

FOURIER SERIES

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A self-contained Tutorial Module for learning
the technique of Fourier series analysis

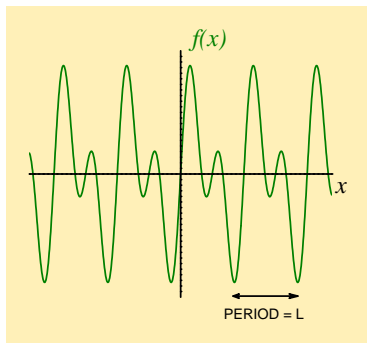
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1. Theory

● A graph of **periodic** function $f(x)$ that has period L exhibits the same pattern every L units along the x -axis, so that $f(x + L) = f(x)$ for every value of x . If we know what the function looks like over one complete period, we can thus sketch a graph of the function over a wider interval of x (that may contain many periods)



● This property of repetition defines a **fundamental spatial frequency** $k = \frac{2\pi}{L}$ that can be used to give a **first approximation** to the periodic pattern $f(x)$:

$$f(x) \simeq c_1 \sin(kx + \alpha_1) = a_1 \cos(kx) + b_1 \sin(kx),$$

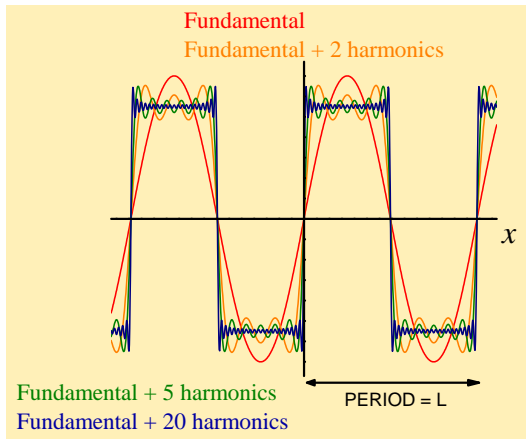
where symbols with subscript 1 are constants that determine the amplitude and phase of this first approximation

● A much **better approximation** of the periodic pattern $f(x)$ can be built up by adding an appropriate combination of **harmonics** to this fundamental (sine-wave) pattern. For example, adding

$$\begin{aligned} c_2 \sin(2kx + \alpha_2) &= a_2 \cos(2kx) + b_2 \sin(2kx) && \text{(the 2nd harmonic)} \\ c_3 \sin(3kx + \alpha_3) &= a_3 \cos(3kx) + b_3 \sin(3kx) && \text{(the 3rd harmonic)} \end{aligned}$$

Here, symbols with subscripts are constants that determine the amplitude and phase of each harmonic contribution

One can even approximate a square-wave pattern with a suitable sum that involves a fundamental sine-wave plus a combination of harmonics of this fundamental frequency. This sum is called a **Fourier series**



● In this Tutorial, we consider working out Fourier series for functions $f(x)$ with period $L = 2\pi$. Their fundamental frequency is then $k = \frac{2\pi}{L} = 1$, and their Fourier series representations involve terms like

$$\begin{aligned}a_1 \cos x &, & b_1 \sin x \\a_2 \cos 2x &, & b_2 \sin 2x \\a_3 \cos 3x &, & b_3 \sin 3x\end{aligned}$$

We also include a constant term $a_0/2$ in the Fourier series. This allows us to represent functions that are, for example, entirely above the x -axis. With a sufficient number of harmonics included, our approximate series can exactly represent a given function $f(x)$

$$\begin{aligned}f(x) = a_0/2 &+ a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + \dots \\&+ b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + \dots\end{aligned}$$

A more compact way of writing the Fourier series of a function $f(x)$, with period 2π , uses the variable subscript $n = 1, 2, 3, \dots$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx]$$

● We need to work out the **Fourier coefficients** (a_0 , a_n and b_n) for given functions $f(x)$. This process is broken down into three steps

STEP ONE

$$a_0 = \frac{1}{\pi} \int_{2\pi} f(x) dx$$

STEP TWO

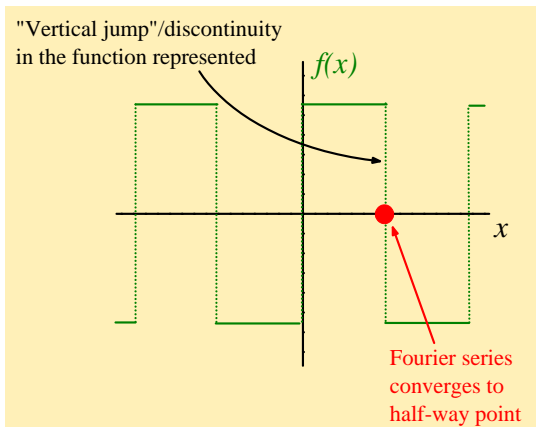
$$a_n = \frac{1}{\pi} \int_{2\pi} f(x) \cos nx dx$$

STEP THREE

$$b_n = \frac{1}{\pi} \int_{2\pi} f(x) \sin nx dx$$

where integrations are over a single interval in x of $L = 2\pi$

● Finally, specifying a particular value of $x = x_1$ in a Fourier series, gives a series of constants that should equal $f(x_1)$. However, if $f(x)$ is discontinuous at this value of x , then the series converges to a value that is **half-way** between the two possible function values



2. Exercises

Click on [EXERCISE](#) links for full worked solutions (7 exercises in total).

EXERCISE 1.

Let $f(x)$ be a function of period 2π such that

$$f(x) = \begin{cases} 1, & -\pi < x < 0 \\ 0, & 0 < x < \pi. \end{cases}$$

- a) Sketch a graph of $f(x)$ in the interval $-2\pi < x < 2\pi$
- b) Show that the Fourier series for $f(x)$ in the interval $-\pi < x < \pi$ is

$$\frac{1}{2} - \frac{2}{\pi} \left[\sin x + \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x + \dots \right]$$

- c) By giving an appropriate value to x , show that

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$$

● [THEORY](#) ● [ANSWERS](#) ● [INTEGRALS](#) ● [TRIG](#) ● [NOTATION](#)

EXERCISE 2.

Let $f(x)$ be a function of period 2π such that

$$f(x) = \begin{cases} 0, & -\pi < x < 0 \\ x, & 0 < x < \pi. \end{cases}$$

- a) Sketch a graph of $f(x)$ in the interval $-3\pi < x < 3\pi$
- b) Show that the Fourier series for $f(x)$ in the interval $-\pi < x < \pi$ is

$$\begin{aligned} \frac{\pi}{4} - \frac{2}{\pi} \left[\cos x + \frac{1}{3^2} \cos 3x + \frac{1}{5^2} \cos 5x + \dots \right] \\ + \left[\sin x - \frac{1}{2} \sin 2x + \frac{1}{3} \sin 3x - \dots \right] \end{aligned}$$

- c) By giving appropriate values to x , show that

$$(i) \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots \quad \text{and} \quad (ii) \frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots$$

● THEORY ● ANSWERS ● INTEGRALS ● TRIG ● NOTATION

EXERCISE 3.

Let $f(x)$ be a function of period 2π such that

$$f(x) = \begin{cases} x, & 0 < x < \pi \\ \pi, & \pi < x < 2\pi . \end{cases}$$

- a) Sketch a graph of $f(x)$ in the interval $-2\pi < x < 2\pi$
- b) Show that the Fourier series for $f(x)$ in the interval $0 < x < 2\pi$ is

$$\begin{aligned} \frac{3\pi}{4} - \frac{2}{\pi} \left[\cos x + \frac{1}{3^2} \cos 3x + \frac{1}{5^2} \cos 5x + \dots \right] \\ - \left[\sin x + \frac{1}{2} \sin 2x + \frac{1}{3} \sin 3x + \dots \right] \end{aligned}$$

c) By giving appropriate values to x , show that

$$(i) \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots \quad \text{and} \quad (ii) \frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots$$

● THEORY ● ANSWERS ● INTEGRALS ● TRIG ● NOTATION

EXERCISE 4.

Let $f(x)$ be a function of period 2π such that

$$f(x) = \frac{x}{2} \text{ over the interval } 0 < x < 2\pi.$$

- a) Sketch a graph of $f(x)$ in the interval $0 < x < 4\pi$
- b) Show that the Fourier series for $f(x)$ in the interval $0 < x < 2\pi$ is

$$\frac{\pi}{2} - \left[\sin x + \frac{1}{2} \sin 2x + \frac{1}{3} \sin 3x + \dots \right]$$

- c) By giving an appropriate value to x , show that

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \dots$$

EXERCISE 5.

