

INTEGRATION BY PARTS

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A self-contained Tutorial Module for learning
the technique of integration by parts

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1. Theory

To differentiate a product of two functions of x , one uses the product rule:

$$\frac{d}{dx}(uv) = u \frac{dv}{dx} + \frac{du}{dx}v$$

where $u = u(x)$ and $v = v(x)$ are two functions of x . A slight rearrangement of the product rule gives

$$u \frac{dv}{dx} = \frac{d}{dx}(uv) - \frac{du}{dx}v$$

Now, integrating both sides with respect to x results in

$$\int u \frac{dv}{dx} dx = uv - \int \frac{du}{dx}v dx$$

This gives us a rule for integration, called **INTEGRATION BY PARTS**, that allows us to integrate many products of functions of x . We take one factor in this product to be u (this also appears on the right-hand-side, along with $\frac{du}{dx}$). The other factor is taken to be $\frac{dv}{dx}$ (on the right-hand-side only v appears – i.e. the other factor integrated with respect to x).

2. Usage

We highlight here four different types of products for which integration by parts can be used (as well as which factor to label u and which one to label $\frac{dv}{dx}$). These are:

$$(i) \quad \int x^n \cdot \left\{ \begin{array}{c} \sin bx \\ \text{or} \\ \cos bx \end{array} \right\} dx \quad (ii) \quad \int x^n \cdot e^{ax} dx$$

$$\begin{array}{ccc} \uparrow & \uparrow & \uparrow \quad \uparrow \\ u & \frac{dv}{dx} & u \quad \frac{dv}{dx} \end{array}$$

$$(iii) \quad \int x^r \cdot \ln(ax) dx \quad (iv) \quad \int e^{ax} \cdot \left\{ \begin{array}{c} \sin bx \\ \text{or} \\ \cos bx \end{array} \right\} dx$$

$$\begin{array}{ccc} \uparrow & \uparrow & \uparrow \quad \uparrow \\ \frac{dv}{dx} & u & u \quad \frac{dv}{dx} \end{array}$$

where a, b and r are given constants and n is a positive integer.

3. Exercises

Click on [EXERCISE](#) links for full worked solutions (there are 14 exercises in total)

EXERCISE 1. $\int x \cos x \, dx$

EXERCISE 2. $\int x^2 \sin x \, dx$

EXERCISE 3. $\int x e^x \, dx$

EXERCISE 4. $\int x^2 e^{4x} \, dx$

EXERCISE 5. $\int x^2 \ln x \, dx$

EXERCISE 6. $\int (x + 1)^2 \ln 3x \, dx$

EXERCISE 7. $\int e^{2x} \cos x \, dx$

EXERCISE 8. $\int e^{-x} \sin 4x \, dx$

EXERCISE 9. $\int_0^{1/2} x e^{2x} \, dx$

EXERCISE 10. $\int_0^{\pi/4} x \sin 2x \, dx$

EXERCISE 11. $\int_{1/2}^1 x^4 \ln 2x \, dx$

EXERCISE 12. $\int_0^{\pi} 3x^2 \cos\left(\frac{x}{2}\right) \, dx$

EXERCISE 13. $\int x^3 e^x \, dx$

EXERCISE 14. $\int e^{3x} \cos x \, dx$

4. Final solutions

1. $x \sin x + \cos x + C$,

2. $-x^2 \cos x + 2x \sin x + 2 \cos x + C$,

3. $(x - 1)e^x + C$,

4. $\frac{1}{32}e^{4x}(8x^2 - 4x + 1) + C$,

5. $\frac{1}{9}x^3(3 \ln x - 1) + C$,

6. $\frac{1}{3}(x + 1)^3 \ln 3x - \frac{1}{9}x^3 - \frac{1}{2}x^2 - x - \frac{1}{3} \ln x + C$,

