = e^{-\gamma t/2}[A_0\sin(\omega t + \phi)] + A\sin(\Omega t - \delta) \qquad (47)$$

where A_0 and ϕ are arbitrary constants, $\omega = \sqrt{\omega_0^2 - \dfrac{\gamma^2}{4}}$ (48)

$$A = \frac{a_0}{\sqrt{(\omega_0^2 - \Omega^2)^2 + (\gamma\Omega)^2}} \text{ and } \delta = \arctan\left(\frac{\gamma\Omega}{\omega_0^2 - \Omega^2}\right) \qquad (49)$$

Note that as t increases the relative importance of the first term on the right in Equation 47

$$x(t) = e^{-\gamma t/2}[A_0 \sin(\omega t + \phi)] + A \sin(\Omega t - \delta) \qquad \text{(Eqn 47)}$$

decreases, so in the steady state the solution is effectively

$$x(t) = A \sin(\Omega t - \delta) \qquad (50)$$

where Ω is the driving angular frequency, and δ is the extent to which the phase of the oscillator lags behind that of its driver. Once again, note that A and δ are *not* arbitrary constants but are determined by the given values of γ, ω_0, Ω and a_0.

✦ A series circuit of negligible total resistance consists of a switch, a 2.4 V battery, a capacitor of capacitance 0.01 F and an inductor of inductance of 5.0 H. Write down the differential equation that determines the current $I(t)$ at a time t after the circuit is completed. Write down an expression for $I(t)$ assuming that the capacitor is initially uncharged.

2.5 Electrical impedance

Figure 6 shows three simple circuits. Each contains a single component, and is driven by an externally applied voltage $V(t) = V_0 \sin(\Omega t)$. We will now determine the steady state current that flows in each of these circuits.

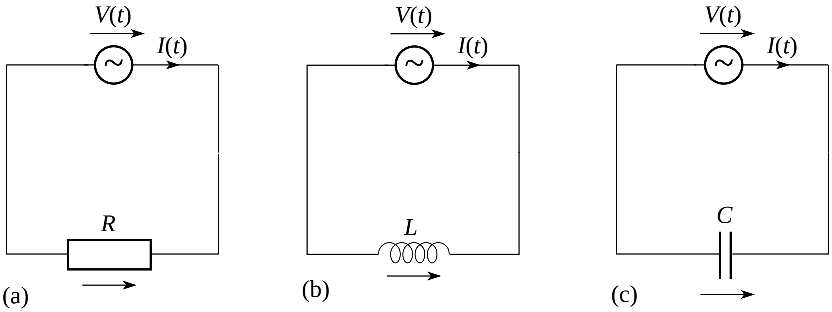


Figure 6 Some simple circuits.

(a) For the circuit containing the resistor (Figure 6a), Equation 14

$$V(t) = RI(t) + \frac{q(t)}{C} + L\frac{dI(t)}{dt} \qquad \text{(Eqn 14)}$$

gives

$$\underbrace{V(t)}_{\substack{\text{potential difference}\\ \text{supplied by the}\\ \text{voltage generator}}} = \underbrace{RI(t)}_{\substack{\text{potential difference}\\ \text{across the resistor}}}$$

so that $V_0 \sin(\Omega t) = RI(t)$ and therefore

$$I(t) = \frac{V_0}{R}\sin(\Omega t) \qquad (57)$$

Notice that the current $I(t)$ and the applied voltage are in phase in this case.

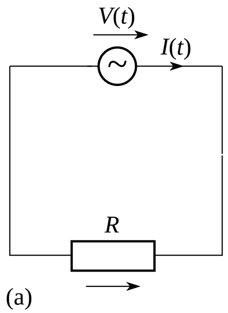


Figure 6a A simple circuit containing a resistor

(b) For the circuit containing the inductor (Figure 6b), Equation 14

$$V(t) = RI(t) + \frac{q(t)}{C} + L\frac{dI(t)}{dt} \tag{Eqn 14}$$

gives

$$\underbrace{V(t)}_{\substack{\text{potential difference}\\\text{supplied by the}\\\text{voltage generator}}} = \underbrace{L\frac{dI(t)}{dt}}_{\substack{\text{potential difference}\\\text{across the inductor}}}$$

so that $V_0 \sin(\Omega t) = L\dfrac{dI(t)}{dt}$ implying that $\dfrac{dI(t)}{dt} = \dfrac{V_0}{L}\sin(\Omega t)$

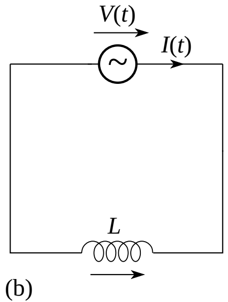


Figure 6b A simple circuit containing an inductor

Hence, in the steady state integration gives us

$$I(t) = -\frac{V_0}{\Omega L}\cos(\Omega t) = \frac{V_0}{\Omega L}\sin(\Omega t - \pi/2) \qquad (58)$$

Comparing Equations 58 and 57

$$I(t) = \frac{V_0}{R}\sin(\Omega t) \qquad \text{(Eqn 57)}$$

we see that an inductor behaves rather like a resistor with effective resistance ΩL, but the phase of the current lags that of the voltage by $\pi/2$.

(c) For the circuit containing the capacitor (Figure 6c), Equation 19

$$L\frac{d^2I(t)}{dt^2} + R\frac{dI(t)}{dt} + \frac{1}{C}I(t) = \frac{dV(t)}{dt} \tag{Eqn 19}$$

gives

$$\frac{I(t)}{C} = \frac{dV(t)}{dt}$$

so that

$$I(t) = C\frac{dV(t)}{dt} = \Omega C V_0\cos(\Omega t) = \frac{V_0}{1/(\Omega C)}\sin(\Omega t + \pi/2) \tag{59}$$

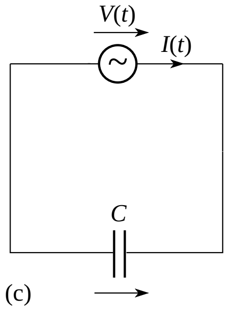


Figure 6c A simple circuit containing a capacitor.

Comparing Equations 59 and 57

$$I(t) = \frac{V_0}{R}\sin(\Omega t) \tag{Eqn 57}$$

we see that a capacitor behaves rather like a resistor with effective resistance $1/(\Omega C)$, but the phase of the current leads that of the voltage by $\pi/2$.

The frequency dependent ‘effective resistance’ of an inductor, $X_L = \Omega L$ is known as its **inductive reactance**, while the corresponding quantity for a capacitor, $X_C = 1/(\Omega C)$, is known as its **capacitive reactance**. These quantities, together with the resistance R of a resistor, play an important part in determining the current $I(t)$ that will flow through a component when a voltage $V(t)$ is applied across it. We can summarize these relationships contained in Equations 57, 58 and 59

$$I(t) = \frac{V_0}{R}\sin(\Omega t) \qquad \text{(Eqn 57)}$$

$$I(t) = -\frac{V_0}{\Omega L}\cos(\Omega t) = \frac{V_0}{\Omega L}\sin(\Omega t - \pi/2) \qquad \text{(Eqn 58)}$$

$$I(t) = C\frac{dV(t)}{dt} = \Omega C V_0\cos(\Omega t) = \frac{V_0}{1/(\Omega C)}\sin(\Omega t + \pi/2) \qquad \text{(Eqn 59)}$$

as follows.

If $V(t) = V_0\sin(\Omega t)$ and $I(t) = I_0\sin(\Omega t - \delta)$, then;

for an inductor	$V_0/I_0 = X_L = \Omega L$	and	$\delta = \pi/2$
for a resistor	$V_0/I_0 = R$	and	$\delta = 0$
for a capacitor	$V_0/I_0 = X_C = 1/(\Omega C)$	and	$\delta = -\pi/2$ ☞

The ability of inductors and capacitors to ‘react’ to an applied voltage by altering their ‘effective resistance’ according to its frequency is part of the reason for their significance in electronics. In particular, it allows them to play an important role in *filter circuits* designed to pass signals (varying voltages) in certain frequency ranges while inhibiting the passage of others. The differences in behaviour between resistors, capacitors and inductors mean that appropriately designed combinations of these components can be used to manipulate signals in a variety of ways.

The analysis of circuits in terms of the differential equations that represent them is a sophisticated study in its own right, but provided we are only concerned with the steady state behaviour of networks of passive components (resistors, capacitors and inductors), driven by applied voltages that vary sinusoidally with time, the subject can be greatly simplified. As an example we will state without proof four more results for simple series circuits. (These can be established by finding particular solutions for the appropriate versions of Equation 19.)

$$L\frac{d^2I(t)}{dt^2} + R\frac{dI(t)}{dt} + \frac{1}{C}I(t) = \frac{dV(t)}{dt} \qquad \text{(Eqn 19)}$$

In each case the result consists of a description of the current $I(t) = I_0 \sin(\Omega t - \delta)$ that flows in response to an applied voltage $V(t) = V_0 \sin(\Omega t)$, and in each case this requires that we provide an explicit expression relating I_0 to the known quantities V_0, R, C and L. In order to do this each result provides an explicit expression for the quantity $Z = V_0/I_0$, which is known as the **impedance** of the circuit. The impedance is measured in ohm (Ω), and represents a generalization of resistance and reactance.

