Integration

SUBSTITUTION III .. $[f(x)]^n \cdot f'(x)$

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A self-contained Tutorial Module for practising the integration of expressions of the form $[f(x)]^n \cdot f'(x)$, where $n \neq -1$

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1. Theory

Consider an integral of the form

$$\int [f(x)]^n f'(x) dx$$

Letting $u = f(x)$ gives $\dfrac{du}{dx} = f'(x)$ and $du = f'(x)dx$

$$\therefore \int [f(x)]^n f'(x) dx = \int u^n du = \frac{u^{n+1}}{n+1} + C = \frac{[f(x)]^{n+1}}{n+1} + C$$

For example, when $n = 1$,

$$\int f(x) f'(x) dx = \int u\, du = \frac{u^2}{2} + C = \frac{[f(x)]^2}{2} + C$$

2. Exercises

Click on EXERCISE links for full worked solutions (10 exercises in total).

Perform the following integrations:

Exercise 1.

$$\int \sin x \cos x \, dx$$

Exercise 2.

$$\int \sinh x \cosh x \, dx$$

Exercise 3.

$$\int \tan x \sec^2 x \, dx$$

Exercise 4.

$$\int \sin 2x \cos 2x \, dx$$

Exercise 5.

$$\int \sinh 3x \cosh 3x \, dx$$

Exercise 6.

$$\int \frac{1}{x} \ln x \, dx \quad , \quad \text{x} > 0$$

Exercise 7.

$$\int \sin^4 x \cos x \, dx$$

Exercise 8.

$$\int \sinh^3 x \cosh x \, dx$$

Exercise 9.

$$\int \cos^3 x \sin x \, dx$$

Exercise 10.

$$\int \frac{2x}{(x^2-4)^2} \, dx$$

3. Answers

1. $\frac{1}{2}\sin^2 x + C$,
2. $\frac{1}{2}\sinh^2 x + C$,
3. $\frac{1}{2}\tan^2 x + C$,
4. $\frac{1}{4}\sin^2 2x + C$,
5. $\frac{1}{6}\sinh^2 3x + C$,
6. $\frac{1}{2}\ln^2 x + C$
7. $\frac{\sin^5 x}{5} + C$,
8. $\frac{1}{4}\sinh^4 x + C$,
9. $-\frac{\cos^4 x}{4} + C$,
10. $-\frac{1}{x^2-4} + C$.

4. Standard integrals

$f(x)$	$\int f(x)dx$	$f(x)$	$\int f(x)dx$
x^n	$\frac{x^{n+1}}{n+1} \quad (n \neq -1)$	$[g(x)]^n g'(x)$	$\frac{[g(x)]^{n+1}}{n+1} \quad (n \neq -1)$
$\frac{1}{x}$	$\ln\|x\|$	$\frac{g'(x)}{g(x)}$	$\ln\|g(x)\|$
e^x	e^x	a^x	$\frac{a^x}{\ln a} \quad (a > 0)$
$\sin x$	$-\cos x$	$\sinh x$	$\cosh x$
$\cos x$	$\sin x$	$\cosh x$	$\sinh x$
$\tan x$	$-\ln\|\cos x\|$	$\tanh x$	$\ln\cosh x$
$\operatorname{cosec} x$	$\ln\left\|\tan\frac{x}{2}\right\|$	$\operatorname{cosech} x$	$\ln\left\|\tanh\frac{x}{2}\right\|$
$\sec x$	$\ln\|\sec x + \tan x\|$	$\operatorname{sech} x$	$2\tan^{-1} e^x$
$\sec^2 x$	$\tan x$	$\operatorname{sech}^2 x$	$\tanh x$
$\cot x$	$\ln\|\sin x\|$	$\coth x$	$\ln\|\sinh x\|$
$\sin^2 x$	$\frac{x}{2} - \frac{\sin 2x}{4}$	$\sinh^2 x$	$\frac{\sinh 2x}{4} - \frac{x}{2}$
$\cos^2 x$	$\frac{x}{2} + \frac{\sin 2x}{4}$	$\cosh^2 x$	$\frac{\sinh 2x}{4} + \frac{x}{2}$