7. $\frac{1}{5}e^{2x}(\sin x + 2 \cos x) + C$,

8. $-\frac{1}{17}e^{-x}(4\cos 4x - \sin 4x) + C,$

9. $\frac{1}{4},$

10. $\frac{1}{4},$

11. $\frac{1}{5}\ln 2 - \frac{31}{800},$

12. $6(\pi^2 - 8),$

13. $e^x(x^3 - 3x^2 + 6x - 6) + C,$

14. $\frac{1}{10}e^{3x}(\sin x + 3\cos x) + C.$

5. Standard integrals

$f(x)$	$\int f(x)dx$	$f(x)$	$\int f(x)dx$
x^n	$\frac{x^{n+1}}{n+1} \quad (n \neq -1)$	$[g(x)]^n g'(x)$	$\frac{[g(x)]^{n+1}}{n+1} \quad (n \neq -1)$
$\frac{1}{x}$	$\ln x $	$\frac{g'(x)}{g(x)}$	$\ln g(x) $
e^x	e^x	a^x	$\frac{a^x}{\ln a} \quad (a > 0)$
$\sin x$	$-\cos x$	$\sinh x$	$\cosh x$
$\cos x$	$\sin x$	$\cosh x$	$\sinh x$
$\tan x$	$-\ln \cos x $	$\tanh x$	$\ln \cosh x$
$\operatorname{cosec} x$	$\ln\left \tan \frac{x}{2}\right $	$\operatorname{cosech} x$	$\ln\left \tanh \frac{x}{2}\right $
$\sec x$	$\ln \sec x + \tan x $	$\operatorname{sech} x$	$2 \tan^{-1} e^x$
$\sec^2 x$	$\tan x$	$\operatorname{sech}^2 x$	$\tanh x$
$\cot x$	$\ln \sin x $	$\operatorname{coth} x$	$\ln \sinh x $
$\sin^2 x$	$\frac{x}{2} - \frac{\sin 2x}{4}$	$\sinh^2 x$	$\frac{\sinh 2x}{4} - \frac{x}{2}$
$\cos^2 x$	$\frac{x}{2} + \frac{\sin 2x}{4}$	$\cosh^2 x$	$\frac{\sinh 2x}{4} + \frac{x}{2}$

$f(x)$	$\int f(x) dx$	$f(x)$	$\int f(x) dx$
$\frac{1}{a^2+x^2}$	$\frac{1}{a} \tan^{-1} \frac{x}{a}$ $(a > 0)$	$\frac{1}{a^2-x^2}$	$\frac{1}{2a} \ln \left \frac{a+x}{a-x} \right $ ($0 < x < a$) $\frac{1}{2a} \ln \left \frac{x-a}{x+a} \right $ ($ x > a > 0$)
$\frac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1} \frac{x}{a}$ $(-a < x < a)$	$\frac{1}{\sqrt{a^2+x^2}}$	$\ln \left \frac{x+\sqrt{a^2+x^2}}{a} \right $ ($a > 0$) $\ln \left \frac{x+\sqrt{x^2-a^2}}{a} \right $ ($x > a > 0$)
$\sqrt{a^2-x^2}$	$\frac{a^2}{2} \left[\sin^{-1} \left(\frac{x}{a} \right) + \frac{x\sqrt{a^2-x^2}}{a^2} \right]$	$\sqrt{a^2+x^2}$	$\frac{a^2}{2} \left[\sinh^{-1} \left(\frac{x}{a} \right) + \frac{x\sqrt{a^2+x^2}}{a^2} \right]$ $\frac{a^2}{2} \left[-\cosh^{-1} \left(\frac{x}{a} \right) + \frac{x\sqrt{x^2-a^2}}{a^2} \right]$

6. Tips on using solutions

- When looking at the THEORY, INTEGRALS, FINAL SOLUTIONS, TIPS or NOTATION pages, use the [Back](#) button (at the bottom of the page) to return to the exercises

- Use the solutions intelligently. For example, they can help you get started on an exercise, or they can allow you to check whether your intermediate results are correct

- Try to make less use of the full solutions as you work your way through the Tutorial

7. Alternative notation

In this Tutorial, we express the rule for integration by parts using the formula:

$$\int u \frac{dv}{dx} dx = uv - \int \frac{du}{dx} v dx$$

But you may also see other forms of the formula, such as:

$$\int f(x)g(x)dx = F(x)g(x) - \int F(x)\frac{dg}{dx}dx$$

where

$$\frac{dF}{dx} = f(x)$$

Of course, this is simply different notation for the same rule. To see this, make the identifications: $u = g(x)$ and $v = F(x)$.