Let $f(x)$ be a function of period 2π such that

$$f(x) = \begin{cases} \pi - x, & 0 < x < \pi \\ 0, & \pi < x < 2\pi \end{cases}$$

a) Sketch a graph of $f(x)$ in the interval $-2\pi < x < 2\pi$

b) Show that the Fourier series for $f(x)$ in the interval $0 < x < 2\pi$ is

$$\begin{aligned} \frac{\pi}{4} + \frac{2}{\pi} \left[\cos x + \frac{1}{3^2} \cos 3x + \frac{1}{5^2} \cos 5x + \dots \right] \\ + \sin x + \frac{1}{2} \sin 2x + \frac{1}{3} \sin 3x + \frac{1}{4} \sin 4x + \dots \end{aligned}$$

c) By giving an appropriate value to x , show that

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots$$

● THEORY ● ANSWERS ● INTEGRALS ● TRIG ● NOTATION

EXERCISE 6.

Let $f(x)$ be a function of period 2π such that

$$f(x) = x \text{ in the range } -\pi < x < \pi.$$

- a) Sketch a graph of $f(x)$ in the interval $-3\pi < x < 3\pi$
- b) Show that the Fourier series for $f(x)$ in the interval $-\pi < x < \pi$ is

$$2 \left[\sin x - \frac{1}{2} \sin 2x + \frac{1}{3} \sin 3x - \dots \right]$$

- c) By giving an appropriate value to x , show that

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$$

EXERCISE 7.

Let $f(x)$ be a function of period 2π such that

$$f(x) = x^2 \text{ over the interval } -\pi < x < \pi.$$

- a) Sketch a graph of $f(x)$ in the interval $-3\pi < x < 3\pi$
- b) Show that the Fourier series for $f(x)$ in the interval $-\pi < x < \pi$ is

$$\frac{\pi^2}{3} - 4 \left[\cos x - \frac{1}{2^2} \cos 2x + \frac{1}{3^2} \cos 3x - \dots \right]$$

- c) By giving an appropriate value to x , show that

$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots$$

● THEORY ● ANSWERS ● INTEGRALS ● TRIG ● NOTATION

3. Answers

The sketches asked for in part (a) of each exercise are given within the full worked solutions – click on the [EXERCISE](#) links to see these solutions

The answers below are suggested values of x to get the series of constants quoted in part (c) of each exercise

1. $x = \frac{\pi}{2}$,

2. (i) $x = \frac{\pi}{2}$, (ii) $x = 0$,

3. (i) $x = \frac{\pi}{2}$, (ii) $x = 0$,

4. $x = \frac{\pi}{2}$,

5. $x = 0$,

6. $x = \frac{\pi}{2}$,

7. $x = \pi$.

4. Integrals

Formula for integration by parts: $\int_a^b u \frac{dv}{dx} dx = [uv]_a^b - \int_a^b \frac{du}{dx} v dx$

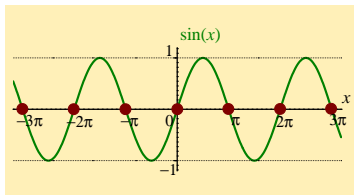
$f(x)$	$\int f(x) dx$	$f(x)$	$\int f(x) dx$
x^n	$\frac{x^{n+1}}{n+1} \quad (n \neq -1)$	$[g(x)]^n g'(x)$	$\frac{[g(x)]^{n+1}}{n+1} \quad (n \neq -1)$
$\frac{1}{x}$	$\ln x $	$\frac{g'(x)}{g(x)}$	$\ln g(x) $
e^x	e^x	a^x	$\frac{a^x}{\ln a} \quad (a > 0)$
$\sin x$	$-\cos x$	$\sinh x$	$\cosh x$
$\cos x$	$\sin x$	$\cosh x$	$\sinh x$
$\tan x$	$-\ln \cos x $	$\tanh x$	$\ln \cosh x$
$\operatorname{cosec} x$	$\ln \left \tan \frac{x}{2} \right $	$\operatorname{cosech} x$	$\ln \left \tanh \frac{x}{2} \right $
$\sec x$	$\ln \sec x + \tan x $	$\operatorname{sech} x$	$2 \tan^{-1} e^x$
$\sec^2 x$	$\tan x$	$\operatorname{sech}^2 x$	$\tanh x$
$\cot x$	$\ln \sin x $	$\operatorname{coth} x$	$\ln \sinh x $
$\sin^2 x$	$\frac{x}{2} - \frac{\sin 2x}{4}$	$\sinh^2 x$	$\frac{\sinh 2x}{4} - \frac{x}{2}$
$\cos^2 x$	$\frac{x}{2} + \frac{\sin 2x}{4}$	$\cosh^2 x$	$\frac{\sinh 2x}{4} + \frac{x}{2}$

$f(x)$	$\int f(x) dx$	$f(x)$	$\int f(x) dx$
$\frac{1}{a^2+x^2}$	$\frac{1}{a} \tan^{-1} \frac{x}{a}$ $(a > 0)$	$\frac{1}{a^2-x^2}$	$\frac{1}{2a} \ln \left \frac{a+x}{a-x} \right $ ($0 < x < a$) $\frac{1}{2a} \ln \left \frac{x-a}{x+a} \right $ ($ x > a > 0$)
$\frac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1} \frac{x}{a}$ $(-a < x < a)$	$\frac{1}{\sqrt{a^2+x^2}}$	$\ln \left \frac{x+\sqrt{a^2+x^2}}{a} \right $ ($a > 0$) $\ln \left \frac{x+\sqrt{x^2-a^2}}{a} \right $ ($x > a > 0$)
$\sqrt{a^2-x^2}$	$\frac{a^2}{2} \left[\sin^{-1} \left(\frac{x}{a} \right) + \frac{x\sqrt{a^2-x^2}}{a^2} \right]$	$\sqrt{a^2+x^2}$	$\frac{a^2}{2} \left[\sinh^{-1} \left(\frac{x}{a} \right) + \frac{x\sqrt{a^2+x^2}}{a^2} \right]$ $\frac{a^2}{2} \left[-\cosh^{-1} \left(\frac{x}{a} \right) + \frac{x\sqrt{x^2-a^2}}{a^2} \right]$

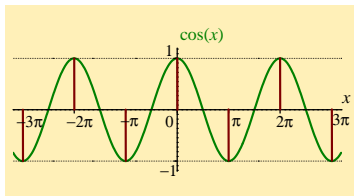
5. Useful trig results

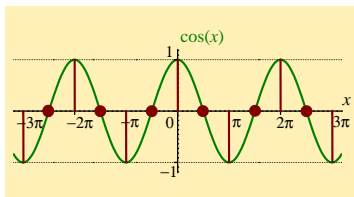
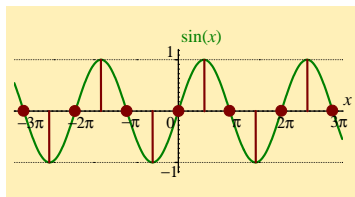
When calculating the Fourier coefficients a_n and b_n , for which $n = 1, 2, 3, \dots$, the following trig. results are useful. Each of these results, which are also true for $n = 0, -1, -2, -3, \dots$, can be deduced from the graph of $\sin x$ or that of $\cos x$

● $\sin n\pi = 0$



● $\cos n\pi = (-1)^n$





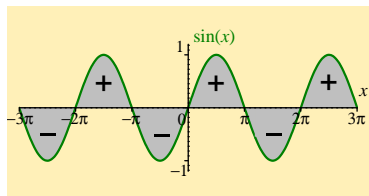
$$\bullet \sin n \frac{\pi}{2} = \begin{cases} 0 & , n \text{ even} \\ 1 & , n = 1, 5, 9, \dots \\ -1 & , n = 3, 7, 11, \dots \end{cases}$$

$$\bullet \cos n \frac{\pi}{2} = \begin{cases} 0 & , n \text{ odd} \\ 1 & , n = 0, 4, 8, \dots \\ -1 & , n = 2, 6, 10, \dots \end{cases}$$

Areas cancel when
when integrating
over whole periods

$$\bullet \int_{2\pi} \sin nx \, dx = 0$$

$$\bullet \int_{2\pi} \cos nx \, dx = 0$$



6. Alternative notation

- For a waveform $f(x)$ with period $L = \frac{2\pi}{k}$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos nkx + b_n \sin nkx]$$

The corresponding Fourier coefficients are

STEP ONE

$$a_0 = \frac{2}{L} \int_L f(x) dx$$

STEP TWO

$$a_n = \frac{2}{L} \int_L f(x) \cos nkx dx$$

STEP THREE

$$b_n = \frac{2}{L} \int_L f(x) \sin nkx dx$$

and integrations are over a single interval in x of L

● For a waveform $f(x)$ with period $2L = \frac{2\pi}{k}$, we have that $k = \frac{2\pi}{2L} = \frac{\pi}{L}$ and $nkx = \frac{n\pi x}{L}$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right]$$