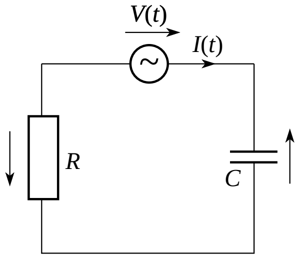
Resistor and capacitor in series

For the circuit shown in Figure 7a:

If $V(t) = V_0 \sin(\Omega t)$ then $I(t) = \dfrac{V_0}{Z}\sin(\Omega t - \delta)$

where $$Z^2 = R^2 + \frac{1}{(\Omega C)^2} \text{ and } \tan\delta = \frac{-1}{\Omega CR} \qquad (60)$$

✥ Given that $V_0 = 3\,\text{V}$, $\Omega = 5\,\text{s}^{-1}$, $R = 2\,\Omega$ and $C = 0.2\,\text{F}$ for the circuit shown in Figure 7a, calculate the current $I(t)$.



(a)

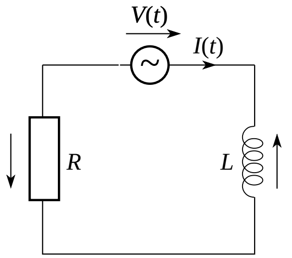
Figure 7a Circuits with two components: resistor and capacitor in series.

Resistor and inductor in series

For the circuit shown in Figure 7b:

If $V(t) = V_0 \sin(\Omega t)$ then $I(t) = \dfrac{V_0}{Z}\sin(\Omega t - \delta)$

where $Z^2 = R^2 + (\Omega L)^2$ and $\tan\delta = \Omega L/R$ (61)



(b)

Figure 7b Circuits with two components: resistor and inductor in series.

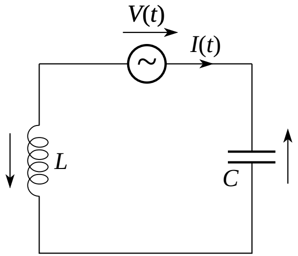
Inductor and capacitor in series

For the circuit shown in Figure 7c:

If $V(t) = V_0 \sin(\Omega t)$ then $$I(t) = \frac{V_0}{Z}\sin(\Omega t - \delta)$$

where

$$Z = \left|\frac{1}{\Omega C} - \Omega L\right| \text{ and } \delta = \pi/2 \text{ if } \frac{1}{\Omega C} < \Omega L\text{, or } \delta = -\pi/2 \text{ if } \frac{1}{\Omega C} > \Omega L \qquad (62)$$



(c)

Figure 7c Circuits with two components: inductor and capacitor in series.

Resistor, inductor and capacitor in series

For the circuit shown in Figure 3b:

If $V(t) = V_0 \sin(\Omega t)$

then $$I(t) = \frac{V_0}{Z}\sin(\Omega t - \delta) \quad (63a)$$

where $$Z^2 = R^2 + \left(\frac{1}{\Omega C} - \Omega L\right)^2 \text{ and } \tan\delta = \frac{1}{R}\left(\Omega L - \frac{1}{\Omega C}\right) \quad (63b)$$

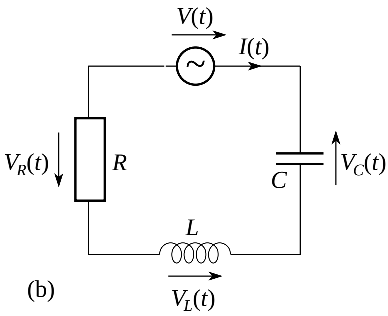


Figure 3b A simple LCR circuit containing a resistor, a capacitor and an inductor connected in series. At the instant shown the current is increasing in the direction shown and the directions (polarity) of the voltages are shown by arrows.

The above results can be neatly summarized and generalized if we introduce a total reactance $X = X_L - X_C$, for we can then say:

When a voltage $V(t) = V_0 \sin(\Omega t)$ is applied across any series circuit of components of total resistance R and total reactance X, the resulting steady state current will have the form $I(t) = I_0 \sin(\Omega t - \delta)$

where $V_0 = I_0 Z$ and $\delta = \arctan(X/R)$

and the impedance Z is given by $Z = \sqrt{R^2 + X^2}$.

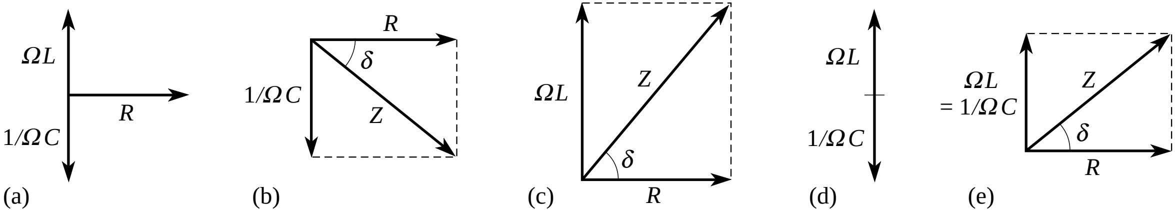


Figure 8 Geometrical interpretations of Z and δ in terms of X_L, X_C and R.

It is possible to give a simple geometric interpretation to the relationship between impedance, resistance and reactance. This is indicated in Figure 8a, where X_L is treated as a 'vector quantity' directed vertically upwards, X_C is treated as a vector directed vertically downwards, and R is treated as a vector directed to the right. (The length of each 'vector' represents the relevant value of resistance or reactance.)

As Figures 8b to 8e indicate, the value of Z in each of the cases discussed above will be represented by the length of the 'vector sum' of X_L, X_C and R, and the value of δ will be given by the angle (measured in the anticlockwise direction) from the horizontal axis to the 'vector' representing Z ☞. We will return to this geometric interpretation of impedance in Section 3.

Example 2 Calculate the impedance of a circuit which consists of a resistor of 10 Ω, a capacitor of 0.05 F and an inductor of 2.0 H in series. If a voltage $V(t) = 3\sin[(2\,\text{s}^{-1})t]$ V is applied to the circuit, then by how much do the current and voltage differ in phase in the steady state? Write down an expression for the steady state current.

Solution From Equation 63b $Z^2 = R^2 + \left(\dfrac{1}{\Omega C} - \Omega L\right)^2$

so in this case, with $\Omega = 2$ Hz,

$$Z = \sqrt{10^2 + \left(\frac{1}{0.05\times 2} - 2\times 2\right)^2}\;\Omega \approx 11.66\,\Omega$$

and $$\tan\delta = \frac{1}{R}\left(\Omega L - \frac{1}{\Omega C}\right) = \frac{1}{10}\left(2\times 2 - \frac{1}{2\times 0.05}\right) = -0.60$$

so $\delta \approx -0.54$.

Thus $$I(t) = I_0\sin(\Omega t - \delta) = \frac{3}{11.66}\sin[(2\,\text{s}^{-1})t + 0.54]\,\text{A}$$

$$I(t) \approx (0.26)\sin[(2\,\text{s}^{-1})t + 0.54]\,\text{A}$$ ❏

Question T5

When the series *LCR* circuit of Figure 3b is driven by an applied voltage $V(t) = V_0 \sin(\Omega t)$, the current $I(t)$ satisfies Equation 19.

$$L\frac{d^2I(t)}{dt^2} + R\frac{dI(t)}{dt} + \frac{1}{C}I(t) = \frac{dV(t)}{dt} \qquad \text{(Eqn 19)}$$

Show that Equation 63

$$I(t) = \frac{V_0}{Z}\sin(\Omega t - \delta) \qquad \text{(Eqn 63a)}$$

$$Z^2 = R^2 + \left(\frac{1}{\Omega C} - \Omega L\right)^2 \text{ and } \tan\delta = \frac{1}{R}\left(\Omega L - \frac{1}{\Omega C}\right) \qquad \text{(Eqn 63b)}$$

really does provide a particular solution of this equation, as claimed above. ☞ ❑

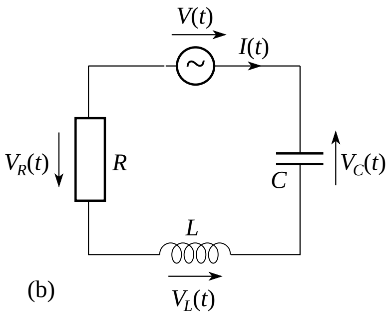


Figure 3b A simple LCR circuit containing a resistor, a capacitor and an inductor connected in series. At the instant shown the current is increasing in the direction shown and the directions (polarity) of the voltages are shown by arrows.

Question T6

Calculate the impedance of a circuit which consists of a resistor of 5.0 Ω, a capacitor of 1/6 F and an inductor of 1.0 H in series.

A voltage $V(t) = \frac{3}{5}\sin[(5\,\mathrm{s}-1)t]\,\mathrm{V}$ is applied to the circuit and the current is allowed to reach its steady state. By how much do the steady state current and voltage differ in phase? Write down an expression for the steady state current. Compare your answer with that of Example 1. ❏

2.6 Mechanical impedance

We saw earlier that the mathematical description of a harmonically driven series LCR circuit is essentially identical to that of a harmonically driven, linearly damped mechanical oscillator. In particular we saw that charge oscillations in the circuit are directly analogous to the displacement oscillations of the mechanical system. However, we have just seen that the circuit also displays current oscillations, the amplitude of which can be expressed in terms of an impedance that depends on the angular frequency of the driving voltage. Does the mechanical oscillator exhibit oscillations analogous to the current oscillations? If so, what are they, how do they behave and what is the mechanical analogue of the impedance?