Full worked solutions

Exercise 1. We evaluate by integration by parts:

$$\int x \cos x \, dx = x \cdot \sin x - \int (1) \cdot \sin x \, dx, \text{ i.e. take } u = x$$

giving $\frac{du}{dx} = 1$ (by differentiation)

and take $\frac{dv}{dx} = \cos x$

giving $v = \sin x$ (by integration),

$$= x \sin x - \int \sin x \, dx$$

$= x \sin x - (-\cos x) + C$, where C is an arbitrary

$= x \sin x + \cos x + C$ constant of integration.

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Exercise 2.

$$\int x^2 \sin x \, dx = x^2 \cdot (-\cos x) - \int (2x) \cdot (-\cos x) \, dx ,$$

i.e. take $u = x^2$

giving $\frac{du}{dx} = 2x$

and take $\frac{dv}{dx} = \sin x$

giving $v = -\cos x$,

$$= -x^2 \cos x + 2 \underbrace{\int x \cos x \, dx}$$

\hookrightarrow we need to use integration
by parts again!

$$= -x^2 \cos x + 2 \left\{ x \sin x - \int (1) \cdot \sin x dx \right\},$$

as in question 1,

$$= -x^2 \cos x + 2x \sin x - 2 \int \sin x dx$$

$$= -x^2 \cos x + 2x \sin x - 2 \cdot (-\cos x) + C$$

$$= -x^2 \cos x + 2x \sin x + 2 \cos x + C .$$

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Exercise 3.

$$\int x e^x dx = x \cdot e^x - \int (1) \cdot e^x dx, \quad \text{i.e. take } u = x$$

$$\text{giving } \frac{du}{dx} = 1$$

$$\text{and take } \frac{dv}{dx} = e^x$$

$$\text{giving } v = e^x,$$

$$= x e^x - \int e^x dx$$

$$= x e^x - e^x + C$$

$$= (x - 1) e^x + C.$$

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Exercise 4.

$$\int x^2 e^{4x} dx = x^2 \cdot \frac{1}{4} e^{4x} - \int 2x \cdot \frac{1}{4} e^{4x} dx ,$$

i.e. take $u = x^2$

$$\frac{du}{dx} = 2x$$

$$\frac{dv}{dx} = e^{4x}$$

$$v = \frac{1}{4} e^{4x} ,$$

$$= \frac{1}{4} x^2 e^{4x} - \frac{1}{2} \underbrace{\int x e^{4x} dx}$$

\hookrightarrow now use integration by parts again

$$= \frac{1}{4}x^2e^{4x} - \frac{1}{2} \left\{ x \cdot \frac{1}{4}e^{4x} - \int 1 \cdot \frac{1}{4}e^{4x} dx \right\},$$

i.e. this time $u = x$

$$\frac{du}{dx} = 1$$

$$\frac{dv}{dx} = e^{4x}$$

$$v = \frac{1}{4}e^{4x},$$

$$= \frac{1}{4}x^2e^{4x} - \frac{1}{8}xe^{4x} - \frac{1}{2} \cdot \left(-\frac{1}{4}\right) \cdot \int e^{4x} dx$$

$$= \frac{1}{4}x^2e^{4x} - \frac{1}{8}xe^{4x} + \frac{1}{8} \int e^{4x} dx$$

$$= \frac{1}{4}x^2e^{4x} - \frac{1}{8}xe^{4x} + \frac{1}{8} \cdot \frac{1}{4}e^{4x} + C$$

$$\begin{aligned} &= \frac{8}{32}x^2e^{4x} - \frac{4}{32}xe^{4x} + \frac{1}{32}e^{4x} + C \\ &= \frac{1}{32}e^{4x} (8x^2 - 4x + 1) + C. \end{aligned}$$

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Exercise 5.