The corresponding Fourier coefficients are

STEP ONE

$$a_0 = \frac{1}{2L} \int_{2L} f(x) dx$$

STEP TWO

$$a_n = \frac{1}{2L} \int_{2L} f(x) \cos \frac{n\pi x}{L} dx$$

STEP THREE

$$b_n = \frac{1}{2L} \int_{2L} f(x) \sin \frac{n\pi x}{L} dx$$

and integrations are over a single interval in x of $2L$

- For a waveform $f(t)$ with period $T = \frac{2\pi}{\omega}$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos n\omega t + b_n \sin n\omega t]$$

The corresponding Fourier coefficients are

STEP ONE

$$a_0 = \frac{2}{T} \int_T f(t) dt$$

STEP TWO

$$a_n = \frac{2}{T} \int_T f(t) \cos n\omega t dt$$

STEP THREE

$$b_n = \frac{2}{T} \int_T f(t) \sin n\omega t dt$$

and integrations are over a single interval in t of T

7. Tips on using solutions

- When looking at the THEORY, ANSWERS, INTEGRALS, TRIG or NOTATION pages, use the [Back](#) button (at the bottom of the page) to return to the exercises

- Use the solutions intelligently. For example, they can help you get started on an exercise, or they can allow you to check whether your intermediate results are correct

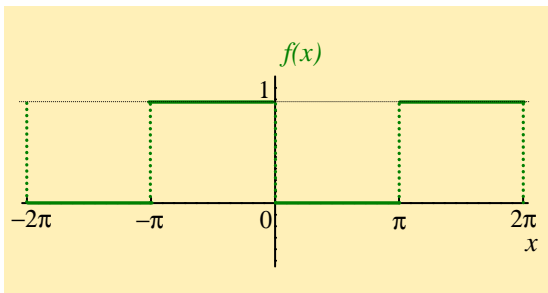
- Try to make less use of the full solutions as you work your way through the Tutorial

Full worked solutions

Exercise 1.

$$f(x) = \begin{cases} 1, & -\pi < x < 0 \\ 0, & 0 < x < \pi, \end{cases} \text{ and has period } 2\pi$$

a) Sketch a graph of $f(x)$ in the interval $-2\pi < x < 2\pi$



b) Fourier series representation of $f(x)$

STEP ONE

$$\begin{aligned}a_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{\pi} \int_{-\pi}^0 f(x) dx + \frac{1}{\pi} \int_0^{\pi} f(x) dx \\&= \frac{1}{\pi} \int_{-\pi}^0 1 \cdot dx + \frac{1}{\pi} \int_0^{\pi} 0 \cdot dx \\&= \frac{1}{\pi} \int_{-\pi}^0 dx \\&= \frac{1}{\pi} [x]_{-\pi}^0 \\&= \frac{1}{\pi} (0 - (-\pi)) \\&= \frac{1}{\pi} \cdot (\pi) \\ \text{i.e. } a_0 &= 1.\end{aligned}$$

STEP TWO

$$\begin{aligned}a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx = \frac{1}{\pi} \int_{-\pi}^0 f(x) \cos nx \, dx + \frac{1}{\pi} \int_0^{\pi} f(x) \cos nx \, dx \\&= \frac{1}{\pi} \int_{-\pi}^0 1 \cdot \cos nx \, dx + \frac{1}{\pi} \int_0^{\pi} 0 \cdot \cos nx \, dx \\&= \frac{1}{\pi} \int_{-\pi}^0 \cos nx \, dx \\&= \frac{1}{\pi} \left[\frac{\sin nx}{n} \right]_{-\pi}^0 = \frac{1}{n\pi} [\sin nx]_{-\pi}^0 \\&= \frac{1}{n\pi} (\sin 0 - \sin(-n\pi)) \\&= \frac{1}{n\pi} (0 + \sin n\pi) \\ \text{i.e. } a_n &= \frac{1}{n\pi} (0 + 0) = 0.\end{aligned}$$

STEP THREE

$$\begin{aligned}
 b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx \\
 &= \frac{1}{\pi} \int_{-\pi}^0 f(x) \sin nx \, dx + \frac{1}{\pi} \int_0^{\pi} f(x) \sin nx \, dx \\
 &= \frac{1}{\pi} \int_{-\pi}^0 1 \cdot \sin nx \, dx + \frac{1}{\pi} \int_0^{\pi} 0 \cdot \sin nx \, dx
 \end{aligned}$$

$$\begin{aligned}
 \text{i.e. } b_n &= \frac{1}{\pi} \int_{-\pi}^0 \sin nx \, dx = \frac{1}{\pi} \left[\frac{-\cos nx}{n} \right]_{-\pi}^0 \\
 &= -\frac{1}{n\pi} [\cos nx]_{-\pi}^0 = -\frac{1}{n\pi} (\cos 0 - \cos(-n\pi)) \\
 &= -\frac{1}{n\pi} (1 - \cos n\pi) = -\frac{1}{n\pi} (1 - (-1)^n), \text{ see TRIG}
 \end{aligned}$$

$$\text{i.e. } b_n = \begin{cases} 0 & , n \text{ even} \\ -\frac{2}{n\pi} & , n \text{ odd} \end{cases}, \text{ since } (-1)^n = \begin{cases} 1 & , n \text{ even} \\ -1 & , n \text{ odd} \end{cases}$$

We now have that

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx]$$

with the three steps giving

$$a_0 = 1, \quad a_n = 0, \quad \text{and} \quad b_n = \begin{cases} 0 & , n \text{ even} \\ -\frac{2}{n\pi} & , n \text{ odd} \end{cases}$$

It may be helpful to construct a table of values of b_n

n	1	2	3	4	5
b_n	$-\frac{2}{\pi}$	0	$-\frac{2}{\pi} \left(\frac{1}{3}\right)$	0	$-\frac{2}{\pi} \left(\frac{1}{5}\right)$

Substituting our results now gives the required series

$$f(x) = \frac{1}{2} - \frac{2}{\pi} \left[\sin x + \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x + \dots \right]$$

c) Pick an appropriate value of x , to show that

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$$

Comparing this series with

$$f(x) = \frac{1}{2} - \frac{2}{\pi} \left[\sin x + \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x + \dots \right],$$

we need to introduce a minus sign in front of the constants $\frac{1}{3}, \frac{1}{7}, \dots$

So we need $\sin x = 1, \sin 3x = -1, \sin 5x = 1, \sin 7x = -1$, etc

The first condition of $\sin x = 1$ suggests trying $x = \frac{\pi}{2}$.

This choice gives

$$\sin \frac{\pi}{2} + \frac{1}{3} \sin 3 \frac{\pi}{2} + \frac{1}{5} \sin 5 \frac{\pi}{2} + \frac{1}{7} \sin 7 \frac{\pi}{2}$$

i.e.

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7}$$

Looking at the graph of $f(x)$, we also have that $f(\frac{\pi}{2}) = 0$.

Picking $x = \frac{\pi}{2}$ thus gives

$$0 = \frac{1}{2} - \frac{2}{\pi} \left[\sin \frac{\pi}{2} + \frac{1}{3} \sin \frac{3\pi}{2} + \frac{1}{5} \sin \frac{5\pi}{2} \right. \\ \left. + \frac{1}{7} \sin \frac{7\pi}{2} + \dots \right]$$

$$\text{i.e. } 0 = \frac{1}{2} - \frac{2}{\pi} \left[\begin{array}{cccc} 1 & - & \frac{1}{3} & + & \frac{1}{5} \\ & & & - & \frac{1}{7} & + \dots \end{array} \right]$$

A little manipulation then gives a series representation of $\frac{\pi}{4}$

$$\frac{2}{\pi} \left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots \right] = \frac{1}{2}$$

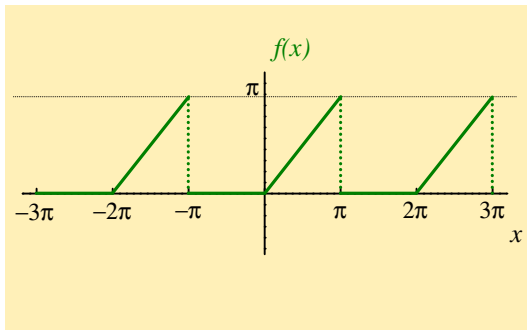
$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots = \frac{\pi}{4}.$$

[Return to Exercise 1](#)

Exercise 2.

