

SUBSTITUTION II .. $\frac{1}{f(x)} \cdot f'(x)$

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A Tutorial Module for practising the integration of expressions of the form $\frac{1}{f(x)} \cdot f'(x)$

- [Table of contents](#)
- [Begin Tutorial](#)

Table of contents

1. Theory
 2. Exercises
 3. Answers
 4. Standard integrals
 5. Tips
- Full worked solutions

1. Theory

Consider an integral of the form

$$\int \frac{f'(x)}{f(x)} dx$$

Letting $u = f(x)$ then $\frac{du}{dx} = f'(x)$, and this gives $du = f'(x)dx$

So

$$\begin{aligned} \int \frac{f'(x)}{f(x)} dx &= \int \frac{1}{f(x)} \cdot f'(x) dx \\ &= \int \frac{du}{u} \\ &= \ln |u| + C \end{aligned}$$

The general result is

$$\int \frac{f'(x)}{f(x)} dx = \ln |f(x)| + C$$

This is a useful generalisation of the standard integral

$$\int \frac{1}{x} dx = \ln |x| + C,$$

that applies straightforwardly when the top line of what we are integrating is a multiple of the derivative of the bottom line

2. Exercises

Click on [EXERCISE](#) links for full worked solutions (there are 11 exercises in total).

Perform the following integrations:

[EXERCISE 1.](#)

$$\int \frac{10x + 3}{5x^2 + 3x - 1} dx$$

[EXERCISE 2.](#)

$$\int \frac{3x^2}{x^3 - 6} dx$$

[EXERCISE 3.](#)

$$\int \frac{4x^2}{x^3 - 1} dx$$

● [THEORY](#) ● [STANDARD INTEGRALS](#) ● [ANSWERS](#) ● [TIPS](#)

EXERCISE 4.

$$\int \frac{2x - 1}{x^2 - x - 7} dx$$

EXERCISE 5.

$$\int \frac{x - 1}{x^2 - 2x + 7} dx$$

EXERCISE 6.

$$\int \frac{8x}{x^2 - 4} dx$$

EXERCISE 7.

$$\int \frac{x^2}{x^3 + 2} dx$$

EXERCISE 8.

$$\int \frac{\sin x}{\cos x} dx$$

EXERCISE 9.

$$\int \frac{\cos x}{\sin x} dx$$

EXERCISE 10.

$$\int \frac{\cos x}{5 + \sin x} dx$$

EXERCISE 11.

$$\int \frac{\sinh x}{1 - \cosh x} dx ,$$

● THEORY ● STANDARD INTEGRALS ● ANSWERS ● TIPS

3. Answers

1. $\ln |5x^2 + 3x - 1| + C$,
2. $\ln |x^3 - 6| + C$,
3. $\frac{4}{3} \ln |x^3 - 1| + C$,
4. $\ln |x^2 - x - 7| + C$,
5. $\frac{1}{2} \ln |x^2 - 2x + 7| + C$,
6. $4 \ln |x^2 - 4| + C$,
7. $\frac{1}{3} \ln |x^3 + 2| + C$
8. $-\ln |\cos x| + C$,
9. $\ln |\sin x| + C$,
10. $\ln |5 + \sin x| + C$,
11. $-\ln |1 - \cosh x| + C$.

4. Standard integrals

$f(x)$	$\int f(x)dx$	$f(x)$	$\int f(x)dx$
x^n	$\frac{x^{n+1}}{n+1} \quad (n \neq -1)$	$[g(x)]^n g'(x)$	$\frac{[g(x)]^{n+1}}{n+1} \quad (n \neq -1)$
$\frac{1}{x}$	$\ln x $	$\frac{g'(x)}{g(x)}$	$\ln g(x) $
e^x	e^x	a^x	$\frac{a^x}{\ln a} \quad (a > 0)$
$\sin x$	$-\cos x$	$\sinh x$	$\cosh x$
$\cos x$	$\sin x$	$\cosh x$	$\sinh x$
$\tan x$	$-\ln \cos x $	$\tanh x$	$\ln \cosh x$
$\operatorname{cosec} x$	$\ln\left \tan \frac{x}{2}\right $	$\operatorname{cosech} x$	$\ln\left \tanh \frac{x}{2}\right $
$\sec x$	$\ln \sec x + \tan x $	$\operatorname{sech} x$	$2 \tan^{-1} e^x$
$\sec^2 x$	$\tan x$	$\operatorname{sech}^2 x$	$\tanh x$
$\cot x$	$\ln \sin x $	$\operatorname{coth} x$	$\ln \sinh x $
$\sin^2 x$	$\frac{x}{2} - \frac{\sin 2x}{4}$	$\sinh^2 x$	$\frac{\sinh 2x}{4} - \frac{x}{2}$
$\cos^2 x$	$\frac{x}{2} + \frac{\sin 2x}{4}$	$\cosh^2 x$	$\frac{\sinh 2x}{4} + \frac{x}{2}$