$f(x)$	$\int f(x)\,dx$	$f(x)$	$\int f(x)\,dx$
$\frac{1}{a^2+x^2}$	$\frac{1}{a}\tan^{-1}\frac{x}{a}$	$\frac{1}{a^2-x^2}$	$\frac{1}{2a}\ln\left\lvert\frac{a+x}{a-x}\right\rvert \quad (0<\lvert x\rvert<a)$
	$(a>0)$	$\frac{1}{x^2-a^2}$	$\frac{1}{2a}\ln\left\lvert\frac{x-a}{x+a}\right\rvert \quad (\lvert x\rvert>a>0)$
$\frac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1}\frac{x}{a}$	$\frac{1}{\sqrt{a^2+x^2}}$	$\ln\left\lvert\frac{x+\sqrt{a^2+x^2}}{a}\right\rvert \quad (a>0)$
	$(-a<x<a)$	$\frac{1}{\sqrt{x^2-a^2}}$	$\ln\left\lvert\frac{x+\sqrt{x^2-a^2}}{a}\right\rvert \quad (x>a>0)$
$\sqrt{a^2-x^2}$	$\frac{a^2}{2}\left[\sin^{-1}\left(\frac{x}{a}\right)\right.$	$\sqrt{a^2+x^2}$	$\frac{a^2}{2}\left[\sinh^{-1}\left(\frac{x}{a}\right)+\frac{x\sqrt{a^2+x^2}}{a^2}\right]$
	$\left.+\frac{x\sqrt{a^2-x^2}}{a^2}\right]$	$\sqrt{x^2-a^2}$	$\frac{a^2}{2}\left[-\cosh^{-1}\left(\frac{x}{a}\right)+\frac{x\sqrt{x^2-a^2}}{a^2}\right]$

5. Tips

● STANDARD INTEGRALS are provided. Do not forget to use these tables when you need to

● When looking at the THEORY, STANDARD INTEGRALS, ANSWERS or TIPS pages, use the Back button (at the bottom of the page) to return to the exercises

● Use the solutions intelligently. For example, they can help you get started on an exercise, or they can allow you to check whether your intermediate results are correct

● Try to make less use of the full solutions as you work your way through the Tutorial

Full worked solutions

Exercise 1.

$$\int \sin x \cos x \, dx \quad \text{is of the form} \quad \int f(x) f'(x) dx = \frac{1}{2}\left[f(x)\right]^2 + C$$

To see this, set $u = \sin x$, to find $\dfrac{du}{dx} = \cos x$ and $du = \cos x \, dx$

$$\begin{aligned} \therefore \int \sin x \cos x \, dx &= \int u \, du \\ &= \frac{1}{2}u^2 + C \\ &= \frac{1}{2}\sin^2 x + C. \end{aligned}$$

Return to Exercise 1

Exercise 2.

$$\int \sinh x \cosh x \, dx \text{ is of the form } \int f(x)f'(x)dx = [f(x)]^2 + C$$

To see this, set $u = \sinh x$ then $\dfrac{du}{dx} = \cosh x$ and $du = \cosh x \, dx$

$$\begin{aligned} \therefore \int \sinh x \cosh x \, dx &= \int u \, du \\ &= \frac{1}{2}u^2 + C \\ &= \frac{1}{2}\sinh^2 x + C. \end{aligned}$$

Return to Exercise 2

Exercise 3.

$$\int \tan x \sec^2 x \, dx \text{ is of the form } \int f(x) f'(x) dx = [f(x)]^2 + C$$

Let $u = \tan x$ then $\dfrac{du}{dx} = \sec^2 x$ and $du = \sec^2 x \, dx$

$$\begin{aligned} \therefore \int \tan x \sec^2 x \, dx &= \int u \, du \\ &= \frac{1}{2} u^2 + C \\ &= \frac{1}{2} \tan^2 x + C. \end{aligned}$$

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Exercise 4.

$$\int \sin 2x \cos 2x \, dx \text{ is close to the form } \int f(x)f'(x)dx = [f(x)]^2 + C$$

Let $u = \sin 2x$ then $\frac{du}{dx} = 2\cos 2x$ and $\frac{du}{2} = \cos 2x \, dx$

$$\begin{aligned} \therefore \int \sin 2x \cos 2x \, dx &= \int u \frac{du}{2} \\ &= \frac{1}{2} \int u \, du \\ &= \frac{1}{2}\frac{1}{2}u^2 + C \\ &= \frac{1}{4}\sin^2 2x + C. \end{aligned}$$

Return to Exercise 4

Exercise 5.