$$f(x) = \begin{cases} 0, & -\pi < x < 0 \\ x, & 0 < x < \pi, \end{cases} \quad \text{and has period } 2\pi$$

a) Sketch a graph of $f(x)$ in the interval $-3\pi < x < 3\pi$



b) Fourier series representation of $f(x)$

STEP ONE

$$\begin{aligned}a_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{\pi} \int_{-\pi}^0 f(x) dx + \frac{1}{\pi} \int_0^{\pi} f(x) dx \\&= \frac{1}{\pi} \int_{-\pi}^0 0 \cdot dx + \frac{1}{\pi} \int_0^{\pi} x dx \\&= \frac{1}{\pi} \left[\frac{x^2}{2} \right]_0^{\pi} \\&= \frac{1}{\pi} \left(\frac{\pi^2}{2} - 0 \right) \\ \text{i.e. } a_0 &= \frac{\pi}{2} .\end{aligned}$$

STEP TWO

$$\begin{aligned}
 a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx = \frac{1}{\pi} \int_{-\pi}^0 f(x) \cos nx \, dx + \frac{1}{\pi} \int_0^{\pi} f(x) \cos nx \, dx \\
 &= \frac{1}{\pi} \int_{-\pi}^0 0 \cdot \cos nx \, dx + \frac{1}{\pi} \int_0^{\pi} x \cos nx \, dx
 \end{aligned}$$

$$\text{i.e. } a_n = \frac{1}{\pi} \int_0^{\pi} x \cos nx \, dx = \frac{1}{\pi} \left\{ \left[x \frac{\sin nx}{n} \right]_0^{\pi} - \int_0^{\pi} \frac{\sin nx}{n} \, dx \right\}$$

(using [integration by parts](#))

$$\begin{aligned}
 \text{i.e. } a_n &= \frac{1}{\pi} \left\{ \left(\pi \frac{\sin n\pi}{n} - 0 \right) - \frac{1}{n} \left[-\frac{\cos nx}{n} \right]_0^{\pi} \right\} \\
 &= \frac{1}{\pi} \left\{ (0 - 0) + \frac{1}{n^2} [\cos nx]_0^{\pi} \right\} \\
 &= \frac{1}{\pi n^2} \{ \cos n\pi - \cos 0 \} = \frac{1}{\pi n^2} \{ (-1)^n - 1 \}
 \end{aligned}$$

$$\text{i.e. } a_n = \begin{cases} 0 & , n \text{ even} \\ -\frac{2}{\pi n^2} & , n \text{ odd} \end{cases}, \text{ see } \text{TRIG.}$$

STEP THREE

$$\begin{aligned} b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx = \frac{1}{\pi} \int_{-\pi}^0 f(x) \sin nx \, dx + \frac{1}{\pi} \int_0^{\pi} f(x) \sin nx \, dx \\ &= \frac{1}{\pi} \int_{-\pi}^0 0 \cdot \sin nx \, dx + \frac{1}{\pi} \int_0^{\pi} x \sin nx \, dx \\ \text{i.e. } b_n &= \frac{1}{\pi} \int_0^{\pi} x \sin nx \, dx = \frac{1}{\pi} \left\{ \left[x \left(-\frac{\cos nx}{n} \right) \right]_0^{\pi} - \int_0^{\pi} \left(-\frac{\cos nx}{n} \right) dx \right\} \\ &\quad \text{(using integration by parts)} \\ &= \frac{1}{\pi} \left\{ -\frac{1}{n} [x \cos nx]_0^{\pi} + \frac{1}{n} \int_0^{\pi} \cos nx \, dx \right\} \\ &= \frac{1}{\pi} \left\{ -\frac{1}{n} (\pi \cos n\pi - 0) + \frac{1}{n} \left[\frac{\sin nx}{n} \right]_0^{\pi} \right\} \\ &= -\frac{1}{n} (-1)^n + \frac{1}{\pi n^2} (0 - 0), \text{ see TRIG} \\ &= -\frac{1}{n} (-1)^n \end{aligned}$$

$$\text{i.e. } b_n = \begin{cases} -\frac{1}{n} & , n \text{ even} \\ +\frac{1}{n} & , n \text{ odd} \end{cases}$$

We now have

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx]$$

$$\text{where } a_0 = \frac{\pi}{2}, \quad a_n = \begin{cases} 0 & , n \text{ even} \\ -\frac{2}{\pi n^2} & , n \text{ odd} \end{cases}, \quad b_n = \begin{cases} -\frac{1}{n} & , n \text{ even} \\ \frac{1}{n} & , n \text{ odd} \end{cases}$$

Constructing a table of values gives

n	1	2	3	4	5
a_n	$-\frac{2}{\pi}$	0	$-\frac{2}{\pi} \cdot \frac{1}{3^2}$	0	$-\frac{2}{\pi} \cdot \frac{1}{5^2}$
b_n	1	$-\frac{1}{2}$	$\frac{1}{3}$	$-\frac{1}{4}$	$\frac{1}{5}$

This table of coefficients gives

$$\begin{aligned}
 f(x) &= \frac{1}{2} \left(\frac{\pi}{2} \right) + \left(-\frac{2}{\pi} \right) \cos x + 0 \cdot \cos 2x \\
 &+ \left(-\frac{2}{\pi} \cdot \frac{1}{3^2} \right) \cos 3x + 0 \cdot \cos 4x \\
 &+ \left(-\frac{2}{\pi} \cdot \frac{1}{5^2} \right) \cos 5x + \dots \\
 &+ \sin x - \frac{1}{2} \sin 2x + \frac{1}{3} \sin 3x - \dots
 \end{aligned}$$

$$\begin{aligned}
 \text{i.e. } f(x) &= \frac{\pi}{4} - \frac{2}{\pi} \left[\cos x + \frac{1}{3^2} \cos 3x + \frac{1}{5^2} \cos 5x + \dots \right] \\
 &+ \left[\sin x - \frac{1}{2} \sin 2x + \frac{1}{3} \sin 3x - \dots \right]
 \end{aligned}$$

and we have found the required series!

c) Pick an appropriate value of x , to show that

$$(i) \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$$

Comparing this series with

$$f(x) = \frac{\pi}{4} - \frac{2}{\pi} \left[\cos x + \frac{1}{3^2} \cos 3x + \frac{1}{5^2} \cos 5x + \dots \right] \\ + \left[\sin x - \frac{1}{2} \sin 2x + \frac{1}{3} \sin 3x - \dots \right],$$

the required series of constants does not involve terms like $\frac{1}{3^2}, \frac{1}{5^2}, \frac{1}{7^2}, \dots$

So we need to pick a value of x that sets the $\cos nx$ terms to zero.

The **TRIG** section shows that $\cos n\frac{\pi}{2} = 0$ when n is odd, and note also that $\cos nx$ terms in the Fourier series all have odd n

$$\text{i.e. } \cos x = \cos 3x = \cos 5x = \dots = 0 \quad \text{when } x = \frac{\pi}{2},$$

$$\text{i.e. } \cos \frac{\pi}{2} = \cos 3\frac{\pi}{2} = \cos 5\frac{\pi}{2} = \dots = 0$$

Setting $x = \frac{\pi}{2}$ in the series for $f(x)$ gives

$$\begin{aligned}
 f\left(\frac{\pi}{2}\right) &= \frac{\pi}{4} - \frac{2}{\pi} \left[\cos \frac{\pi}{2} + \frac{1}{3^2} \cos \frac{3\pi}{2} + \frac{1}{5^2} \cos \frac{5\pi}{2} + \dots \right] \\
 &\quad + \left[\sin \frac{\pi}{2} - \frac{1}{2} \sin \frac{2\pi}{2} + \frac{1}{3} \sin \frac{3\pi}{2} - \frac{1}{4} \sin \frac{4\pi}{2} + \frac{1}{5} \sin \frac{5\pi}{2} - \dots \right] \\
 &= \frac{\pi}{4} - \frac{2}{\pi} [0 + 0 + 0 + \dots] \\
 &\quad + \left[1 - \frac{1}{2} \underbrace{\sin \pi}_{=0} + \frac{1}{3} \cdot (-1) - \frac{1}{4} \underbrace{\sin 2\pi}_{=0} + \frac{1}{5} \cdot (1) - \dots \right]
 \end{aligned}$$

The graph of $f(x)$ shows that $f\left(\frac{\pi}{2}\right) = \frac{\pi}{2}$, so that

$$\begin{aligned}
 \frac{\pi}{2} &= \frac{\pi}{4} + 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots \\
 \text{i.e. } \frac{\pi}{4} &= 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots
 \end{aligned}$$

Pick an appropriate value of x , to show that

$$(ii) \quad \frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots$$

Compare this series with

$$f(x) = \frac{\pi}{4} - \frac{2}{\pi} \left[\cos x + \frac{1}{3^2} \cos 3x + \frac{1}{5^2} \cos 5x + \dots \right] \\ + \left[\sin x - \frac{1}{2} \sin 2x + \frac{1}{3} \sin 3x - \dots \right].$$

This time, we want to use the coefficients of the $\cos nx$ terms, and the same choice of x needs to set the $\sin nx$ terms to zero