$f(x)$	$\int f(x) dx$	$f(x)$	$\int f(x) dx$
$\frac{1}{a^2+x^2}$	$\frac{1}{a} \tan^{-1} \frac{x}{a}$ $(a > 0)$	$\frac{1}{a^2-x^2}$	$\frac{1}{2a} \ln \left \frac{a+x}{a-x} \right $ ($0 < x < a$) $\frac{1}{2a} \ln \left \frac{x-a}{x+a} \right $ ($ x > a > 0$)
$\frac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1} \frac{x}{a}$ $(-a < x < a)$	$\frac{1}{\sqrt{a^2+x^2}}$	$\ln \left \frac{x+\sqrt{a^2+x^2}}{a} \right $ ($a > 0$) $\ln \left \frac{x+\sqrt{x^2-a^2}}{a} \right $ ($x > a > 0$)
$\sqrt{a^2-x^2}$	$\frac{a^2}{2} \left[\sin^{-1} \left(\frac{x}{a} \right) + \frac{x\sqrt{a^2-x^2}}{a^2} \right]$	$\sqrt{a^2+x^2}$	$\frac{a^2}{2} \left[\sinh^{-1} \left(\frac{x}{a} \right) + \frac{x\sqrt{a^2+x^2}}{a^2} \right]$ $\frac{a^2}{2} \left[-\cosh^{-1} \left(\frac{x}{a} \right) + \frac{x\sqrt{x^2-a^2}}{a^2} \right]$

5. Tips

- STANDARD INTEGRALS are provided. Do not forget to use these tables when you need to

- When looking at the THEORY, STANDARD INTEGRALS, ANSWERS or TIPS pages, use the [Back](#) button (at the bottom of the page) to return to the exercises

- Use the solutions intelligently. For example, they can help you get started on an exercise, or they can allow you to check whether your intermediate results are correct

- Try to make less use of the full solutions as you work your way through the Tutorial

Full worked solutions

Exercise 1.

$$\int \frac{10x + 3}{5x^2 + 3x - 1} dx \text{ is of the form } \int \frac{f'(x)}{f(x)} dx = \ln |f(x)| + C.$$

Let $u = 5x^2 + 3x - 1$ then $\frac{du}{dx} = 10x + 3$, and $du = (10x + 3)dx$

$$\begin{aligned} \therefore \int \frac{10x + 3}{5x^2 + 3x - 1} dx &= \int \frac{du}{u} \\ &= \ln |u| + C \\ &= \ln |5x^2 + 3x - 1| + C. \end{aligned}$$

[Return to Exercise 1](#)

Exercise 2.

$$\int \frac{3x^2}{x^3 - 6} dx \text{ is of the form } \int \frac{f'(x)}{f(x)} dx = \ln |f(x)| + C.$$

$$\text{Let } u = x^3 - 6 \text{ then } \frac{du}{dx} = 3x^2 \text{ and } du = 3x^2 dx$$

$$\begin{aligned} \therefore \int \frac{3x^2}{x^3 - 6} dx &= \int \frac{1}{u} du \\ &= \ln |u| + C \\ &= \ln |x^3 - 6| + C. \end{aligned}$$

[Return to Exercise 2](#)

Exercise 3.

$\int \frac{4x^2}{x^3 - 1} dx$ is of the slightly more general form

$$\int k \frac{f'(x)}{f(x)} dx = k \int \frac{f'(x)}{f(x)} dx = k \ln |f(x)| + C,$$

where k is a constant

i.e. the top line is a **multiple of the derivative** of the bottom line.