The current in the series LCR circuit is related to the charge on the capacitor by

$$I = \frac{dq}{dt} \qquad \text{(Eqn 8)}$$

Since the mechanical analogue of the charge q is the displacement x, we should expect the mechanical analogue of the current to be the velocity

$$v_x = \frac{dx}{dt}$$

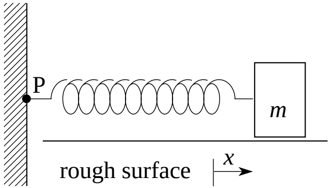
✥ According to Equation 19

$$L\frac{d^2I(t)}{dt^2} + R\frac{dI(t)}{dt} + \frac{1}{C}I(t) = \frac{dV(t)}{dt} \qquad \text{(Eqn 19)}$$

sinusoidally driven current oscillations satisfy a differential equation of the form

$$L\frac{d^2I(t)}{dt^2} + R\frac{dI(t)}{dt} + \frac{I(t)}{C} = \Omega V_0\cos(\Omega t)$$

Write down the analogous differential equation that you might expect mechanical velocity oscillations to satisfy, and show that the mechanical oscillator of Figure 3a does in fact obey such an equation.



(a)

Figure 3a A mass subject to restoring, damping and driving forces.

❖ Using the description of the current oscillations given in the last subsection, write down the corresponding description of the velocity oscillations in the driven mechanical oscillator, and hence identify the mechanical impedance $Z_{\rm m}$.

By analogy with the electrical case, it is possible to identify the mass m and the spring constant k as 'reactive' parts of the mechanical oscillator, since their contribution to the mechanical impedance depends on the angular frequency of the driving force.

2.7 Resonance and driven oscillations

The displacement oscillations described by Equations 49 and 50,

$$A = \frac{a_0}{\sqrt{(\omega_0^2 - \Omega^2)^2 + (\gamma\Omega)^2}} \text{ and } \delta = \arctan\left(\frac{\gamma\Omega}{\omega_0^2 - \Omega^2}\right) \quad \text{(Eqn 49)}$$

$$x(t) = A\sin(\Omega t - \delta) \quad \text{(Eqn 50)}$$

and the velocity oscillations described by Equations 65 and 66,

$$v_x = v_0 \sin(\Omega t - \delta) \quad \text{(Eqn 65)}$$

$$\tan\delta = \frac{1}{b}\left(\Omega m - \frac{k}{\Omega}\right) \quad \text{(Eqn 66)}$$

both have an amplitude that depends sensitively on the angular frequency Ω of the driving force. The charge and current oscillations in the driven LCR circuit show a similar sensitivity to the angular frequency of the driving voltage.

As an example of this behaviour, Figure 9 shows the way in which the amplitude I_0 of the steady state current varies with driving frequency for fixed values of C, L and V_0 at a variety of values of R. It is clear from Equation 63

$$I(t) = \frac{V_0}{Z}\sin(\Omega t - \delta) \qquad \text{(Eqn 63a)}$$

$$Z^2 = R^2 + \left(\frac{1}{\Omega C} - \Omega L\right)^2 \text{ and } \tan\delta = \frac{1}{R}\left(\Omega L - \frac{1}{\Omega C}\right) \qquad \text{(Eqn 63b)}$$

that in this case, for any fixed value of R, the impedance is a minimum and the current amplitude a maximum when $\Omega = 1/\sqrt{(LC)}$.

This is an example of the phenomenon of **resonance**, the production of a large response in a driven oscillator by driving it at a frequency close to the natural frequency it would have in the absence of any driving or damping. As you can see, the smaller the value of R, the taller and narrower the peak, i.e. the sharper the resonance. In this particular case the **resonant frequency** at which the response is a maximum is *identical* to the natural frequency $\omega_0 = 1/\sqrt{(LC)}$, but that is not always the case.

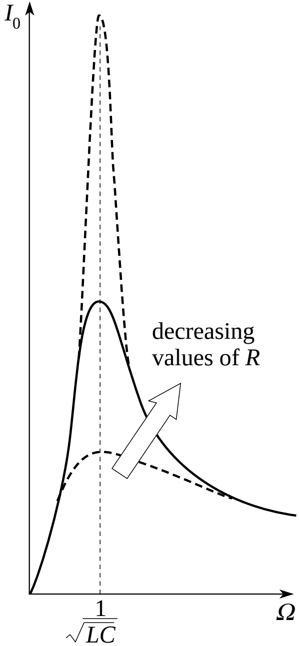


Figure 9 The amplitude of the current I_0 as a function of the driving angular frequency Ω.

✥ The current and velocity oscillations have a resonant frequency that is independent of the resistance or damping (R and b respectively). Is the same true of the resonant frequency of the charge and displacement oscillations?
If not, what is the relationship between the resonant frequency, the natural frequency and the resistance or damping in this case?

Question T7

A radio aerial for BBC Radio 4 contains an inductor $L = 0.001\,\text{H}$ (i.e. 1.0 mH) and a variable capacitor C in series. The transmitter induces a voltage in the aerial, and produces a potential difference $V(t) = V_0 \sin(\omega t)$ across the open circuit, where $\omega = 2\pi \times 198\,\text{kHz}$. To what value should the capacitor be set in order to maximize the amplitude of the current? ❏

3 Oscillations and complex numbers

Complex numbers are often used in the analysis of sinusoidal oscillations. They can greatly simplify many problems, so much so that they constitute the standard approach in most advanced work. As you pursue your studies of physics it is inevitable that you will frequently encounter discussions of oscillatory phenomena based on complex methods. This is particularly true in quantum physics, where complex numbers are not just useful, but essentially unavoidable.

3.1 Complex numbers — a brief review

1 Any **complex number**, z, may be written as

$$z = x + iy$$

where x and y are real numbers and i satisfies $i^2 = -1$.

2 If $z = x + iy$ (with x and y real) then x is known as the **real part** of z, written as $\text{Re}(z)$, and y is known as the **imaginary part** of z, written as $\text{Im}(z)$. ☞ Thus,

$$z = \text{Re}(z) + i\text{Im}(z)$$

3 Complex numbers obey the rules of normal algebra except that i^2 can be replaced by -1 whenever it appears.

4 The **complex conjugate** of z (written z^*) is defined by

$$z^* = x - iy = \mathrm{Re}(z) - i\mathrm{Im}(z)$$

5 The **modulus** of $z = x + iy$ (written as $|z|$) is defined by

$$|z| = \sqrt{x^2 + y^2} = \sqrt{[\mathrm{Re}(z)]^2 + [\mathrm{Im}(z)]^2}$$

6 For arbitrary complex numbers z and w

$$\mathrm{Re}(z) = \frac{1}{2}(z + z^*) \qquad \mathrm{Im}(z) = \frac{1}{2i}(z - z^*)$$

$$(zw)^* = z^*w^* \qquad (z^*)^* = z \quad \text{and} \quad |z|^2 = zz^*$$

7 A complex number, $z = x + iy$, is said to be in **Cartesian form** or a *Cartesian representation*.

Such a complex number may also be written in the form,

$$z = r(\cos\theta + i\sin\theta)$$

where r and θ are real numbers, in which case it is said to be in **polar form** or a *polar representation*. When written in polar form, the modulus of z is then given by $|z| = r$, and θ is referred to as the **argument** of z (written as $\arg(z)$). Adding 2π to the argument of a complex number does not change that complex number. The **principal value** of the argument of a complex number is the value θ which lies in the range $-\pi < \theta \le \pi$.

8 We can convert from Cartesian to polar form using

$$r = \sqrt{x^2 + y^2}\,, \ \cos\theta = \frac{x}{\sqrt{x^2 + y^2}} \text{ and } \sin\theta = \frac{y}{\sqrt{x^2 + y^2}}$$

and from polar to Cartesian form by means of

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta$$

9 A complex number can be represented by a point on an **Argand diagram** (*complex plane*) by using (x, y) as the Cartesian coordinates or (r, θ) as the polar coordinates of the point. (By convention, θ is measured anticlockwise from the positive x-axis.)

10 A complex number may also be written in **exponential form** or *exponential representation* by using **Euler's formula**

$$\mathrm{e}^{i\theta} = \cos\theta + i\sin\theta$$

If $z = r\,\mathrm{e}^{i\theta}$, then $|z| = r$, $\arg(z) = \theta$ and $z^* = r\,\mathrm{e}^{-i\theta}$.

11 Addition and subtraction of complex numbers is simplest in Cartesian form:

$$z + w = (x + iy) + (u + iv) = (x + u) + i(y + v)$$

$$z - w = (x + iy) - (u + iv) = (x - u) + i(y - v)$$

Multiplication and division of complex numbers is simplest in exponential form:

$$zw = (r\,e^{i\theta})(s\,e^{i\phi}) = (rs)\,e^{i(\theta + \phi)}$$

$$z/w = (r\,e^{i\theta})/(s\,e^{i\phi}) = (r/s)\,e^{i(\theta - \phi)}$$

The following result, known as **Demoivre's theorem** is valid for *any* real value of n

$$e^{ni\theta} = (\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$$

3.2 Complex impedance

Study comment It is important to appreciate that in the following discussion of electrical circuits we are only interested in the steady state, and we are therefore assuming that sufficient time has elapsed for the transient part of the current to be negligible.

In this subsection when we refer to 'the current' we always mean the steady state current. Since we are only concerned with the steady state, the phase of the applied voltage is unimportant — it is the phase difference between the applied voltage and the current that is critical. Previously it was convenient to assume that the applied voltage was of the form $V_0 \sin(\Omega t)$, but here it is more conventional to choose $V_0 \cos(\omega t)$, and to describe the steady state current by $I_0 \cos(\omega t - \delta)$. The effect is minimal and serves only to make the mathematics a little easier, and more standard.