Picking $x = 0$ gives

$$\sin x = \sin 2x = \sin 3x = 0 \quad \text{and} \quad \cos x = \cos 3x = \cos 5x = 1$$

Note also that the graph of $f(x)$ gives $f(x) = 0$ when $x = 0$

So, picking $x = 0$ gives

$$0 = \frac{\pi}{4} - \frac{2}{\pi} \left[\cos 0 + \frac{1}{3^2} \cos 0 + \frac{1}{5^2} \cos 0 + \frac{1}{7^2} \cos 0 + \dots \right] \\ + \sin 0 - \frac{\sin 0}{2} + \frac{\sin 0}{3} - \dots$$

$$\text{i.e. } 0 = \frac{\pi}{4} - \frac{2}{\pi} \left[1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots \right] + 0 - 0 + 0 - \dots$$

We then find that

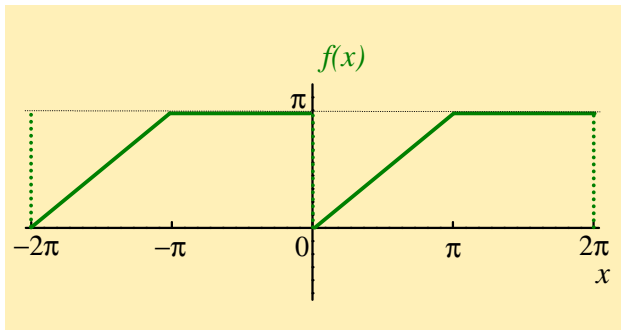
$$\frac{2}{\pi} \left[1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots \right] = \frac{\pi}{4} \\ \text{and} \quad 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots = \frac{\pi^2}{8}.$$

[Return to Exercise 2](#)

Exercise 3.

$$f(x) = \begin{cases} x, & 0 < x < \pi \\ \pi, & \pi < x < 2\pi, \end{cases} \text{ and has period } 2\pi$$

a) Sketch a graph of $f(x)$ in the interval $-2\pi < x < 2\pi$



b) Fourier series representation of $f(x)$

STEP ONE

$$\begin{aligned}a_0 &= \frac{1}{\pi} \int_0^{2\pi} f(x) dx = \frac{1}{\pi} \int_0^{\pi} f(x) dx + \frac{1}{\pi} \int_{\pi}^{2\pi} f(x) dx \\&= \frac{1}{\pi} \int_0^{\pi} x dx + \frac{1}{\pi} \int_{\pi}^{2\pi} \pi \cdot dx \\&= \frac{1}{\pi} \left[\frac{x^2}{2} \right]_0^{\pi} + \frac{\pi}{\pi} \left[x \right]_{\pi}^{2\pi} \\&= \frac{1}{\pi} \left(\frac{\pi^2}{2} - 0 \right) + \left(2\pi - \pi \right) \\&= \frac{\pi}{2} + \pi \\&\text{i.e. } a_0 = \frac{3\pi}{2}.\end{aligned}$$

STEP TWO

$$\begin{aligned}a_n &= \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx \, dx \\&= \frac{1}{\pi} \int_0^{\pi} x \cos nx \, dx + \frac{1}{\pi} \int_{\pi}^{2\pi} \pi \cdot \cos nx \, dx \\&= \frac{1}{\pi} \underbrace{\left[\left[x \frac{\sin nx}{n} \right]_0^{\pi} - \int_0^{\pi} \frac{\sin nx}{n} dx \right]}_{\text{using integration by parts}} + \frac{\pi}{\pi} \left[\frac{\sin nx}{n} \right]_{\pi}^{2\pi} \\&= \frac{1}{\pi} \left[\frac{1}{n} \left(\pi \sin n\pi - 0 \cdot \sin n0 \right) - \left[\frac{-\cos nx}{n^2} \right]_0^{\pi} \right] \\&\qquad\qquad\qquad + \frac{1}{n} (\sin n2\pi - \sin n\pi)\end{aligned}$$

$$\begin{aligned}\text{i.e. } a_n &= \frac{1}{\pi} \left[\frac{1}{n} (0 - 0) + \left(\frac{\cos n\pi}{n^2} - \frac{\cos 0}{n^2} \right) \right] + \frac{1}{n} (0 - 0) \\ &= \frac{1}{n^2\pi} (\cos n\pi - 1), \quad \text{see TRIG} \\ &= \frac{1}{n^2\pi} ((-1)^n - 1),\end{aligned}$$

$$\text{i.e. } a_n = \begin{cases} -\frac{2}{n^2\pi} & , n \text{ odd} \\ 0 & , n \text{ even.} \end{cases}$$

STEP THREE

$$\begin{aligned}
 b_n &= \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx \, dx \\
 &= \frac{1}{\pi} \int_0^{\pi} x \sin nx \, dx + \frac{1}{\pi} \int_{\pi}^{2\pi} \pi \cdot \sin nx \, dx \\
 &= \frac{1}{\pi} \left[\underbrace{\left[x \left(-\frac{\cos nx}{n} \right) \right]_0^{\pi} - \int_0^{\pi} \left(-\frac{\cos nx}{n} \right) dx}_{\text{using integration by parts}} \right] + \frac{\pi}{\pi} \left[\frac{-\cos nx}{n} \right]_{\pi}^{2\pi} \\
 &= \frac{1}{\pi} \left[\left(\frac{-\pi \cos n\pi}{n} + 0 \right) + \left[\frac{\sin nx}{n^2} \right]_0^{\pi} \right] - \frac{1}{n} (\cos 2n\pi - \cos n\pi) \\
 &= \frac{1}{\pi} \left[\frac{-\pi(-1)^n}{n} + \left(\frac{\sin n\pi - \sin 0}{n^2} \right) \right] - \frac{1}{n} (1 - (-1)^n) \\
 &= -\frac{1}{n}(-1)^n + 0 - \frac{1}{n}(1 - (-1)^n)
 \end{aligned}$$

$$\text{i.e. } b_n = -\frac{1}{n}(-1)^n - \frac{1}{n} + \frac{1}{n}(-1)^n$$

$$\text{i.e. } b_n = -\frac{1}{n}.$$

We now have

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx]$$

$$\text{where } a_0 = \frac{3\pi}{2}, \quad a_n = \begin{cases} 0 & , n \text{ even} \\ -\frac{2}{n^2\pi} & , n \text{ odd} \end{cases}, \quad b_n = -\frac{1}{n}$$

Constructing a table of values gives

n	1	2	3	4	5
a_n	$-\frac{2}{\pi}$	0	$-\frac{2}{\pi} \left(\frac{1}{3^2}\right)$	0	$-\frac{2}{\pi} \left(\frac{1}{5^2}\right)$
b_n	-1	$-\frac{1}{2}$	$-\frac{1}{3}$	$-\frac{1}{4}$	$-\frac{1}{5}$

This table of coefficients gives

$$f(x) = \frac{1}{2} \left(\frac{3\pi}{2} \right) + \left(-\frac{2}{\pi} \right) \left[\cos x + 0 \cdot \cos 2x + \frac{1}{3^2} \cos 3x + \dots \right]$$

$$+ \left(-1 \right) \left[\sin x + \frac{1}{2} \sin 2x + \frac{1}{3} \sin 3x + \dots \right]$$

$$\text{i.e. } f(x) = \frac{3\pi}{4} - \frac{2}{\pi} \left[\cos x + \frac{1}{3^2} \cos 3x + \frac{1}{5^2} \cos 5x + \dots \right]$$

$$- \left[\sin x + \frac{1}{2} \sin 2x + \frac{1}{3} \sin 3x + \dots \right]$$

and we have found the required series.

c) Pick an appropriate value of x , to show that

$$\boxed{\text{(i) } \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots}$$

Compare this series with

$$f(x) = \frac{3\pi}{4} - \frac{2}{\pi} \left[\cos x + \frac{1}{3^2} \cos 3x + \frac{1}{5^2} \cos 5x + \dots \right] \\ - \left[\sin x + \frac{1}{2} \sin 2x + \frac{1}{3} \sin 3x + \dots \right]$$

Here, we want to set the $\cos nx$ terms to zero (since their coefficients are $1, \frac{1}{3^2}, \frac{1}{5^2}, \dots$). Since $\cos n\frac{\pi}{2} = 0$ when n is odd, we will try setting $x = \frac{\pi}{2}$ in the series. Note also that $f(\frac{\pi}{2}) = \frac{\pi}{2}$

