Integration

PPLATO PROMOTING PHYSICS LEARNING & TEACHING OPPORTUNITIES

SUBSTITUTION I .. $f(ax+b)$

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A Tutorial Module for practising the integration of expressions of the form $f(ax+b)$

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1. Theory

Consider an integral of the form

$$\int f(ax+b)dx$$

where a and b are constants. We have here an unspecified function f of a linear function of x

Letting $\quad u = ax + b \quad$ then $\quad \frac{du}{dx} = a\,, \quad$ and this gives $\quad dx = \frac{du}{a}$

This allows us to change the integration variable from x to u

$$\int f(ax+b)dx = \int f(u)\frac{du}{a}$$

The final result is

$$\boxed{\int f(ax+b)dx = \frac{1}{a}\int f(u)\,du}$$

where $u = ax + b$

This is a general result for integrating functions of a linear function of x

Each application of this result involves **dividing by the coefficient of** x and then integrating

2. Exercises

Click on EXERCISE links for full worked solutions (10 exercises in total).

Perform the following integrations:

EXERCISE 1.

$$\int (2x-1)^3 dx$$

EXERCISE 2.

$$\int \cos(3x+5) dx$$

EXERCISE 3.

$$\int e^{5x+2} dx$$

Exercise 4.

$$\int \sinh 3x \, dx$$

Exercise 5.

$$\int \frac{dx}{2x-1}$$

Exercise 6.

$$\int \frac{dx}{1+(5x)^2}$$

Exercise 7.

$$\int \sec^2(7x+1)dx$$

Exercise 8.

$$\int \sin(3x-1)dx$$

Exercise 9.

$$\int \cosh(1+2x)dx$$

Exercise 10.

$$\int \tan(9x-1)dx$$

3. Answers

1. $\frac{1}{8}(2x-1)^4 + C$,
2. $\frac{1}{3}\sin(3x+5) + C$,
3. $\frac{1}{5}e^{5x+2} + C$,
4. $\frac{1}{3}\cosh 3x + C$,
5. $\frac{1}{2}\ln|2x-1| + C$,
6. $\frac{1}{5}\tan^{-1} 5x + C$,
7. $\frac{1}{7}\tan(7x+1) + C$,
8. $-\frac{1}{3}\cos(3x-1) + C$,
9. $\frac{1}{2}\sinh(1+2x) + C$,
10. $-\frac{1}{9}\ln|\cos(9x-1)| + C$.

4. Standard integrals

$f(x)$	$\int f(x)dx$	$f(x)$	$\int f(x)dx$
x^n	$\frac{x^{n+1}}{n+1} \quad (n \neq -1)$	$[g(x)]^n \, g'(x)$	$\frac{[g(x)]^{n+1}}{n+1} \quad (n \neq -1)$
$\frac{1}{x}$	$\ln \lvert x \rvert$	$\frac{g'(x)}{g(x)}$	$\ln \lvert g(x) \rvert$
e^x	e^x	a^x	$\frac{a^x}{\ln a} \quad (a > 0)$
$\sin x$	$-\cos x$	$\sinh x$	$\cosh x$
$\cos x$	$\sin x$	$\cosh x$	$\sinh x$
$\tan x$	$-\ln \lvert \cos x \rvert$	$\tanh x$	$\ln \cosh x$
$\operatorname{cosec} x$	$\ln \left\lvert \tan \frac{x}{2} \right\rvert$	$\operatorname{cosech} x$	$\ln \left\lvert \tanh \frac{x}{2} \right\rvert$
$\sec x$	$\ln \lvert \sec x + \tan x \rvert$	$\operatorname{sech} x$	$2 \tan^{-1} e^x$
$\sec^2 x$	$\tan x$	$\operatorname{sech}^2 x$	$\tanh x$
$\cot x$	$\ln \lvert \sin x \rvert$	$\coth x$	$\ln \lvert \sinh x \rvert$
$\sin^2 x$	$\frac{x}{2} - \frac{\sin 2x}{4}$	$\sinh^2 x$	$\frac{\sinh 2x}{4} - \frac{x}{2}$
$\cos^2 x$	$\frac{x}{2} + \frac{\sin 2x}{4}$	$\cosh^2 x$	$\frac{\sinh 2x}{4} + \frac{x}{2}$