The impedance Z and phase lag δ determine the relationship between the voltage that drives the LCR circuit of Figure 3b, and the steady state current it eventually produces. There is however a very simple method of determining these quantities in terms of the values of R, C and L, and, as we will see, this new method may be used to analyse far more complicated circuits.

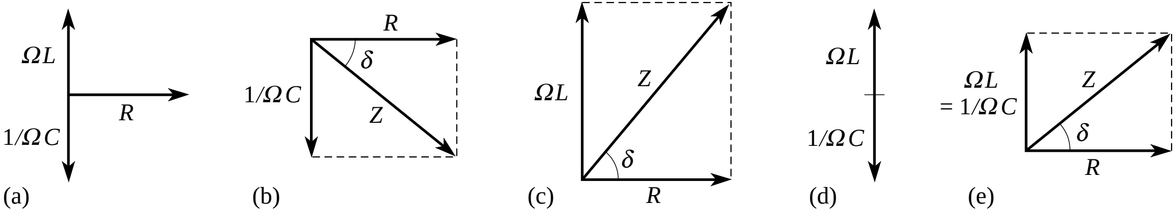


Figure 8 Geometrical interpretations of Z and δ in terms of X_L, X_C and R.

We begin by referring once again to Figure 8, the geometric interpretation of impedance. If we interpret the direction assigned to the 'vector' representing the resistance R as the real axis of an Argand diagram, and if we allow the heads of the various 'vectors' in Figure 8 to denote complex numbers, then the values of both Z and δ can be found very easily as the modulus and argument of the sum of the various complex numbers involved.

Adopting this approach, and noting that in this case we are dealing with an applied voltage with angular frequency ω, we can identify the following complex quantities from Figure 8a

- the **complex inductive reactance** $i\omega L$
- the **complex capacitive reactance** $-i/\omega C$
- the resistance (a real quantity) R

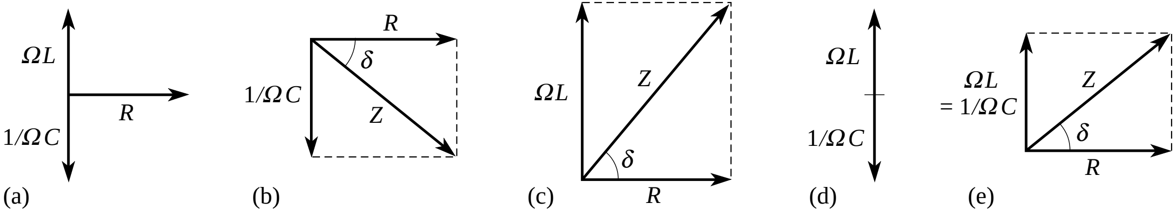


Figure 8 Geometrical interpretations of Z and δ in terms of X_L, X_C and R.

We can then define a new quantity, the **complex impedance** $\mathcal{Z}$ of a series LCR circuit by the relation

$$\mathcal{Z} = R + i\omega L + \left(\frac{-i}{\omega C}\right) = R + i\left[\omega L - \frac{1}{\omega C}\right] \tag{67}$$

From Figures 8b to 8e it can be seen that this complex impedance has the following properties:

- its modulus is equal to the impedance, so $|\mathcal{Z}| = Z$
- its argument is equal to the phase lag, so $\arg(\mathcal{Z}) = \delta$

Thus, the complex impedance can be written in the form $\mathcal{Z} = Z\,\mathrm{e}^{i\delta}$

We can take this representation further by expressing the applied voltage $V(t)$ as the real part of a complex voltage defined by

$$\mathcal{V}(t) = V_0\,\mathrm{e}^{i\omega t} \tag{68}$$

so that $V(t) = \mathrm{Re}\big(\mathcal{V}(t)\big) = \mathrm{Re}(V_0\mathrm{e}^{i\omega t}) = V_0\cos(\omega t)$

It then follows that the instantaneous current $I(t)$ in the series LCR circuit is given by the real part of the complex current $\mathcal{I}(t)$ defined by

$$\mathcal{I}(t) = \frac{\mathcal{V}(t)}{\mathcal{Z}} \tag{69}$$

since $\mathrm{Re}[\mathcal{I}(t)] = \mathrm{Re}\left(\dfrac{\mathcal{V}(t)}{\mathcal{Z}}\right) = \mathrm{Re}\left(\dfrac{V_0\mathrm{e}^{i\omega t}}{Z\mathrm{e}^{i\delta}}\right) = \mathrm{Re}\left(\dfrac{V_0}{Z}\,\mathrm{e}^{i(\omega t-\delta)}\right) = I_0\cos(\omega t - \delta)$

i.e. $\mathrm{Re}[\mathcal{I}(t)] = I(t)$

$$\mathscr{I}(t) = \frac{\mathscr{V}(t)}{\mathscr{Z}} \qquad \text{(Eqn 69)}$$

Equation 69 plays an important role in the analysis of circuits carrying sinusoidally varying currents. Such currents are generally referred to as **alternating currents**, or a.c. currents, and Equation 69 is sometimes described as the complex or a.c. form of Ohm's law.

Example 3 Use Equation 67

$$\mathscr{Z} = R + i\omega L + \left(\frac{-i}{\omega C}\right) = R + i\left[\omega L - \frac{1}{\omega C}\right] \qquad \text{(Eqn 67)}$$

to calculate the impedance of a circuit which consists of a resistor of 10 Ω, a capacitor of 0.05 F and an inductor of 2 H in series. If a voltage $V(t) = 3\cos[(2\,\text{s}^{-1})t]$ V is applied to the circuit, what is the phase difference between the steady state current and the applied voltage? Use the complex form of Ohm's law to write down an expression for the steady state current. Plot the complex numbers representing $\mathscr{V}(t)$ and $\mathscr{I}(t)$ at some arbitrary time t on an Argand diagram. Indicate the quantities corresponding to $V(t)$ and $I(t)$ on your diagram.

Solution From Equation 67

$$\mathcal{Z} = R + i\omega L + \left(\frac{-i}{\omega C}\right) = R + i\left[\omega L - \frac{1}{\omega C}\right] \qquad \text{(Eqn 67)}$$

we have

$$\mathcal{Z} = R + i\omega L - \frac{i}{\omega C} = \left(10 + (2\times 2)i - \frac{i}{(2\times 0.05)}\right)\Omega = (10 - 6i)\,\Omega$$

then $\quad Z = |\mathcal{Z}| = |10 - 6i| = \sqrt{10^2 + 6^2} \approx 11.66\,\Omega$

and $\quad \delta = \arg(\mathcal{Z}) = \arctan(-6/10) \approx -0.54$

and it follows that $\quad \mathcal{Z} \approx 11.66\,\mathrm{e}^{-0.54\,i}\,\Omega$

In this case $\quad \mathcal{V}(t) = 3\exp[(2\,\mathrm{s}^{-1})i\,t]\,\mathrm{V}$

so from Equation 69 $\quad \mathcal{I}(t) = \dfrac{\mathcal{V}(t)}{\mathcal{Z}} = \dfrac{3\exp[(2\,\mathrm{s}^{-1})it]}{10 - 6i}\,\mathrm{A} = \dfrac{3\exp[(2\,\mathrm{s}^{-1})it]}{11.66\mathrm{e}^{-0.54\,i}}\,\mathrm{A}$

It follows that

$$I(t) \approx \mathrm{Re}\left\{\left(3\exp[(2\,\mathrm{s}^{-1})it]\right)\left(\frac{e^{0.54i}}{11.66}\right)\right\}\mathrm{A} = \mathrm{Re}\left\{\frac{3}{11.66}\exp[(2\,\mathrm{s}^{-1})it + 0.54i]\right\}$$

i.e. $I(t) = \mathrm{Re}\left\{0.26\exp[(2\,\mathrm{s}^{-1})it + 0.54i]\right\}\mathrm{A} = (0.26)\cos[(2\,\mathrm{s}^{-1})t + 0.54]\,\mathrm{A}$

If we plot the complex numbers $\mathcal{V}(t)$/V and $\mathcal{I}(t)$/A ☞ on the same Argand diagram, as in Figure 10 (not drawn to scale), then they will lie on two circles, of radius 3 units and 0.26 units, respectively. The real parts of these quantities, indicated on the horizontal axis, represent the instantaneous voltage and current. The essential point to notice is that, although the two points move around the circles as t increases, they are fixed in relation to each other. The magnitude of the angle ϕ in Figure 10 is 0.54 radians, which is the magnitude of the argument of $\mathcal{Z}$. In this particular case we can see immediately from the diagram that the current *leads* the voltage by this amount. ❑

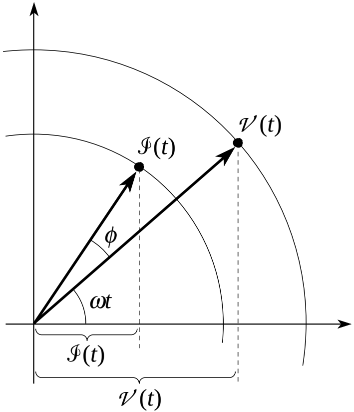


Figure 10 Argand diagram for the two complex numbers $\mathcal{V}(t)$/V and $\mathcal{I}(t)$/A. (Not drawn to scale.) The units have been omitted from the diagram for clarity.