$$\int x^2 \ln x \, dx = (\ln x) \cdot \left(\frac{1}{3}x^3\right) - \int \frac{1}{x} \cdot \left(\frac{1}{3}x^3\right) dx, \quad \text{i.e. } u = \ln x$$

$$\frac{du}{dx} = \frac{1}{x}$$

$$\frac{dv}{dx} = x^2$$

$$v = \frac{1}{3}x^3,$$

$$\begin{aligned} &= \frac{1}{3}x^3 \ln x - \frac{1}{3} \int x^2 dx \\ &= \frac{1}{3}x^3 \ln x - \frac{1}{3} \cdot \left(\frac{1}{3}x^3\right) + C \\ &= \frac{1}{3}x^3 \ln x - \frac{1}{9}x^3 + C \\ &= \frac{1}{9}x^3 (3 \ln x - 1) + C. \end{aligned}$$

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Exercise 6.

$$\int (x+1)^2 \ln 3x \, dx = (\ln 3x) \cdot \left(\frac{1}{3}\right) (x+1)^3 - \int \frac{1}{3x} \cdot (3) \cdot \frac{1}{3} (x+1)^3 \, dx$$

i.e. $u = \ln 3x$ gives $\frac{du}{dx} = \frac{1}{3x} \cdot \frac{d}{dx}(3x) = \frac{1}{3x} \cdot (3)$,

using the chain rule, and $\frac{dv}{dx} = (x+1)^2$ gives $v = \frac{1}{1} \cdot \frac{(x+1)^3}{3}$,

where we have used the result that if

$$\frac{dv}{dx} = (ax+b)^n \text{ then } v = \frac{1}{a} \frac{(ax+b)^{n+1}}{(n+1)},$$

$$\begin{aligned}\therefore \int (x+1)^2 \ln 3x \, dx &= \frac{1}{3} (x+1)^3 \ln 3x - \frac{1}{3} \int \frac{(x+1)^3}{x} \, dx \\ &= \frac{1}{3} (x+1)^3 \ln 3x - \frac{1}{3} \int \frac{x^3 + 3x^2 + 3x + 1}{x} \, dx,\end{aligned}$$

where we have used the binomial theorem,

or just multiplied out $(x+1)^3$,

$$\begin{aligned}&= \frac{1}{3} (x+1)^3 \ln 3x - \frac{1}{3} \int x^2 + 3x + 3 + \frac{1}{x} \, dx \\ &= \frac{1}{3} (x+1)^3 \ln 3x - \frac{1}{3} \left[\frac{x^3}{3} + \frac{3}{2}x^2 + 3x + \ln x \right] + C \\ &= \frac{1}{3} (x+1)^3 \ln 3x - \frac{x^3}{9} - \frac{1}{2}x^2 - x - \frac{1}{3} \ln x + C.\end{aligned}$$

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Exercise 7.

$$\int e^{2x} \cos x \, dx$$

Set $u = e^{2x}$ and $\frac{dv}{dx} = \cos x$, to give $\frac{du}{dx} = 2e^{2x}$ and $v = \sin x$.

Let $I = \int e^{2x} \cos x \, dx$, since we will eventually get I on the right-hand-side for this type of integral

$$\text{i.e. } I = e^{2x} \cdot \sin x - \int 2e^{2x} \cdot \sin x \, dx$$

$$\text{i.e. } I = e^{2x} \sin x - 2 \int e^{2x} \sin x \, dx.$$

Use integration by parts again, with $u = e^{2x}$ and $\frac{dv}{dx} = \sin x$, giving $\frac{du}{dx} = 2e^{2x}$ and $v = -\cos x$

$$\text{i.e. } I = e^{2x} \sin x - 2 \left\{ e^{2x} \cdot (-\cos x) - \int 2e^{2x} \cdot (-\cos x) dx \right\}$$

$$\text{i.e. } I = e^{2x} \sin x - 2 \left\{ -e^{2x} \cos x + 2 \int e^{2x} \cos x dx \right\}$$

$$\text{i.e. } I = e^{2x} \sin x + 2e^{2x} \cos x - \underbrace{4 \int e^{2x} \cos x dx}_{=4I}$$

$$\text{i.e. } 5I = e^{2x} \sin x + 2e^{2x} \cos x + C_1$$

$$\therefore I = \frac{1}{5} e^{2x} (\sin x + 2 \cos x) + C \quad , \text{ where } C = \frac{1}{5} C_1 \text{ (another arbitrary constant).}$$

Note It is customary to introduce the arbitrary constant after the last integration is performed, though strictly one could accommodate arbitrary constants arising from each $\int \frac{du}{dx} \cdot v dx$ (indefinite) integration and these would add up to give a single arbitrary constant in the final answer.