This gives

$$\frac{\pi}{2} = \frac{3\pi}{4} - \frac{2}{\pi} \left[\cos \frac{\pi}{2} + \frac{1}{3^2} \cos 3\frac{\pi}{2} + \frac{1}{5^2} \cos 5\frac{\pi}{2} + \dots \right] \\ - \left[\sin \frac{\pi}{2} + \frac{1}{2} \sin 2\frac{\pi}{2} + \frac{1}{3} \sin 3\frac{\pi}{2} + \frac{1}{4} \sin 4\frac{\pi}{2} + \frac{1}{5} \sin 5\frac{\pi}{2} + \dots \right]$$

and

$$\begin{aligned} \frac{\pi}{2} &= \frac{3\pi}{4} - \frac{2}{\pi} [0 + 0 + 0 + \dots] \\ &\quad - [(1) + \frac{1}{2} \cdot (0) + \frac{1}{3} \cdot (-1) + \frac{1}{4} \cdot (0) + \frac{1}{5} \cdot (1) + \dots] \end{aligned}$$

then

$$\frac{\pi}{2} = \frac{3\pi}{4} - (1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots)$$

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots = \frac{3\pi}{4} - \frac{\pi}{2}$$

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots = \frac{\pi}{4}, \quad \text{as required.}$$

To show that

$$(ii) \quad \frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots,$$

We want zero $\sin nx$ terms and to use the coefficients of $\cos nx$

Setting $x = 0$ eliminates the $\sin nx$ terms from the series, and also gives

$$\cos x + \frac{1}{3^2} \cos 3x + \frac{1}{5^2} \cos 5x + \frac{1}{7^2} \cos 7x + \dots = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots$$

(i.e. the desired series).

The graph of $f(x)$ shows a discontinuity (a “vertical jump”) at $x = 0$

The Fourier series converges to a value that is **half-way** between the two values of $f(x)$ around this discontinuity. That is the series will converge to $\frac{\pi}{2}$ at $x = 0$

$$\begin{aligned} \text{i.e. } \frac{\pi}{2} &= \frac{3\pi}{4} - \frac{2}{\pi} \left[\cos 0 + \frac{1}{3^2} \cos 0 + \frac{1}{5^2} \cos 0 + \frac{1}{7^2} \cos 0 + \dots \right] \\ &\quad - \left[\sin 0 + \frac{1}{2} \sin 0 + \frac{1}{3} \sin 0 + \dots \right] \end{aligned}$$

$$\text{and } \frac{\pi}{2} = \frac{3\pi}{4} - \frac{2}{\pi} \left[1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots \right] - [0 + 0 + 0 + \dots]$$

Finally, this gives

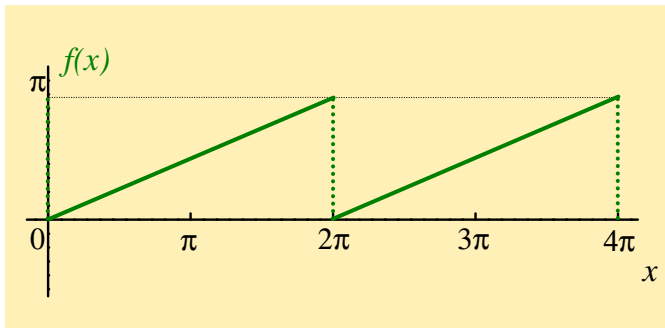
$$\begin{aligned} -\frac{\pi}{4} &= -\frac{2}{\pi} \left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots \right) \\ \text{and } \frac{\pi^2}{8} &= 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots \end{aligned}$$

[Return to Exercise 3](#)

Exercise 4.

$$f(x) = \frac{x}{2}, \text{ over the interval } 0 < x < 2\pi \text{ and has period } 2\pi$$

a) Sketch a graph of $f(x)$ in the interval $0 < x < 4\pi$



b) Fourier series representation of $f(x)$

STEP ONE

$$\begin{aligned} a_0 &= \frac{1}{\pi} \int_0^{2\pi} f(x) \, dx \\ &= \frac{1}{\pi} \int_0^{2\pi} \frac{x}{2} \, dx \\ &= \frac{1}{\pi} \left[\frac{x^2}{4} \right]_0^{2\pi} \\ &= \frac{1}{\pi} \left[\frac{(2\pi)^2}{4} - 0 \right] \end{aligned}$$

$$\text{i.e. } a_0 = \pi.$$

STEP TWO

$$\begin{aligned}a_n &= \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx \, dx \\&= \frac{1}{\pi} \int_0^{2\pi} \frac{x}{2} \cos nx \, dx \\&= \frac{1}{2\pi} \left\{ \underbrace{\left[x \frac{\sin nx}{n} \right]_0^{2\pi} - \frac{1}{n} \int_0^{2\pi} \sin nx \, dx}_{\text{using integration by parts}} \right\} \\&= \frac{1}{2\pi} \left\{ \left(2\pi \frac{\sin n2\pi}{n} - 0 \cdot \frac{\sin n \cdot 0}{n} \right) - \frac{1}{n} \cdot 0 \right\} \\&= \frac{1}{2\pi} \left\{ (0 - 0) - \frac{1}{n} \cdot 0 \right\}, \text{ see TRIG} \\ \text{i.e. } a_n &= 0.\end{aligned}$$

STEP THREE

$$\begin{aligned}b_n &= \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx \, dx = \frac{1}{\pi} \int_0^{2\pi} \left(\frac{x}{2}\right) \sin nx \, dx \\&= \frac{1}{2\pi} \int_0^{2\pi} x \sin nx \, dx \\&= \frac{1}{2\pi} \underbrace{\left\{ \left[x \left(\frac{-\cos nx}{n} \right) \right]_0^{2\pi} - \int_0^{2\pi} \left(\frac{-\cos nx}{n} \right) dx \right\}}_{\text{using integration by parts}} \\&= \frac{1}{2\pi} \left\{ \frac{1}{n} (-2\pi \cos n2\pi + 0) + \frac{1}{n} \cdot 0 \right\}, \text{ see TRIG} \\&= \frac{-2\pi}{2\pi n} \cos(n2\pi) \\&= -\frac{1}{n} \cos(2n\pi) \\ \text{i.e. } b_n &= -\frac{1}{n}, \text{ since } 2n \text{ is even (see TRIG)}\end{aligned}$$

We now have

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx]$$

where $a_0 = \pi$, $a_n = 0$, $b_n = -\frac{1}{n}$

These Fourier coefficients give

$$f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \left(0 - \frac{1}{n} \sin nx \right)$$

$$\text{i.e. } f(x) = \frac{\pi}{2} - \left\{ \sin x + \frac{1}{2} \sin 2x + \frac{1}{3} \sin 3x + \dots \right\}.$$

c) Pick an appropriate value of x , to show that

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \dots$$

Setting $x = \frac{\pi}{2}$ gives $f(x) = \frac{\pi}{4}$ and

$$\frac{\pi}{4} = \frac{\pi}{2} - \left[1 + 0 - \frac{1}{3} + 0 + \frac{1}{5} + 0 - \dots \right]$$

$$\frac{\pi}{4} = \frac{\pi}{2} - \left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \dots \right]$$

$$\left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \dots \right] = \frac{\pi}{4}$$

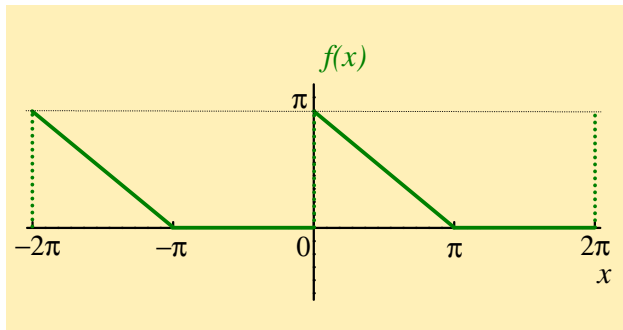
i.e. $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \dots = \frac{\pi}{4}$.