Aside It is worth noting that the conversion of complex numbers from Cartesian to exponential (or polar) form that was at the heart of this example is available as a standard function on many modern calculators. My calculator stores a complex number $3 + 5i$ as (3 5) so to check this example I keyed in:

(10, 0) for the 10 Ω resistance and then stored it as R,

$(0, 2 \times 2)$ for the $i\omega L$ term and stored it as Z1,

then $\left(0, -\dfrac{1}{2 \times 0.05}\right)$ for the $1/(i\omega C)$ term and stored it as Z2.

Then I calculated 3/(R + Z1 + Z2) and finally converted it into polar form using a function labelled c → p on my calculator. You may find that you can perform similar calculations on your own calculator.

If you compare the solution to Example 3 with the solution to Example 2, you will find that the answers are the same (apart from a change of sin to cos), but this method is simpler.

Question T8

Use Equation 67

$$\mathcal{Z} = R + i\omega L + \left(\frac{-i}{\omega C}\right) = R + i\left[\omega L - \frac{1}{\omega C}\right] \qquad \text{(Eqn 67)}$$

to calculate the impedance of a resistor $R = 15\,\Omega$, a capacitor of $C = 5\,\mu\text{F}$ and an inductor $L = 4\,\text{mH}$ in series. Find the complex impedance, $\mathcal{Z}$, and hence find $1/\mathcal{Z}$, when a voltage $V(t) = 10\cos[(10^4\,\text{s}^{-1})t]\,\text{V}$ is applied to the circuit. By how much do the steady state current and the applied voltage differ in phase?

Write down an expression for the steady state current. ❏

Generalizing the complex method

We will now illustrate the power of the complex method by applying it to the circuit shown in Figure 11 in which the three components are connected in *parallel*.

This is the first time we have considered a parallel circuit in this module, and we will need to use the principles (obtained from *Kirchhoff's laws*) that

1 The current drawn from the voltage generator is equal to the sum of the currents through the separate components.

2 The voltage across each component is the same.

The first of these principles implies that $I(t) = I_R(t) + I_L(t) + I_C(t)$, which may be regarded as the real part of the following complex equation

$$\mathscr{I}(t) = \mathscr{I}_R(t) + \mathscr{I}_L(t) + \mathscr{I}_C(t) \tag{70}$$

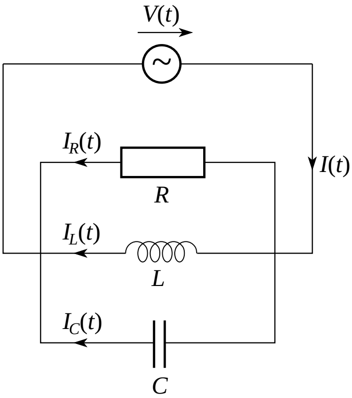


Figure 11 Three components in parallel.

and the common voltage across each component may be regarded as the real part of a common complex voltage $\mathcal{V}(t)$, which will be related to the complex current in each component and its impedance by Equation 69,

$$\mathcal{I}(t) = \frac{\mathcal{V}(t)}{\mathcal{Z}} \qquad \text{(Eqn 69)}$$

as follows

$$\mathcal{V}(t) = R\mathcal{I}_R(t)$$

$$\mathcal{V}(t) = (i\omega L)\mathcal{I}_L(t)$$

$$\mathcal{V}(t) = \frac{-i}{(\omega C)}\mathcal{I}_C(t)$$

☞

Using these relations and recognizing that $-i/(\omega C)$ can be more neatly written as $1/(i\omega C)$, Equation 70

$$\mathcal{I}(t) = \mathcal{I}_R(t) + \mathcal{I}_L(t) + \mathcal{I}_C(t) \qquad \text{(Eqn 70)}$$

becomes

$$\mathcal{I}(t) = \frac{\mathcal{V}(t)}{R} + \frac{\mathcal{V}(t)}{i\omega L} + \frac{\mathcal{V}(t)}{1/(i\omega C)} = \mathcal{V}(t)\left(\frac{1}{R} + \frac{1}{i\omega L} + \frac{1}{1/(i\omega C)}\right) \qquad (71)$$

In other words, if we denote the complex impedance of a resistor, an inductor and a capacitor in parallel by $\mathcal{Z}$, then

$$\frac{1}{\mathcal{Z}} = \frac{1}{R} + \frac{1}{i\omega L} + \frac{1}{1/(i\omega C)} \tag{72}$$

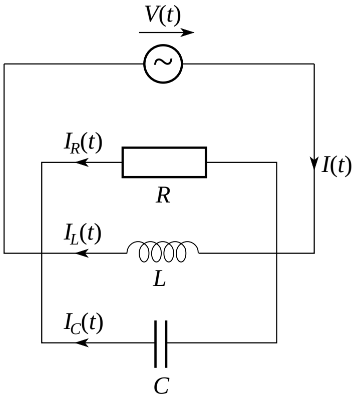


Figure 11 Three components in parallel.

We can use this expression for $\mathcal{Z}$, together with the above relations between $\mathcal{I}_R$, $\mathcal{I}_L$, $\mathcal{I}_C$, $\mathcal{I}$ and $\mathcal{V}$ to solve a wide variety of problems involving parallel LCR circuits.

❖ Use Equation 72 to find an expression for the complex impedance $\mathcal{Z}$ of the components shown in Figure 11, in Cartesian form. Hence determine the (real) impedance of a 10 Ω resistance and a 0.2 H inductance, when connected in parallel and used in a context where the driving voltage has an angular frequency of 50 Hz.

Combining impedances

The expressions we have introduced for the complex impedance of components in series and parallel (Equations 67 and 72)

$$\mathcal{Z} = R + i\omega L + \left(\frac{-i}{\omega C}\right) = R + i\left[\omega L - \frac{1}{\omega C}\right] \qquad \text{(Eqn 67)}$$

$$\frac{1}{\mathcal{Z}} = \frac{1}{R} + \frac{1}{i\omega L} + \frac{1}{1/(i\omega C)} \qquad \text{(Eqn 72)}$$

are in fact particular cases of two general rules for combining complex impedances. With these general rules we can analyse the behaviour of an enormous range of a.c. circuits, though we will not do so in this module.

Given a number of complex impedances $\mathcal{Z}_1$, $\mathcal{Z}_2$ … $\mathcal{Z}_n$, then in series their combined complex impedance $\mathcal{Z}$ is given by

$$\mathcal{Z} = \mathcal{Z}_1 + \mathcal{Z}_2 + \ldots \mathcal{Z}_n \qquad (73)$$

in parallel their combined complex impedance $\mathcal{Z}$ is given by

$$\frac{1}{\mathcal{Z}} = \frac{1}{\mathcal{Z}_1} + \frac{1}{\mathcal{Z}_2} + \ldots + \frac{1}{\mathcal{Z}_n} \qquad (74)$$

Question T9

Find the complex impedance of the circuit shown in Figure 12 in terms of ω, R, C and L. The circuit consists of a resistor in parallel with a capacitor, in series with an inductor. ❏

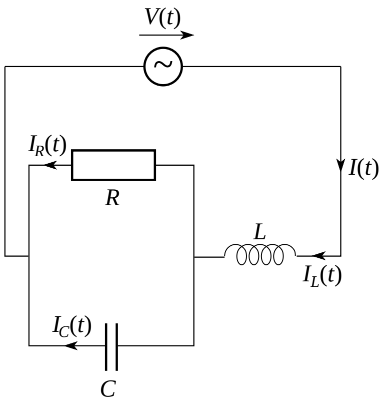


Figure 12 See Question T9.

3.3 The power dissipated

The power dissipated, or rate of energy transfer from devices is often of physical interest. The instantaneous power dissipated by a device in an electrical circuit is the product of the current flowing through it and the potential difference across it, i.e. $P(t) = I(t)V(t)$. We may regard this as the *product of the real parts of a complex current and a complex voltage*, so we can write $P(t) = \text{Re}[\mathscr{I}(t)] \times \text{Re}[\mathscr{V}(t)]$. Such products must be treated with care, as the following question shows.

✦ Given that $z = 2 + 3i$ and $w = 1 - 2i$, calculate $\text{Re}(z) \times \text{Re}(w)$ and $\text{Re}(zw)$. ☞

$P(t)$ varies from moment to moment, but for devices in a.c. circuits it is usually the *average* power dissipated over a full cycle of oscillation, $\langle P \rangle$, which is of interest. This is given by

$$\langle P \rangle = \frac{1}{T}\int_0^T \text{Re}[\mathscr{I}(t)]\,\text{Re}[\mathscr{V}(t)]\,dt \tag{75}$$

where $\mathscr{V}(t)$ is the potential difference across the device and $\mathscr{I}(t)$ is the current flowing through it. $T = 2\pi/\omega$ is the period of an oscillation.

We can simplify this result by introducing two new constants which may be complex, $\mathscr{V}_0$ and $\mathscr{I}_0$. *Any* sinusoidally varying voltage may then be written as the real part of

$$\mathscr{V}(t) = \mathscr{V}_0\,e^{i\omega t} \qquad (76)$$

☞

and *any* sinusoidally varying current as the real part of

$$\mathscr{I}(t) = \mathscr{I}_0\,e^{i\omega t} \qquad (77)$$

With the aid of these generalized complex expressions, it is possible to show that Equation 75

$$\langle P\rangle = \frac{1}{T}\int_0^T \mathrm{Re}[\mathscr{I}(t)]\,\mathrm{Re}[\mathscr{V}(t)]\,dt \qquad \text{(Eqn 75)}$$

leads to the following useful result :

$$\langle P\rangle = \frac{1}{2}\mathrm{Re}(\mathscr{I}_0^{\,*}\mathscr{V}_0) \qquad (78)$$

☞

If you are interested in knowing how this formula is obtained you can answer the following (fairly difficult) question. If not, you can simply use Equation 78 to answer Question T11.