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Exercise 8.

$$\int e^{-x} \sin 4x \, dx. \quad u = e^{-x}, \quad \text{giving} \quad \frac{du}{dx} = -e^{-x},$$
$$\frac{dv}{dx} = \sin 4x, \quad \text{giving} \quad v = \frac{1}{4} \cdot (-\cos 4x)$$
$$\text{i.e.} \quad v = -\frac{1}{4} \cos 4x,$$

$$\text{Let} \quad I = \int e^{-x} \sin 4x \, dx$$

$$\text{i.e.} \quad I = e^{-x} \cdot \left(-\frac{1}{4} \cos 4x\right) - \int (-e^{-x}) \cdot \left(-\frac{1}{4} \cos 4x\right) dx$$

$$\text{i.e. } I = -\frac{1}{4}e^{-x} \cos 4x - \frac{1}{4} \underbrace{\int e^{-x} \cos 4x dx}_{\downarrow}$$

$$u = e^{-x}, \quad \frac{du}{dx} = -e^{-x}$$

$$\frac{dv}{dx} = \cos 4x, \quad v = \frac{1}{4} \sin 4x,$$

$$\text{i.e. } I = -\frac{1}{4}e^{-x} \cos 4x - \frac{1}{4} \left\{ e^{-x} \cdot \left(\frac{1}{4} \sin 4x\right) - \int (-e^{-x}) \cdot \left(\frac{1}{4} \sin 4x\right) dx \right\}$$

$$\therefore I = -\frac{1}{4}e^{-x} \cos 4x - \frac{1}{4} \left\{ \frac{1}{4}e^{-x} \sin 4x + \frac{1}{4} \int e^{-x} \sin 4x dx \right\}$$

$$\text{i.e. } I = -\frac{1}{4}e^{-x} \cos 4x - \frac{1}{16}e^{-x} \sin 4x - \frac{1}{16}I$$

$$\text{i.e. } \left(1 + \frac{1}{16}\right) I = -\frac{1}{4}e^{-x} \cos 4x - \frac{1}{16}e^{-x} \sin 4x$$

$$\left(\frac{17}{16}\right) I = -\frac{1}{4}e^{-x} \cos 4x - \frac{1}{16}e^{-x} \sin 4x$$

$$\text{i.e. } I = -\frac{4}{17}e^{-x} \cos 4x - \frac{1}{17}e^{-x} \sin 4x$$

$$= -\frac{1}{17}e^{-x} (4 \cos 4x - \sin 4x) + C.$$

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Exercise 9.

$$\int_0^{1/2} x e^{2x} dx = \left[x \cdot \left(\frac{1}{2} e^{2x} \right) \right]_0^{1/2} - \int_0^{1/2} 1 \cdot \left(\frac{1}{2} e^{2x} \right) dx, \quad \text{i.e. } u = x$$

$$\frac{du}{dx} = 1$$

$$\frac{dv}{dx} = e^{2x}$$

$$v = \frac{1}{2} e^{2x},$$

$$\begin{aligned} &= \frac{1}{2} [x e^{2x}]_0^{1/2} - \frac{1}{2} \int_0^{1/2} e^{2x} dx \\ &= \frac{1}{2} [x e^{2x}]_0^{1/2} - \frac{1}{2} \left[\frac{1}{2} e^{2x} \right]_0^{1/2} \end{aligned}$$

$$\begin{aligned} &= \frac{1}{2} \left[\frac{1}{2} e^{2 \cdot \frac{1}{2}} - 0 \cdot e^0 \right] - \frac{1}{4} \left[e^{2 \cdot \frac{1}{2}} - e^0 \right] \\ &= \frac{1}{2} \cdot \frac{1}{2} e^1 - \frac{1}{4} e^1 + \frac{1}{4} e^0 \\ &= 0 + \frac{1}{4} e^0 = \frac{1}{4}. \end{aligned}$$