[Return to Exercise 4](#)

Exercise 5.

$$f(x) = \begin{cases} \pi - x & , 0 < x < \pi \\ 0 & , \pi < x < 2\pi, \text{ and has period } 2\pi \end{cases}$$

a) Sketch a graph of $f(x)$ in the interval $-2\pi < x < 2\pi$



b) Fourier series representation of $f(x)$

STEP ONE

$$\begin{aligned}a_0 &= \frac{1}{\pi} \int_0^{2\pi} f(x) \, dx \\&= \frac{1}{\pi} \int_0^{\pi} (\pi - x) \, dx + \frac{1}{\pi} \int_{\pi}^{2\pi} 0 \cdot dx \\&= \frac{1}{\pi} \left[\pi x - \frac{1}{2} x^2 \right]_0^{\pi} + 0 \\&= \frac{1}{\pi} \left[\pi^2 - \frac{\pi^2}{2} - 0 \right] \\ \text{i.e. } a_0 &= \frac{\pi}{2}.\end{aligned}$$

STEP TWO

$$\begin{aligned}
 a_n &= \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx \, dx \\
 &= \frac{1}{\pi} \int_0^{\pi} (\pi - x) \cos nx \, dx + \frac{1}{\pi} \int_{\pi}^{2\pi} 0 \cdot dx \\
 \text{i.e. } a_n &= \frac{1}{\pi} \left\{ \underbrace{\left[(\pi - x) \frac{\sin nx}{n} \right]_0^{\pi} - \int_0^{\pi} (-1) \cdot \frac{\sin nx}{n} \, dx}_{\text{using integration by parts}} \right\} + 0 \\
 &= \frac{1}{\pi} \left\{ (0 - 0) + \int_0^{\pi} \frac{\sin nx}{n} \, dx \right\} \quad , \text{ see TRIG} \\
 &= \frac{1}{\pi n} \left[\frac{-\cos nx}{n} \right]_0^{\pi} \\
 &= -\frac{1}{\pi n^2} (\cos n\pi - \cos 0) \\
 \text{i.e. } a_n &= -\frac{1}{\pi n^2} ((-1)^n - 1) \quad , \text{ see TRIG}
 \end{aligned}$$

$$\text{i.e. } a_n = \begin{cases} 0 & , n \text{ even} \\ \frac{2}{\pi n^2} & , n \text{ odd} \end{cases}$$

STEP THREE

$$\begin{aligned} b_n &= \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx \, dx \\ &= \frac{1}{\pi} \int_0^{\pi} (\pi - x) \sin nx \, dx + \int_{\pi}^{2\pi} 0 \cdot dx \\ &= \frac{1}{\pi} \left\{ \left[(\pi - x) \left(-\frac{\cos nx}{n} \right) \right]_0^{\pi} - \int_0^{\pi} (-1) \cdot \left(-\frac{\cos nx}{n} \right) dx \right\} + 0 \\ &= \frac{1}{\pi} \left\{ \left(0 - \left(-\frac{\pi}{n} \right) \right) - \frac{1}{n} \cdot 0 \right\}, \text{ see TRIG} \\ \text{i.e. } b_n &= \frac{1}{n}. \end{aligned}$$

In summary, $a_0 = \frac{\pi}{2}$ and a table of other Fourier coefficients is

n	1	2	3	4	5
$a_n = \frac{2}{\pi n^2}$ (when n is odd)	$\frac{2}{\pi}$	0	$\frac{2}{\pi} \frac{1}{3^2}$	0	$\frac{2}{\pi} \frac{1}{5^2}$
$b_n = \frac{1}{n}$	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$

$$\begin{aligned} \therefore f(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx] \\ &= \frac{\pi}{4} + \frac{2}{\pi} \cos x + \frac{2}{\pi} \frac{1}{3^2} \cos 3x + \frac{2}{\pi} \frac{1}{5^2} \cos 5x + \dots \\ &\quad + \sin x + \frac{1}{2} \sin 2x + \frac{1}{3} \sin 3x + \frac{1}{4} \sin 4x + \dots \end{aligned}$$

$$\begin{aligned} \text{i.e. } f(x) &= \frac{\pi}{4} + \frac{2}{\pi} \left[\cos x + \frac{1}{3^2} \cos 3x + \frac{1}{5^2} \cos 5x + \dots \right] \\ &\quad + \sin x + \frac{1}{2} \sin 2x + \frac{1}{3} \sin 3x + \frac{1}{4} \sin 4x + \dots \end{aligned}$$

c) To show that

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots,$$

note that, as $x \rightarrow 0$, the series converges to the half-way value of $\frac{\pi}{2}$,

$$\begin{aligned} \text{and then } \frac{\pi}{2} &= \frac{\pi}{4} + \frac{2}{\pi} \left(\cos 0 + \frac{1}{3^2} \cos 0 + \frac{1}{5^2} \cos 0 + \dots \right) \\ &+ \sin 0 + \frac{1}{2} \sin 0 + \frac{1}{3} \sin 0 + \dots \end{aligned}$$

$$\frac{\pi}{2} = \frac{\pi}{4} + \frac{2}{\pi} \left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots \right) + 0$$

$$\frac{\pi}{4} = \frac{2}{\pi} \left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots \right)$$

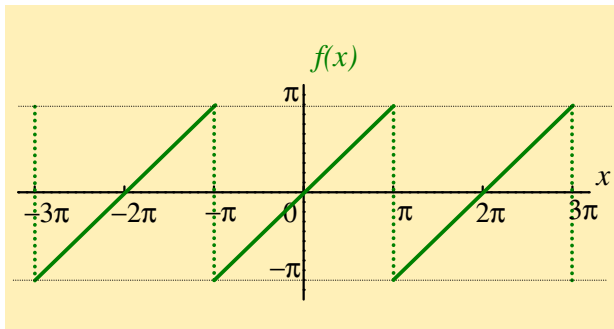
$$\text{giving } \frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots$$

[Return to Exercise 5](#)

Exercise 6.

$$f(x) = x, \text{ over the interval } -\pi < x < \pi \text{ and has period } 2\pi$$

a) Sketch a graph of $f(x)$ in the interval $-3\pi < x < 3\pi$



b) Fourier series representation of $f(x)$

STEP ONE

$$\begin{aligned} a_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \, dx \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} x \, dx \\ &= \frac{1}{\pi} \left[\frac{x^2}{2} \right]_{-\pi}^{\pi} \\ &= \frac{1}{\pi} \left(\frac{\pi^2}{2} - \frac{\pi^2}{2} \right) \end{aligned}$$

$$\text{i.e. } a_0 = 0.$$

STEP TWO

$$\begin{aligned}a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} x \cos nx \, dx \\ &= \frac{1}{\pi} \left\{ \underbrace{\left[x \frac{\sin nx}{n} \right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} \left(\frac{\sin nx}{n} \right) dx}_{\text{using integration by parts}} \right\}\end{aligned}$$

$$\begin{aligned}\text{i.e. } a_n &= \frac{1}{\pi} \left\{ \frac{1}{n} (\pi \sin n\pi - (-\pi) \sin(-n\pi)) - \frac{1}{n} \int_{-\pi}^{\pi} \sin nx \, dx \right\} \\ &= \frac{1}{\pi} \left\{ \frac{1}{n} (0 - 0) - \frac{1}{n} \cdot 0 \right\},\end{aligned}$$

$$\text{since } \sin n\pi = 0 \text{ and } \int_{2\pi} \sin nx \, dx = 0,$$

$$\text{i.e. } a_n = 0.$$

STEP THREE

$$\begin{aligned}b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx \\&= \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin nx \, dx \\&= \frac{1}{\pi} \left\{ \left[\frac{-x \cos nx}{n} \right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} \left(\frac{-\cos nx}{n} \right) dx \right\} \\&= \frac{1}{\pi} \left\{ -\frac{1}{n} [x \cos nx]_{-\pi}^{\pi} + \frac{1}{n} \int_{-\pi}^{\pi} \cos nx \, dx \right\} \\&= \frac{1}{\pi} \left\{ -\frac{1}{n} (\pi \cos n\pi - (-\pi) \cos(-n\pi)) + \frac{1}{n} \cdot 0 \right\} \\&= -\frac{\pi}{n\pi} (\cos n\pi + \cos n\pi) \\&= -\frac{1}{n} (2 \cos n\pi) \\ \text{i.e. } b_n &= -\frac{2}{n} (-1)^n.\end{aligned}$$

We thus have

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx]$$

with $a_0 = 0$, $a_n = 0$, $b_n = -\frac{2}{n}(-1)^n$

and

n	1	2	3
b_n	2	-1	$\frac{2}{3}$

Therefore

$$\begin{aligned} f(x) &= b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + \dots \\ \text{i.e. } f(x) &= 2 \left[\sin x - \frac{1}{2} \sin 2x + \frac{1}{3} \sin 3x - \dots \right] \end{aligned}$$

and we have found the required Fourier series.

c) Pick an appropriate value of x , to show that

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$$

Setting $x = \frac{\pi}{2}$ gives $f(x) = \frac{\pi}{2}$ and

$$\frac{\pi}{2} = 2 \left[\sin \frac{\pi}{2} - \frac{1}{2} \sin \frac{2\pi}{2} + \frac{1}{3} \sin \frac{3\pi}{2} - \frac{1}{4} \sin \frac{4\pi}{2} + \frac{1}{5} \sin \frac{5\pi}{2} - \dots \right]$$

This gives

$$\frac{\pi}{2} = 2 \left[1 + 0 + \frac{1}{3} \cdot (-1) - 0 + \frac{1}{5} \cdot (1) - 0 + \frac{1}{7} \cdot (-1) + \dots \right]$$

$$\frac{\pi}{2} = 2 \left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots \right]$$