Question T10

Prove this claim.

i.e. that $$\langle P \rangle = \frac{1}{2}\mathrm{Re}(\mathscr{I}_0^{\,*}\mathcal{V}_0)$$ (Eqn 78)

(*Hint*: First show that $\int\limits_0^T \mathrm{e}^{ni\omega t}\,dt = 0$ for any non-zero integer, n, and then use that result to derive Equation 78 from Equation 75.)

$$\langle P \rangle = \frac{1}{T}\int\limits_0^T \mathrm{Re}[\mathscr{I}(t)]\,\mathrm{Re}[\mathcal{V}(t)]\,dt$$ (Eqn 75) ❏

Question T11

Suppose that for the series LCR circuit shown in Figure 3b, we are told that the current is $I(t) = I_0 \cos(\omega t)$ (where I_0 is real). Find the average power, $\langle P \rangle$,

$$\langle P \rangle = \frac{1}{2}\mathrm{Re}(\mathscr{I}_0^{\,*}\mathscr{V}_0) \qquad \text{(Eqn 78)}$$

dissipated by the circuit in terms of L, C, R and I_0. What effect would varying the values of L and C have on the value of $\langle P \rangle$? ❏

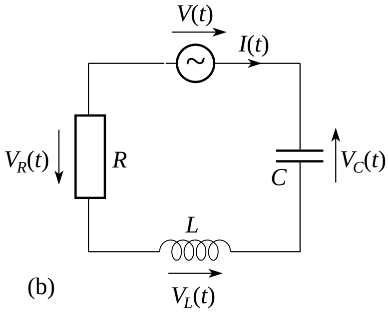


Figure 3b A simple LCR circuit containing a resistor, a capacitor and an inductor connected in series. At the instant shown the current is increasing in the direction shown and the directions (polarity) of the voltages are shown by arrows.

4 Superposed oscillations and complex algebra

Simple harmonic oscillation is common in nature, and situations often arise in which an oscillating system is subject to several independent influences each of which tends to promote SHM. For example, the steady state current in a circuit might be the response of two independent sinusoidal voltage supplies, each of which drives the circuit with a characteristic amplitude, angular frequency and phase constant. In such cases, provided the circuit concerned has the property of **linearity**, i.e. provided its behaviour can be modelled by *linear* differential equations, the response of the circuit at any time will be the sum of the responses it would have shown to each of the applied voltages independently. This general feature of the behaviour of linear systems is enshrined in the **superposition principle**:

> When several oscillations are added, the resulting disturbance at any time is the sum of the disturbances due to each oscillation at that time.

In view of the wide applicability of the superposition principle it is hardly surprising that the problem of adding or rather superposing simple harmonic oscillations is frequently encountered in physics. From a mathematical point of view these additions can be carried out in a number of ways (the use of *phasors* and of *trigonometric identities* are both explored elsewhere in *FLAP*); however when large numbers of oscillations must be added together it is often advantageous to make use of complex methods. It is this process that we will discuss in this section.

In what follows we will make use of the following general result from the arithmetic of complex numbers.

If α_1 and α_2 are arbitrary real numbers, then

$$\begin{aligned}
&\exp(i\alpha_1) + \exp(i\alpha_2)\\
&\quad = \exp[i(\alpha_1 + \alpha_2)/2]\exp[i(\alpha_1 - \alpha_2)/2] + \exp[i(\alpha_1 + \alpha_2)/2]\exp[-i(\alpha_1 - \alpha_2)/2]\\
&\quad = \exp[i(\alpha_1 + \alpha_2)/2]\{\exp[i(\alpha_1 - \alpha_2)/2] + \exp[-i(\alpha_1 - \alpha_2)/2]\}\\
&\quad = \exp[i(\alpha_1 + \alpha_2)/2]\{\cos[(\alpha_1 - \alpha_2)/2] + i\sin[(\alpha_1 - \alpha_2)/2]\\
&\qquad\qquad + \cos[(\alpha_1 - \alpha_2)/2] - i\sin[(\alpha_1 - \alpha_2)/2]\}
\end{aligned}$$

i.e. $$\exp(i\alpha_1) + \exp(i\alpha_2) = 2\exp[i(\alpha_1 + \alpha_2)/2]\cos[(\alpha_1 - \alpha_2)/2] \tag{79}$$

4.1 Superposition of two SHMs differing only in phase constant

Suppose we want to superpose the (real) oscillations $A\cos(\omega t + \phi_1)$ and $A\cos(\omega t + \phi_2)$, which might represent two simultaneous electric currents combining to produce a total current. We already know that each oscillation may be written as the real part of a complex expression of the form $A\exp[i(\omega t + \phi)]$, so we can write the sum of the two oscillations as the real part of the quantity

$$z(t) = A\exp[i(\omega t + \phi_1)] + A\exp[i(\omega t + \phi_2)]$$
$$= A\exp(i\omega t)[\exp(i\phi_1) + \exp(i\phi_2)]$$

Using Equation 79

$$\exp(i\alpha_1) + \exp(i\alpha_2) = 2\exp[i(\alpha_1 + \alpha_2)/2]\cos[(\alpha_1 - \alpha_2)/2] \qquad \text{Eqn (79)}$$

with $\alpha_1 = \phi_1$ and $\alpha_2 = \phi_2$ this may be written

$$z(t) = 2A\cos\left(\frac{\phi_1 - \phi_2}{2}\right)\exp(i\omega t)\exp\left[\frac{i(\phi_1 + \phi_2)}{2}\right]$$

and it follows that

$$\mathrm{Re}[z(t)] = \mathrm{Re}\left\{2A\cos\left(\frac{\phi_1 - \phi_2}{2}\right)\exp(i\omega t)\exp\left[\frac{i(\phi_1 + \phi_2)}{2}\right]\right\}$$

i.e. $$\mathrm{Re}[z(t)] = 2A\cos\left(\frac{\phi_1 - \phi_2}{2}\right)\cos\left(\omega t + \frac{\phi_1 + \phi_2}{2}\right)$$

which implies

$$A\cos(\omega t + \phi_1) + A\cos(\omega t + \phi_2) = \underbrace{2A\cos\left(\frac{\phi_1 - \phi_2}{2}\right)}_{\text{amplitude}}\cos\left(\omega t + \frac{\phi_1 + \phi_2}{2}\right) \quad (80)$$

This result shows that adding two SHMs with the same amplitude A and angular frequency ω, but different phase constants ϕ_1 and ϕ_2 gives a new SHM with:

- the same angular frequency, ω
- a new phase constant $(\phi_1 + \phi_2)/2$
- a new amplitude, $2A\cos[(\phi_1 - \phi_2)/2]$.

The particular case $A = 2.0\,\text{m}$, $\omega = 3.0\,\text{Hz}$, $\phi_1 = -1.07$ and $\phi_2 = 2.57$ is illustrated in Figure 13. The dotted and dashed curves combine to give the solid lined curve.

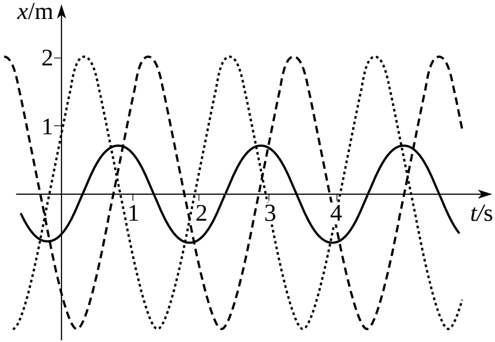


Figure 13 Combining two SHMs that differ only in phase.

Question T12

Use the complex method to find the sum of two SHMs which have the form $A\sin(\omega t)$ and $A\cos(\omega t)$, where A and ω are (real) constants. ❏

4.2 Superposition of two SHMs differing in angular frequency and phase constant

Suppose now that we want to sum $A\cos(\omega_1 t + \phi_1)$ and $A\cos(\omega_2 t + \phi_2)$. We can do this by determining the real part of

$$z(t) = A\exp[i(\omega_1 t + \phi_1)] + A\exp[i(\omega_2 t + \phi_2)]$$

and using Equation 79,

$$\exp(i\alpha_1) + \exp(i\alpha_2) = 2\exp[i(\alpha_1 + \alpha_2)/2]\cos[(\alpha_1 - \alpha_2)/2] \qquad \text{Eqn (79)}$$

with $\alpha_1 = \omega_1 t + \phi_1$ and $\alpha_2 = \omega_2 t + \phi_2$,

to obtain

$$z(t) = A\exp[i(\omega_1 t + \phi_1)] + A\exp[i(\omega_2 t + \phi_2)]$$

$$= 2A\exp\{i[(\omega_1 + \omega_2)t + (\phi_1 + \phi_2)]/2\}\cos\left(\frac{\omega_1 - \omega_2}{2}t + \frac{\phi_1 - \phi_2}{2}\right)$$

Taking the real part we obtain

$$A\cos(\omega_1 t + \phi_1) + A\cos(\omega_2 t + \phi_2) = \mathrm{Re}[z(t)]$$

$$= 2A\cos\left[\frac{(\omega_1 - \omega_2)t + (\phi_1 - \phi_2)}{2}\right]\cos\left[\frac{(\omega_1 + \omega_2)t + (\phi_1 + \phi_2)}{2}\right] \qquad (81)$$

This general expression is fairly complicated and so it is useful to consider a particular case as in the following exercise.