$f(x)$	$\int f(x)\,dx$	$f(x)$	$\int f(x)\,dx$
$\frac{1}{a^2+x^2}$	$\frac{1}{a}\tan^{-1}\frac{x}{a}$	$\frac{1}{a^2-x^2}$	$\frac{1}{2a}\ln\left\lvert\frac{a+x}{a-x}\right\rvert \quad (0<\lvert x\rvert<a)$
	$(a>0)$	$\frac{1}{x^2-a^2}$	$\frac{1}{2a}\ln\left\lvert\frac{x-a}{x+a}\right\rvert \quad (\lvert x\rvert>a>0)$
$\frac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1}\frac{x}{a}$	$\frac{1}{\sqrt{a^2+x^2}}$	$\ln\left\lvert\frac{x+\sqrt{a^2+x^2}}{a}\right\rvert \quad (a>0)$
	$(-a<x<a)$	$\frac{1}{\sqrt{x^2-a^2}}$	$\ln\left\lvert\frac{x+\sqrt{x^2-a^2}}{a}\right\rvert \quad (x>a>0)$
$\sqrt{a^2-x^2}$	$\frac{a^2}{2}\left[\sin^{-1}\left(\frac{x}{a}\right)\right.$	$\sqrt{a^2+x^2}$	$\frac{a^2}{2}\left[\sinh^{-1}\left(\frac{x}{a}\right)+\frac{x\sqrt{a^2+x^2}}{a^2}\right]$
	$\left.+\frac{x\sqrt{a^2-x^2}}{a^2}\right]$	$\sqrt{x^2-a^2}$	$\frac{a^2}{2}\left[-\cosh^{-1}\left(\frac{x}{a}\right)+\frac{x\sqrt{x^2-a^2}}{a^2}\right]$

5. Tips

● STANDARD INTEGRALS are provided. Do not forget to use these tables when you need to

● When looking at the THEORY, STANDARD INTEGRALS, ANSWERS or TIPS pages, use the Back button (at the bottom of the page) to return to the exercises

● Use the solutions intelligently. For example, they can help you get started on an exercise, or they can allow you to check whether your intermediate results are correct

● Try to make less use of the full solutions as you work your way through the Tutorial

Full worked solutions

Exercise 1.

$$\int (2x-1)^3 dx$$

Let $u = 2x - 1$ then $\dfrac{du}{dx} = 2$ and $dx = \dfrac{du}{2}$

$$\begin{aligned} \therefore \int (2x-1)^3 dx = \int u^3 \frac{du}{2} &= \frac{1}{2}\int u^3 du \\ &= \frac{1}{2}\frac{1}{4}u^4 + C \\ &= \frac{1}{8}(2x-1)^4 + C\,. \end{aligned}$$

Note. The final result can also be obtained using the general pattern:

$$\int f(ax+b)\,dx = \frac{1}{a}\int f(u)\,du$$

where $a = 2$ and $\int f(u)du$ is $\int u^3 du$.

Return to Exercise 1

Exercise 2.

$$\int \cos(3x+5)dx$$

Let $u = 3x + 5$ then $\frac{du}{dx} = 3$ and $dx = \frac{du}{3}$

$$\begin{aligned} \therefore \int \cos(3x+5)dx = \int \cos u \cdot \frac{du}{3} &= \frac{1}{3}\int \cos u\, du \\ &= \frac{1}{3}\sin u + C \\ &= \frac{1}{3}\sin(3x+5) + C\,. \end{aligned}$$

Note. The final result can also be obtained using the general pattern:

$$\int f(ax+b)\,dx = \frac{1}{a}\int f(u)\,du$$

where $a = 3$ and $\int f(u)du$ is $\int \cos u du$. Return to Exercise 2

Exercise 3.

$$\int e^{5x+2} dx$$

Let $u = 5x + 2$ then $\frac{du}{dx} = 5$ and $dx = \frac{du}{5}$

$$\therefore \int e^{5x+2} dx = \int e^u \frac{du}{5} = \frac{1}{5} \int e^u du$$
$$= \frac{1}{5} e^u + C$$
$$= \frac{1}{5} e^{5x+2} + C\,.$$

Note. The final result can also be obtained using the general pattern:

$$\int f(ax+b)\, dx = \frac{1}{a} \int f(u)\, du$$

where $a = 5$ and $\int f(u) du$ is $\int e^u du$. Return to Exercise 3

Exercise 4.

$$\int \sinh 3x\, dx$$

Let $u = 3x$ then $\dfrac{du}{dx} = 3$ and $dx = \dfrac{du}{3}$

$$\begin{aligned} \therefore \int \sinh 3x\, dx = \int \sinh\, u \cdot \frac{du}{3} &= \frac{1}{3}\int \sinh\, u\, du \\ &= \frac{1}{3}\cosh\, u + C \\ &= \frac{1}{3}\cosh 3x + C\,. \end{aligned}$$

<u>Note.</u> The final result can also be obtained using the general pattern:

$$\int f(ax+b)\, dx = \frac{1}{a}\int f(u)\, du$$

where $a = 3$ and $\int f(u)du$ is $\int \sinh\, u\, du$. Return to Exercise 4

Exercise 5.