Let $u = x^3 - 1$ then $\frac{du}{dx} = 3x^2$ and $\frac{du}{3} = x^2 dx$

$$\begin{aligned} \therefore 4 \int \frac{x^2}{x^3 - 1} dx &= 4 \int \frac{1}{u} \frac{du}{3} \\ &= \frac{4}{3} \ln |u| + C \\ &= \frac{4}{3} \ln |x^3 - 1| + C. \end{aligned}$$

[Return to Exercise 3](#)

Exercise 4.

$$\int \frac{2x - 1}{x^2 - x - 7} dx \text{ is of the form } \int \frac{f'(x)}{f(x)} dx = \ln |f(x)| + C.$$

Let $u = x^2 - x - 7$ then $\frac{du}{dx} = 2x - 1$ and $du = (2x - 1)dx$

$$\begin{aligned} \therefore \int \frac{2x - 1}{x^2 - x - 7} dx &= \int \frac{du}{u} \\ &= \ln |u| + C \\ &= \ln |x^2 - x - 7| + C. \end{aligned}$$

[Return to Exercise 4](#)

Exercise 5.

$\int \frac{x-1}{x^2-2x+7} dx$ is of the slightly more general form

$$\int k \frac{f'(x)}{f(x)} dx = k \int \frac{f'(x)}{f(x)} dx = k \ln |f(x)| + C,$$

where k is a constant

i.e. the top line is a **multiple of the derivative** of the bottom line.

Let $u = x^2 - 2x + 7$ then $\frac{du}{dx} = 2x - 2$ and $du = (2x - 2)dx$

so that $\frac{du}{2} = (x - 1)dx$

$$\begin{aligned}\therefore \int \frac{x-1}{x^2-2x+7} dx &= \int \frac{1}{u} \frac{du}{2} = \frac{1}{2} \int \frac{du}{u} \\ &= \frac{1}{2} \ln |u| + C \\ &= \frac{1}{2} \ln |x^2 - 2x + 7| + C.\end{aligned}$$

[Return to Exercise 5](#)

Exercise 6.

$$\int \frac{8x}{x^2 - 4} dx$$

Let $u = x^2 - 4$ then $du = 2x dx$ and $\frac{du}{2} = x dx$

$$\begin{aligned} \therefore \int \frac{8x}{x^2 - 4} dx &= 8 \int \frac{1}{u} \frac{du}{2} \\ &= 4 \ln |u| + C \\ &= 4 \ln |x^2 - 4| + C. \end{aligned}$$

[Return to Exercise 6](#)

Exercise 7.

$$\int \frac{x^2}{x^3 + 2} dx$$

Let $u = x^3 + 2$ then $du = 3x^2 dx$ and $\frac{du}{3} = x^2 dx$

$$\begin{aligned} \therefore \int \frac{x^2}{x^3 + 2} dx &= \int \frac{1}{u} \frac{du}{3} \\ &= \frac{1}{3} \ln |u| + C \\ &= \frac{1}{3} \ln |x^3 + 2| + C. \end{aligned}$$

[Return to Exercise 7](#)

Exercise 8.

$$\int \frac{\sin x}{\cos x} dx$$

Let $u = \cos x$ then $\frac{du}{dx} = -\sin x$ and $\frac{du}{(-1)} = \sin x dx$

$$\begin{aligned} \therefore \int \frac{\sin x}{\cos x} dx &= \int \frac{1}{u} \frac{du}{(-1)} \\ &= - \int \frac{du}{u} \\ &= -\ln |u| + C \\ &= -\ln |\cos x| + C. \end{aligned}$$

[Return to Exercise 8](#)

Exercise 9.

$$\int \frac{\cos x}{\sin x} dx$$

Let $u = \sin x$ then $\frac{du}{dx} = \cos x$ and $du = \cos x dx$

$$\begin{aligned} \therefore \int \frac{\cos x}{\sin x} dx &= \int \frac{1}{u} du \\ &= \ln |u| + C \\ &= \ln |\sin x| + C. \end{aligned}$$

[Return to Exercise 9](#)

Exercise 10.

$$\int \frac{\cos x}{5 + \sin x} dx$$

Let $u = 5 + \sin x$ then $\frac{du}{dx} = \cos x$ and $du = \cos x dx$

$$\begin{aligned} \therefore \int \frac{\cos x}{5 + \sin x} dx &= \int \frac{du}{u} \\ &= \ln |u| + C \\ &= \ln |5 + \sin x| + C. \end{aligned}$$

[Return to Exercise 10](#)

Exercise 11.

$$\int \frac{\sinh x}{1 - \cosh x} dx$$

Let $u = 1 - \cosh x$ then $\frac{du}{dx} = -\sinh x$ and $-du = \sinh x dx$

$$\therefore \int \frac{\sinh x}{1 - \cosh x} dx = \int \frac{1}{u} \cdot (-du)$$

$$= -\ln |u| + C$$

$$= -\ln |1 - \cosh x| + C.$$

[Return to Exercise 11](#)