Question T13

Use Equation 79

$$\exp(i\alpha_1) + \exp(i\alpha_2) = 2\exp[i(\alpha_1 + \alpha_2)/2]\cos[(\alpha_1 - \alpha_2)/2] \qquad \text{Eqn (79)}$$

to find the sum of two SHMs which have the form $A\cos(21\omega t + \pi/4)$ and $A\cos(23\omega t - \pi/4)$ where ω and A are real constants. Sketch the resulting function of t and comment on the form of your result. ❏

4.3 Superposition of many SHMs — the diffraction grating

Finally, we consider a problem that is of considerable importance in the study of optics. This concerns the pattern of illumination created on a distant screen when a uniform beam of light of a single colour encounters a **diffraction grating**. For our present purposes a diffraction grating may be thought of as consisting of a large number of narrow parallel slits, each of which acts as a source of light. Two of these slits, S_1 and S_2, together with the screen are indicated in Figure 14, though the figure has not been drawn to scale (D should be so much larger than d that the lines S_1P and S_2P are effectively parallel).

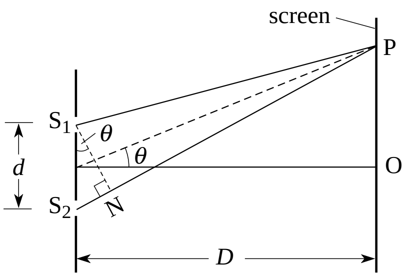


Figure 14 A diffraction grating.

As explained elsewhere in *FLAP*, light may often be treated as a wave phenomenon; so at a given point P on the screen, the effect of the light spreading out from each of the slits is to create an oscillation that may be represented by $a\cos(\omega t + \phi)$, where the angular frequency of the oscillation is determined by the colour of the light, and the phase constant ϕ is determined by the distance between the relevant slit and the point P.

Because each slit is at a different distance from the screen, the oscillation that each slit causes at P will be characterized by a particular value of ϕ, and it may be shown that for oscillations due to any pair of neighbouring slits these phase constants will differ by an amount

$$\mu = \frac{2\pi d}{\lambda}\sin\theta \qquad (82)$$

where d is the separation of adjacent slits on the diffraction grating, λ is another characteristic of the light (its wavelength), and θ is the angle between the points P and O, measured from the diffraction grating.

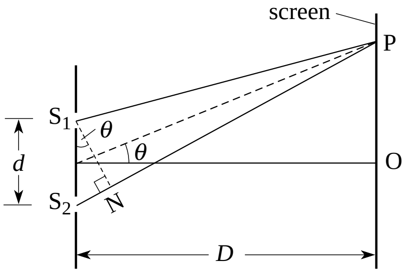


Figure 14 A diffraction grating.

In order to determine the total superposition oscillation occurring at P due to the light arriving from the n slits that make up the grating, we need to determine the sum of n SHMs. In other words we need to evaluate a sum of the form

$$a\cos(\omega t) + a\cos(\omega t + \mu) + a\cos(\omega t + 2\mu) + \dots + a\cos[\omega t + (n-1)\mu]$$

We can do this by finding the real part of the complex quantity

$$z(t) = a\,e^{i\omega t} + a\,e^{i(\omega t + \mu)} + a\,e^{i(\omega t + 2\mu)} + \dots + a\,e^{i[\omega t + (n-1)\mu]} = a\,e^{i\omega t}[1 + e^{i\mu} + e^{i2\mu} + \dots + e^{i(n-1)\mu}]$$

$$z(t) = a\,\mathrm{e}^{i\omega t} + a\,\mathrm{e}^{i(\omega t + \mu)} + a\,\mathrm{e}^{i(\omega t + 2\mu)} + \dots + a\,\mathrm{e}^{i[\omega t + (n-1)\mu]} = a\,\mathrm{e}^{i\omega t}[1 + \mathrm{e}^{i\mu} + \mathrm{e}^{i2\mu} + \dots + \mathrm{e}^{i(n-1)\mu}]$$

The sum in the square brackets is a *geometric series* with *first term* 1, and *common ratio* $\mathrm{e}^{i\mu}$ and (from Question R5) this can be written as

$$1 + \mathrm{e}^{i\mu} + \mathrm{e}^{i2\mu} + \dots + \mathrm{e}^{i(n-1)\mu} = \frac{1 - \mathrm{e}^{in\mu}}{1 - \mathrm{e}^{i\mu}} \tag{83}$$

and the resulting expression for $z(t)$ is

$$z(t) = a\,\mathrm{e}^{i\omega t}\left(\frac{1 - \mathrm{e}^{in\mu}}{1 - \mathrm{e}^{i\mu}}\right)$$

The terms involving μ can be rearranged as follows

$$\frac{1 - \mathrm{e}^{in\mu}}{1 - \mathrm{e}^{i\mu}} = \frac{\mathrm{e}^{in\mu/2}}{\mathrm{e}^{i\mu/2}}\left(\frac{\mathrm{e}^{in\mu/2} - \mathrm{e}^{-in\mu/2}}{\mathrm{e}^{i\mu/2} - \mathrm{e}^{-i\mu/2}}\right) = \frac{\mathrm{e}^{in\mu/2}\sin(n\mu/2)}{\mathrm{e}^{i\mu/2}\sin(\mu/2)} \tag{84}$$

Substituting this in the expression for $z(t)$ and taking the real part gives

$$\mathrm{Re}[z(t)] = a\frac{\sin(n\mu/2)}{\sin(\mu/2)}\cos\left(\omega t + \frac{(n-1)\mu}{2}\right)$$

We can see from this result that:

- the angular frequency is ω;
- the phase constant is $\dfrac{(n-1)\mu}{2}$;
- the amplitude, A say, is given by

$$A = a\,\frac{\sin(n\mu/2)}{\sin(\mu/2)} \tag{85}$$

Substituting the value of μ from Equation 82

$$\mu = \frac{2\pi d}{\lambda}\sin\theta \qquad \text{(Eqn 82)}$$

we can write this as

$$A = a\,\frac{\sin[n\pi d\sin(\theta)/\lambda]}{\sin[\pi d\sin(\theta)/\lambda]}$$

The square of this amplitude will be proportional to the intensity of illumination at any point on the screen, provided that the slits are sufficiently narrow. Thus we can expect to observe an intensity distribution that varies with θ in proportion to

$$I(\theta) = a^2 \frac{\sin^2[n\pi d \sin(\theta)/\lambda]}{\sin^2[\pi d \sin(\theta)/\lambda]}$$

Note The slits must be sufficiently narrow so that light from each slit diffracts to the point of superposition. If the slits are too wide, an additional effect, the diffraction pattern due to a single slit, will modify the intensity pattern, causing the various intensity maxima to reduce in brightness away from the $\theta = 0$ maximum.

This pattern is shown in Figure 15 for the case of 2, 3, 4 and 5 slits. As you can see, increasing the number of slits makes the intensity peaks taller and narrower. In practice, diffraction gratings have a great many slits (~10 000), with the result that the observed pattern of illumination consists of well separated bright lines. This pattern is discussed in more detail in the block of *FLAP* modules devoted to light and optics.

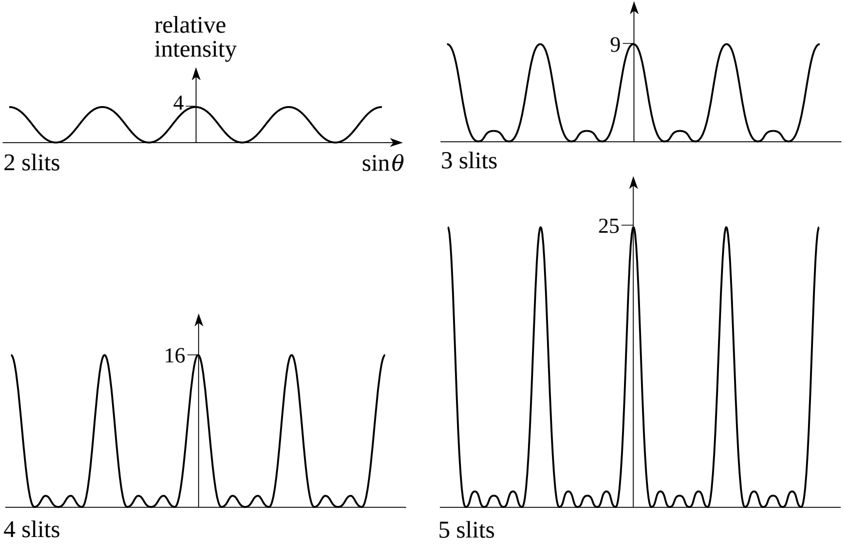


Figure 15 A graph of

$$I(\theta) = a^2 \frac{\sin^2[n\pi d\sin(\theta)/\lambda]}{\sin^2[\pi d\sin(\theta)/\lambda]}$$

for $n = 2, 3, 4$, and 5.

Question T14

Find the sum of

$$a\sin(\omega t) + a\sin(\omega t + \phi) + a\sin(\omega t + 2\phi) + \dots + a\sin[\omega t + (n-1)\phi]$$ ❏

5 Closing items

5.1 Module summary

1 Differential equations of second order with constant coefficients arise from many physical situations, in particular: mechanical and electrical systems.