$$\int \frac{dx}{2x-1}$$

Let $u = 2x - 1$ then $\dfrac{du}{dx} = 2$ and $dx = \dfrac{du}{2}$

$$\begin{aligned} \therefore \int \frac{dx}{2x-1} &= \int \frac{du}{u} \cdot \frac{1}{2} = \frac{1}{2} \int \frac{du}{u} \\ &= \frac{1}{2} \ln |u| + C \\ &= \frac{1}{2} \ln |2x-1| + C\,. \end{aligned}$$

Note. The final result can also be obtained using the general pattern:

$$\int f(ax+b)\,dx = \frac{1}{a} \int f(u)\,du$$

where $a = 2$ and $\int f(u)du$ is $\int \frac{du}{u}$.

Return to Exercise 5

Exercise 6.

$$\int \frac{dx}{1+(5x)^2}$$

Let $u = 5x$ then $\dfrac{du}{dx} = 5$ and $dx = \dfrac{du}{5}$

$$\therefore \int \frac{dx}{1+(5x)^2} = \int \frac{du}{1+u^2} \cdot \frac{1}{5} \quad = \quad \frac{1}{5}\int \frac{du}{1+u^2}$$

$$= \quad \frac{1}{5}\tan^{-1} u + C$$

$$= \quad = \frac{1}{5}\tan^{-1} 5x + C\,.$$

Note. The final result can also be obtained using the general pattern:

$$\int f(ax+b)\,dx = \frac{1}{a}\int f(u)\,du$$

where $a = 5$ and $\int f(u)du$ is $\int \frac{du}{1+u^2} du$.

Return to Exercise 6

Exercise 7.

$$\int \sec^2(7x+1)dx$$

Let $u = 7x + 1$ then $\dfrac{du}{dx} = 7$ and $dx = \dfrac{du}{7}$

$$\begin{aligned} \therefore \int \sec^2(7x+1)dx &= \frac{1}{7}\int \sec^2 u\, du \\ &= \frac{1}{7}\tan u + C \\ &= \frac{1}{7}\tan(7x+1) + C\,. \end{aligned}$$

Note. The final result can also be obtained using the general pattern:

$$\int f(ax+b)\,dx = \frac{1}{a}\int f(u)\,du$$

where $a = 7$ and $\int f(u)du$ is $\int \sec^2 udu$.

Return to Exercise 7

Exercise 8.

$$\int \sin(3x-1)dx$$

Let $u = 3x - 1$ then $\dfrac{du}{dx} = 3$ and $dx = \dfrac{du}{3}$

$$\begin{aligned}\therefore \int \sin(3x-1)dx &= \frac{1}{3}\int \sin u\, du \\ &= -\frac{1}{3}\cos u + C \\ &= -\frac{1}{3}\cos(3x-1) + C\,.\end{aligned}$$

Note. The final result can also be obtained using the general pattern:

$$\int f(ax+b)\,dx = \frac{1}{a}\int f(u)\,du$$

where $a = 3$ and $\int f(u)du$ is $\int \sin u du$.

Return to Exercise 8

Exercise 9.

$$\int \cosh(1+2x)dx$$

Let $u = 1+2x$ then $\dfrac{du}{dx} = 2$ and $dx = \dfrac{du}{2}$

$$\begin{aligned} \therefore \int \cosh(1+2x)dx &= \int \cosh u \frac{du}{2} \\ &= \frac{1}{2}\sinh u + C \\ &= \frac{1}{2}\sinh(1+2x) + C\,. \end{aligned}$$

<u>Note.</u> The final result can also be obtained using the general pattern:

$$\int f(ax+b)\,dx = \frac{1}{a}\int f(u)\,du$$

where $a = 2$ and $\int f(u)du$ is $\int \cosh u\,du$.

Return to Exercise 9

Exercise 10.

$$\int \tan(9x-1)dx$$

Let $u = 9x - 1$ then $\dfrac{du}{dx} = 9$ and $dx = \dfrac{du}{9}$

$$\begin{aligned} \therefore \int \tan(9x-1)\,dx = \int \tan u \cdot \frac{du}{9} &= \frac{1}{9}\int \tan u\,du \\ &= -\frac{1}{9}\ln|\cos u| + C \\ &= -\frac{1}{9}\ln|\cos(9x-1)| + C\,. \end{aligned}$$

Note. The final result can also be obtained using the general pattern:

$$\int f(ax+b)\,dx = \frac{1}{a}\int f(u)\,du$$

where $a = 9$ and $\int f(u)du$ is $\int \tan u\,du$.

Return to Exercise 10