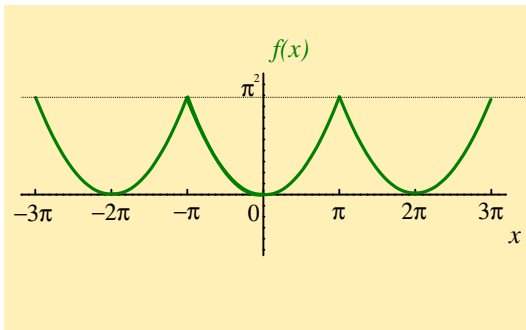
$$\text{i.e. } \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$$

[Return to Exercise 6](#)

Exercise 7.

$$f(x) = x^2, \text{ over the interval } -\pi < x < \pi \text{ and has period } 2\pi$$

a) Sketch a graph of $f(x)$ in the interval $-3\pi < x < 3\pi$



b) Fourier series representation of $f(x)$

STEP ONE

$$\begin{aligned} a_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 dx \\ &= \frac{1}{\pi} \left[\frac{x^3}{3} \right]_{-\pi}^{\pi} \\ &= \frac{1}{\pi} \left(\frac{\pi^3}{3} - \left(-\frac{\pi^3}{3} \right) \right) \\ &= \frac{1}{\pi} \left(\frac{2\pi^3}{3} \right) \\ \text{i.e. } a_0 &= \frac{2\pi^2}{3}. \end{aligned}$$

STEP TWO

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \cos nx \, dx \\ &= \frac{1}{\pi} \underbrace{\left\{ \left[x^2 \frac{\sin nx}{n} \right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} 2x \left(\frac{\sin nx}{n} \right) dx \right\}}_{\text{using integration by parts}} \\ &= \frac{1}{\pi} \left\{ \frac{1}{n} (\pi^2 \sin n\pi - \pi^2 \sin(-n\pi)) - \frac{2}{n} \int_{-\pi}^{\pi} x \sin nx \, dx \right\} \\ &= \frac{1}{\pi} \left\{ \frac{1}{n} (0 - 0) - \frac{2}{n} \int_{-\pi}^{\pi} x \sin nx \, dx \right\}, \text{ see TRIG} \\ &= \frac{-2}{n\pi} \int_{-\pi}^{\pi} x \sin nx \, dx \end{aligned}$$

$$\begin{aligned}
 \text{i.e. } a_n &= \frac{-2}{n\pi} \underbrace{\left\{ \left[x \left(\frac{-\cos nx}{n} \right) \right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} \left(\frac{-\cos nx}{n} \right) dx \right\}}_{\text{using integration by parts again}} \\
 &= \frac{-2}{n\pi} \left\{ -\frac{1}{n} [x \cos nx]_{-\pi}^{\pi} + \frac{1}{n} \int_{-\pi}^{\pi} \cos nx \, dx \right\} \\
 &= \frac{-2}{n\pi} \left\{ -\frac{1}{n} \left(\pi \cos n\pi - (-\pi) \cos(-n\pi) \right) + \frac{1}{n} \cdot 0 \right\} \\
 &= \frac{-2}{n\pi} \left\{ -\frac{1}{n} \left(\pi(-1)^n + \pi(-1)^n \right) \right\} \\
 &= \frac{-2}{n\pi} \left\{ \frac{-2\pi}{n} (-1)^n \right\}
 \end{aligned}$$

$$\begin{aligned}\text{i.e. } a_n &= \frac{-2}{n\pi} \left\{ -\frac{2\pi}{n} (-1)^n \right\} \\ &= \frac{+4\pi}{\pi n^2} (-1)^n \\ &= \frac{4}{n^2} (-1)^n\end{aligned}$$

$$\text{i.e. } a_n = \begin{cases} \frac{4}{n^2} & , n \text{ even} \\ \frac{-4}{n^2} & , n \text{ odd.} \end{cases}$$

STEP THREE

$$\begin{aligned}
 b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \sin nx \, dx \\
 &= \frac{1}{\pi} \left\{ \underbrace{\left[x^2 \left(\frac{-\cos nx}{n} \right) \right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} 2x \cdot \left(\frac{-\cos nx}{n} \right) dx}_{\text{using integration by parts}} \right\} \\
 &= \frac{1}{\pi} \left\{ -\frac{1}{n} [x^2 \cos nx]_{-\pi}^{\pi} + \frac{2}{n} \int_{-\pi}^{\pi} x \cos nx \, dx \right\} \\
 &= \frac{1}{\pi} \left\{ -\frac{1}{n} (\pi^2 \cos n\pi - \pi^2 \cos(-n\pi)) + \frac{2}{n} \int_{-\pi}^{\pi} x \cos nx \, dx \right\} \\
 &= \frac{1}{\pi} \left\{ -\frac{1}{n} \underbrace{(\pi^2 \cos n\pi - \pi^2 \cos(n\pi))}_{=0} + \frac{2}{n} \int_{-\pi}^{\pi} x \cos nx \, dx \right\} \\
 &= \frac{2}{\pi n} \int_{-\pi}^{\pi} x \cos nx \, dx
 \end{aligned}$$

$$\begin{aligned} \text{i.e. } b_n &= \frac{2}{\pi n} \left\{ \underbrace{\left[x \frac{\sin nx}{n} \right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} \frac{\sin nx}{n} dx}_{\text{using integration by parts}} \right\} \\ &= \frac{2}{\pi n} \left\{ \frac{1}{n} (\pi \sin n\pi - (-\pi) \sin(-n\pi)) - \frac{1}{n} \int_{-\pi}^{\pi} \sin nx dx \right\} \\ &= \frac{2}{\pi n} \left\{ \frac{1}{n} (0 + 0) - \frac{1}{n} \int_{-\pi}^{\pi} \sin nx dx \right\} \\ &= \frac{-2}{\pi n^2} \int_{-\pi}^{\pi} \sin nx dx \end{aligned}$$

$$\text{i.e. } b_n = 0.$$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx]$$

where

$$a_0 = \frac{2\pi^2}{3}, \quad a_n = \begin{cases} \frac{4}{n^2} & , n \text{ even} \\ \frac{-4}{n^2} & , n \text{ odd} \end{cases}, \quad b_n = 0$$

n	1	2	3	4
a_n	$-4(1)$	$4\left(\frac{1}{2^2}\right)$	$-4\left(\frac{1}{3^2}\right)$	$4\left(\frac{1}{4^2}\right)$

$$\text{i.e. } f(x) = \frac{1}{2} \left(\frac{2\pi^2}{3} \right) - 4 \left[\cos x - \frac{1}{2^2} \cos 2x + \frac{1}{3^2} \cos 3x - \frac{1}{4^2} \cos 4x \dots \right]$$

$$+ [0 + 0 + 0 + \dots]$$

$$\text{i.e. } f(x) = \frac{\pi^2}{3} - 4 \left[\cos x - \frac{1}{2^2} \cos 2x + \frac{1}{3^2} \cos 3x - \frac{1}{4^2} \cos 4x + \dots \right].$$

c) To show that

$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots,$$

use the fact that $\cos n\pi = \begin{cases} 1 & , n \text{ even} \\ -1 & , n \text{ odd} \end{cases}$

i.e. $\cos x - \frac{1}{2^2} \cos 2x + \frac{1}{3^2} \cos 3x - \frac{1}{4^2} \cos 4x + \dots$ with $x = \pi$

gives $\cos \pi - \frac{1}{2^2} \cos 2\pi + \frac{1}{3^2} \cos 3\pi - \frac{1}{4^2} \cos 4\pi + \dots$

i.e. $(-1) - \frac{1}{2^2} \cdot (1) + \frac{1}{3^2} \cdot (-1) - \frac{1}{4^2} \cdot (1) + \dots$

i.e. $-1 - \frac{1}{2^2} - \frac{1}{3^2} - \frac{1}{4^2} + \dots$

$$= -1 \cdot \underbrace{\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots\right)}_{\text{(the desired series)}}$$

The graph of $f(x)$ gives that $f(\pi) = \pi^2$ and the series converges to this value.

Setting $x = \pi$ in the Fourier series thus gives

$$\pi^2 = \frac{\pi^2}{3} - 4 \left(\cos \pi - \frac{1}{2^2} \cos 2\pi + \frac{1}{3^2} \cos 3\pi - \frac{1}{4^2} \cos 4\pi + \dots \right)$$

$$\pi^2 = \frac{\pi^2}{3} - 4 \left(-1 - \frac{1}{2^2} - \frac{1}{3^2} - \frac{1}{4^2} - \dots \right)$$

$$\pi^2 = \frac{\pi^2}{3} + 4 \left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots \right)$$

$$\frac{2\pi^2}{3} = 4 \left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots \right)$$

$$\text{i.e. } \frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots$$

[Return to Exercise 7](#)