$\int \sinh 3x \cosh 3x \, dx$ is close to the form $\int f(x) f'(x) dx = [f(x)]^2 + C$

Let $u = \sinh 3x$ then $\dfrac{du}{dx} = 3\cosh 3x$ and $\dfrac{du}{3} = \cosh 3x \, dx$

$$\begin{aligned} \therefore \int \sinh 3x \cosh 3x \, dx &= \int u \frac{du}{3} \\ &= \frac{1}{3} \int u \, du \\ &= \frac{1}{3} \frac{1}{2} u^2 + C \\ &= \frac{1}{6} \sinh^2 3x + C. \end{aligned}$$

Return to Exercise 5

Exercise 6.

$$\int \frac{1}{x}\ln x\,dx \text{ is of the form } \int f(x)f'(x)dx = [f(x)]^2 + C$$

Let $u = \ln x$ then $\frac{du}{dx} = \frac{1}{x}$ and $du = \frac{1}{x}\,dx$

$$\begin{aligned} \therefore \int \frac{1}{x}\ln x\,dx &= \int u\,du \\ &= \frac{1}{2}u^2 + C \\ &= \frac{1}{2}\ln^2 x + C\,. \end{aligned}$$

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Exercise 7.

$$\int \sin^4 x \cos x \, dx \text{ is of the form } \int [f(x)]^n \, f'(x) dx = \frac{[f(x)]^{n+1}}{n+1} + C$$

To see this, set $u = \sin x$ to find $du = \cos x \, dx$

$$\begin{aligned} \therefore \int \sin^4 x \cos x \, dx &= \int u^4 \, du \\ &= \frac{u^5}{5} + C \\ &= \frac{\sin^5 x}{5} + C. \end{aligned}$$

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Exercise 8.

$$\int \sinh^3 x \cosh x \, dx \text{ is of the form } \int [f(x)]^n f'(x) dx = \frac{[f(x)]^{n+1}}{n+1} + C$$

To see this, set $u = \sinh x$ then $du = \cosh x \, dx$

$$\begin{aligned} \therefore \int \sinh^3 x \cosh x \, dx &= \int u^3 \, du \\ &= \frac{1}{4} u^4 + C \\ &= \frac{1}{4} \sinh^4 x + C. \end{aligned}$$

Return to Exercise 8

Exercise 9.

$\int \cos^3 x \, \sin x \, dx$ is close to the form $\int [f(x)]^n \, f'(x) dx = \dfrac{[f(x)]^{n+1}}{n+1} + C$

Let $u = \cos x$ then $du = -\sin x \, dx$

$$\begin{aligned} \therefore \int \cos^3 x \, \sin x \, dx &= \int u^3 \cdot (-du) = -\int u^3 du \\ &= -\frac{u^4}{4} + C \\ &= -\frac{\cos^4 x}{4} + C. \end{aligned}$$

Return to Exercise 9

Exercise 10.

$$\int \frac{2x}{(x^2-4)^2}\,dx \;\text{ is of the form }\; \int [f(x)]^n\, f'(x) dx = \frac{[f(x)]^{n+1}}{n+1} + C$$

Let $u = x^2 - 4$ then $du = 2x\,dx$

$$\begin{aligned} \therefore \int \frac{2x}{(x^2-4)^2}\,dx = \int \frac{1}{u^2}\,du &= \int u^{-2}\,du \\ &= -u^{-1} + C = -\frac{1}{u} + C \\ &= -\frac{1}{x^2-4} + C\,. \end{aligned}$$

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