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Exercise 10.

$$\int_0^{\pi/4} x \sin 2x \, dx = \left[x \cdot \left(-\frac{1}{2} \cos 2x \right) \right]_0^{\pi/4} - \int_0^{\pi/4} 1 \cdot \left(-\frac{1}{2} \cos 2x \right) dx,$$

$$u = x$$

$$\frac{dv}{dx} = \sin 2x$$

$$\frac{du}{dx} = 1$$

$$v = -\frac{1}{2} \cos 2x,$$

$$= \left[-\frac{1}{2} x \cos 2x \right]_0^{\pi/4} + \frac{1}{2} \int_0^{\pi/4} \cos 2x \, dx$$

$$\begin{aligned} &= -\frac{1}{2} [x \cos 2x]_0^{\pi/4} + \frac{1}{2} \left[\frac{1}{2} \sin 2x \right]_0^{\pi/4} \\ &= -\frac{1}{2} [x \cos 2x]_0^{\pi/4} + \frac{1}{4} [\sin 2x]_0^{\pi/4} \\ &= -\frac{1}{2} \left\{ \frac{\pi}{4} \cdot \cos \frac{\pi}{2} - 0 \cdot \cos 0 \right\} + \frac{1}{4} \left\{ \sin \frac{\pi}{2} - \sin 0 \right\} \\ &= -\frac{1}{2} \{0 - 0\} + \frac{1}{4} \{1 - 0\}, \quad \text{since } \cos \frac{\pi}{2} = 0, \\ &= \frac{1}{4}. \quad \sin \frac{\pi}{2} = 1, \\ &\quad \text{and } \sin 0 = 0, \end{aligned}$$

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Exercise 11. Note that if a logarithm function is involved then we choose that factor to be u .

$$\begin{aligned} \text{i.e. } \int_{1/2}^1 x^4 \ln 2x \, dx &= \left[(\ln 2x) \cdot \left(\frac{x^5}{5} \right) \right]_{1/2}^1 - \int_{1/2}^1 \left(\frac{1}{x} \right) \cdot \left(\frac{x^5}{5} \right) dx \\ &= \frac{1}{5} [x^5 \ln 2x]_{1/2}^1 - \frac{1}{5} \int_{1/2}^1 x^4 dx \\ &= \frac{1}{5} [x^5 \ln 2x]_{1/2}^1 - \frac{1}{5} \left[\frac{x^5}{5} \right]_{1/2}^1 \\ &= \frac{1}{5} [x^5 \ln 2x]_{1/2}^1 - \frac{1}{25} [x^5]_{1/2}^1 \\ &= \frac{1}{5} \left\{ (1 \cdot \ln 2) - \left(\frac{1}{2} \right)^5 \cdot \ln 1 \right\} - \frac{1}{25} \left\{ 1^5 - \left(\frac{1}{2} \right)^5 \right\} \end{aligned}$$

$$\begin{aligned} &= \frac{1}{5} \ln 2 - \frac{1}{25} \left\{ 1 - \frac{1}{32} \right\}, \quad \text{since } \ln 1 = 0 \text{ and } 2^5 = 32, \\ &= \frac{1}{5} \ln 2 - \frac{1}{25} \cdot \frac{31}{32} \\ &= \frac{1}{5} \ln 2 - \frac{31}{800}. \end{aligned}$$

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Exercise 12. For definite integrals, we can either use integration by parts in the form:

$$\int_a^b u \frac{dv}{dx} dx = [uv]_a^b - \int_a^b \frac{du}{dx} \cdot v dx$$

or we can work out $\int u \frac{dv}{dx} dx$, i.e. without the limits, first and then apply the limits to the final result. We will do that here. So, to work out $\int_0^\pi 3x^2 \cos\left(\frac{x}{2}\right) dx$, we will consider the indefinite integral first:

$$\begin{aligned} \int 3x^2 \cos\left(\frac{x}{2}\right) dx &= 3x^2 \left(\frac{1}{\left(\frac{1}{2}\right)}\right) \sin\left(\frac{x}{2}\right) - \int 6x \cdot \left(\frac{1}{\left(\frac{1}{2}\right)} \sin\left(\frac{x}{2}\right)\right) dx \\ &= 6x^2 \sin\left(\frac{x}{2}\right) - 12 \int x \sin\left(\frac{x}{2}\right) dx \\ &\hspace{15em} \text{(use integration by parts, again)} \\ &= 6x^2 \sin\left(\frac{x}{2}\right) - 12 \left\{ x \left(-2 \cos\left(\frac{x}{2}\right)\right) - \int -2 \cos\left(\frac{x}{2}\right) dx \right\} \end{aligned}$$

$$\begin{aligned} &= 6x^2 \sin\left(\frac{x}{2}\right) - 12 \left\{ -2x \cos\left(\frac{x}{2}\right) + 2 \int \cos\left(\frac{x}{2}\right) dx \right\} \\ \text{i.e. } \int 3x^2 \cos\left(\frac{x}{2}\right) dx &= 6x^2 \sin\left(\frac{x}{2}\right) + 24x \cos\left(\frac{x}{2}\right) - 24 \int \cos\left(\frac{x}{2}\right) dx \\ &= 6x^2 \sin\left(\frac{x}{2}\right) + 24x \cos\left(\frac{x}{2}\right) - 24 \cdot 2 \cdot \sin\left(\frac{x}{2}\right) + C. \end{aligned}$$

On the next page, we will evaluate the definite integral ...

So,

$$\begin{aligned}\int_0^\pi 3x^2 \cos\left(\frac{x}{2}\right) dx &= \left[6x^2 \sin\left(\frac{x}{2}\right) + 24x \cos\left(\frac{x}{2}\right) - 48 \sin\left(\frac{x}{2}\right)\right]_0^\pi \\ &= \left\{6\pi^2 \sin\left(\frac{\pi}{2}\right) - 0\right\} + 24 \left\{\pi \cos\left(\frac{\pi}{2}\right) - 0\right\} \\ &\quad - 48 \left\{\sin\left(\frac{\pi}{2}\right) - 0\right\}, \\ &\quad \text{since } \sin 0 = 0, \\ &= 6\pi^2 - 48, \text{ since } \sin\left(\frac{\pi}{2}\right) = 1 \text{ and } \cos\left(\frac{\pi}{2}\right) = 0, \\ &= 6(\pi^2 - 8).\end{aligned}$$

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Exercise 13.

$$\begin{aligned}\int x^3 e^x dx &= x^3 e^x - \int 3x^2 e^x dx \\ &= x^3 e^x - 3 \left\{ x^2 e^x - \int 2x e^x dx \right\}, && \text{using integration by parts,} \\ &= x^3 e^x - 3x^2 e^x + 6 \int x e^x dx \\ &= x^3 e^x - 3x^2 e^x + 6 \left\{ x e^x - \int e^x dx \right\}, && \text{using integration by parts,} \\ &= x^3 e^x - 3x^2 e^x + 6x e^x - 6e^x + C \\ &= e^x (x^3 - 3x^2 + 6x - 6) + C.\end{aligned}$$

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Exercise 14.

$$\begin{aligned}\text{Let } I &= \int e^{3x} \cos x \, dx \\ &= e^{3x} \sin x - \int 3e^{3x} \sin x \, dx \\ &= e^{3x} \sin x - 3 \int e^{3x} \sin x \, dx \\ &= e^{3x} \sin x - 3 \left\{ e^{3x} \cdot (-\cos x) - \int 3e^{3x} \cdot (-\cos x) \, dx \right\} \\ &= e^{3x} \sin x + 3e^{3x} \cos x - 9 \int e^{3x} \cos x \, dx \\ &= e^{3x} \sin x + 3e^{3x} \cos x - 9I\end{aligned}$$

$$\text{i.e. } 10I = e^{3x} \sin x + 3e^{3x} \cos x + C_1$$

$$\text{i.e. } I = \frac{1}{10} e^{3x} (\sin x + 3 \cos x) + C, \text{ where } C = \frac{C_1}{10}.$$

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