A mass m subject to a sinusoidal *driving force* $F(t) = F_0 \sin(\Omega t)$, a *damping force* proportional to velocity and a *restoring force* proportional to displacement from the origin, will satisfy a differential equation of the form

$$m\frac{d^2x(t)}{dt^2} + b\frac{dx(t)}{dt} + kx(t) = F_0 \sin(\Omega t) \qquad \text{(Eqn 13)}$$

The charge $q(t)$ on a capacitor in a series LCR circuit containing a resistance R, a capacitance C and an inductance L, subject to an externally applied voltage $V(t) = V_0 \sin(\Omega t)$, will satisfy a similar differential equation

$$L\frac{d^2q(t)}{dt^2} + R\frac{dq(t)}{dt} + \frac{1}{C}q(t) = V_0 \sin(\Omega t) \qquad \text{(Eqn 17)}$$

Both equations give rise to harmonically driven, linearly damped harmonic oscillations.

2 In the absence of damping and driving, Equation 13

$$m\frac{d^2x(t)}{dt^2} + b\frac{dx(t)}{dt} + kx(t) = F_0\sin(\Omega t) \qquad \text{(Eqn 13)}$$

reduces to the equation of simple harmonic motion ☞ (SHM) $\dfrac{d^2x(t)}{dt^2} + \omega_0^2 x(t) = 0$, where $\omega_0 = \sqrt{k/m}$.

This has the general solution $x(t) = A_0\sin(\omega_0 t + \phi)$ (Eqn 34)

where A_0 and ϕ are arbitrary constants that are determined by the initial conditions. A_0 is the *amplitude*, ϕ the *phase constant* and ω_0 is the (*natural) angular frequency* while the constants $T = 2\pi/\omega_0$ and $f = 1/T = \omega_0/2\pi$ are known, respectively, as the *period* and the *frequency* of the motion.

3 In the absence of any driving term, Equation 13 reduces to the equation of damped harmonic motion $\dfrac{d^2x(t)}{dt^2} + \gamma\dfrac{dx(t)}{dt} + \omega_0^2 x(t) = 0$, where $\omega_0 = \sqrt{k/m}$ and $\gamma = b/m$. The general solution depends on the relative values of γ and ω_0, but in the physically important case of *underdamping* it takes the form

$$x(t) = e^{-\gamma t/2}[A_0\sin(\omega t + \phi)] \qquad \text{(Eqn 42)}$$

where A_0 and ϕ are arbitrary constants and $\omega = \sqrt{\omega_0^2 - \dfrac{\gamma^2}{4}}$.

4 The equation of harmonically driven linearly damped oscillation may be written in the form

$$\frac{d^2x(t)}{dt^2} + \gamma\frac{dx(t)}{dt} + \omega_0^2 x(t) = a_0 \sin(\Omega t)$$

and has a general solution that is the sum of a transient term and a steady state term. In the case of underdamping, this general solution takes the form

$$x(t) = e^{-\gamma t/2}[A_0 \sin(\omega t + \phi)] + A\sin(\Omega t - \delta) \qquad \text{(Eqn 47)}$$

A_0 and ϕ are arbitrary constants, $\omega = \sqrt{\omega_0^2 - \dfrac{\gamma^2}{4}}$ (Eqn 48)

$$A = \frac{a_0}{\sqrt{(\omega_0^2 - \Omega^2)^2 + (\gamma\Omega)^2}} \text{ and } \delta = \arctan\left(\frac{\gamma\Omega}{\omega_0^2 - \Omega^2}\right) \qquad \text{(Eqn 49)}$$

In the steady state this reduces to $x(t) = A\sin(\Omega t - \delta)$.

5 When a voltage $V(t) = V_0 \sin(\Omega t)$ is applied across a series LCR circuit, the resulting steady state current has the form $I(t) = I_0 \sin(\Omega t - \delta)$

where $V_0 = I_0 Z$ and $\delta = \arctan(X/R)$

Z being the *impedance* given by

$$Z = \sqrt{R^2 + X^2} \text{ and } X = X_L - X_C = \Omega L - 1/(\Omega C) \text{ the total } \textit{reactance}.$$

6 In the case of the mechanical system described by Equation 13 the *mechanical impedance* is defined by $Z_{\rm m} = F_0/v_0$ where v_0 is the amplitude of the *velocity oscillation* and $Z_{\rm m} = \sqrt{b^2 + \left(\dfrac{k}{\Omega} - \Omega m\right)^2}$.

7 *Resonance* is the phenomenon whereby a driven oscillator exhibits large amplitude oscillations when driven at a frequency close to the natural frequency it would have in the absence of any driving or damping.

8 When analysing a.c. circuits in their steady state, it is convenient to represent the applied voltage $V(t)$ as the real part of a complex quantity $\mathcal{V}(t) = \mathcal{V}_0\,e^{i\omega t}$ and the current $I(t)$ as the real part of a complex quantity $\mathcal{I}(t) = \mathcal{I}_0\,e^{i\omega t}$, where $\mathcal{V}_0$ and $\mathcal{I}_0$ are complex constants. The complex form of Ohm's law is then the equation $\mathcal{V}(t) = \mathcal{I}(t)\,\mathcal{Z}$ where $\mathcal{Z}$ is the *complex impedance*. The value $Z = |\mathcal{Z}|$ is the *impedance*, and $\delta = \arg(\mathcal{Z})$ is the extent by which the phase of the current lags behind that of the driving voltage.

9 For complex impedances combined in series

$$\mathcal{Z} = \mathcal{Z}_1 + \mathcal{Z}_2 + \dots \mathcal{Z}_n \qquad \text{(Eqn 73)}$$

For complex impedances combined in parallel

$$\frac{1}{\mathcal{Z}} = \frac{1}{\mathcal{Z}_1} + \frac{1}{\mathcal{Z}_2} + \dots + \frac{1}{\mathcal{Z}_n} \qquad \text{(Eqn 74)}$$

For a single resistor, $\mathcal{Z}_R = R$, for a single capacitor $\mathcal{Z}_C = 1/(i\omega C)$, and for a single inductor $\mathcal{Z}_L = i\omega L$.

10 The average power dissipated by a circuit carrying alternating current is

$$\langle P \rangle = \frac{1}{2}\mathrm{Re}(\mathcal{I}_0^{\,*}\mathcal{V}_0) \qquad \text{(Eqn 78)}$$

11 The effect of superposing harmonic oscillations may be found by adding together appropriate complex quantities and then taking the real part of the result (or the imaginary part if appropriate).

5.2 Achievements

Having completed this module, you should be able to:

A1 Define the terms that are emboldened and flagged in the margins of the module.

A2 Use second-order differential equations to model a variety of oscillatory problems and to highlight the analogy between mechanical and electrical systems in situations where both are modelled by similar differential equations.

A3 To explain the significance of impedance and resonance in relation to the amplitude of the steady state solution to the sinusoidally driven, linearly damped harmonic oscillator.

A4 Use complex numbers to solve simple problems involving LCR circuits and to display relative magnitudes and phases of currents and voltages by means of an Argand diagram.

A5 To calculate the average power dissipated by a suitable component in a simple a.c. circuit.

A6 Use complex numbers to solve problems involving SHM, including the superposition of two or more SHMs.

Study comment You may now wish to take the *Exit test* for this module which tests these Achievements. If you prefer to study the module further before taking this test then return to the *Module contents* to review some of the topics.

5.3 Exit test

Study comment Having completed this module, you should be able to answer the following questions each of which tests one or more of the Achievements.

Question E1

(A2) The general solution of the differential equation

$$\frac{d^2x(t)}{dt^2} + 2q\omega\frac{dx(t)}{dt} + (n^2 + q^2)\omega^2 x(t) = 0$$

is $x(t) = \mathrm{e}^{-q\omega t}[A\cos(n\omega t) + B\sin(n\omega t)]$

and a particular solution of the differential equation

$$\frac{d^2x(t)}{dt^2} + 2q\omega\frac{dx(t)}{dt} + (n^2 + q^2)\omega^2 x(t) = h\sin(\Omega t)$$

is $x_{\rm p}(t) = c\cos(\Omega t) + d\sin(\Omega t)$

where $c = -2q\omega\Omega h/H$ and $d = [(q^2 + n^2)\omega^2 - \Omega^2]h/H$

with $H = q^4\omega^2 + 2q^2\omega^2(n^2 + 1) + (n^2\omega^2 - \Omega^2)^2$ and $n \neq \Omega/\omega$.

Use this solution to write down the transient current and the steady state current in a series LCR circuit containing a resistor 6 Ω, a capacitor 1/13 F and an inductor 1 H, when it is driven by an applied voltage $V(t) = V_0 \cos(\Omega t + \pi)$, where $V_0 = 1.00\,\text{V}$ and $\Omega = 1.00\,\text{s}^{-1}$. Describe an analogous mechanical system.

Question E2

(*A*2) What is the resonant frequency of the electrical circuit described in Question E1? What would you expect to happen when the angular frequency of the applied voltage is close to this value?

Question E3

(*A4* and *A5*) Calculate the complex impedance of a resistance 6 Ω, a capacitor 1/13 F and an inductor 1 H in series. What is the current through the circuit if a voltage $V(t) = V_0 \cos(\omega t + \pi)$, where $\omega = 1\,\text{s}^{-1}$, is applied to the circuit? Sketch the complex current and voltage on an Argand diagram, and calculate the average power dissipated.

Question E4

(*A6*) Use a complex representation to find the result of adding two SHMs of the form $a\sin(\omega t - kx)$ and $a\sin(2\omega t + 2kx)$.

Study comment This is the final *Exit test* question. When you have completed the *Exit test* go back to Subsection 1.2 and try the *Fast track questions* if you have not already done so.

If you have completed **both** the *Fast track questions* and the *Exit test*, then you have finished the module and may leave it here.