Integration

ALGEBRAIC FRACTIONS

Graham S McDonald
and Silvia C Dalla

A self-contained Tutorial Module for practising the integration of algebraic fractions

- **Table of contents**
- **Begin Tutorial**

Table of contents

1. Theory

The method of partial fractions can be used in the integration of a proper algebraic fraction. This technique allows the integration to be done as a sum of much simpler integrals

A **proper algebraic fraction** is a fraction of two polynomials whose top line is a polynomial of lower degree than the one in the bottom line. Recall that, for a polynomial in x, the degree is the highest power of x. For example

$$\frac{x-1}{x^2+3x+5}$$

is a proper algebraic fraction because the top line is a polynomial of degree 1 and the bottom line is a polynomial of degree 2.

● To integrate an **improper algebraic fraction**, one firstly needs to write the fraction as a sum of proper fractions. This first step can be done by using polynomial division ('P-Division')

● Look out for cases of proper algebraic fractions whose top line is a multiple k of the derivative of the bottom line. Then, the standard integral

$$\int \frac{k\ g'(x)}{g(x)}\,dx \quad = \quad k\ \ln|g(x)| + C$$

can be used (instead of working out partial fractions)

● Otherwise, the bottom line of a proper algebraic fraction needs to be factorised as far as possible. This allows us to identify the form of each partial fraction needed

factor in the bottom line	$\longrightarrow$	form of partial fraction(s)
$(ax+b)$		$\frac{A}{ax+b}$
$(ax+b)^2$		$\frac{A}{ax+b} + \frac{B}{(ax+b)^2}$
(ax^2+bx+c)		$\frac{Ax+B}{ax^2+bx+c}$

where A and B are constants to be determined

2. Exercises

Click on Exercise links for full worked solutions (there are 13 exercises in total)

Perform the following integrations:

Exercise 1. $\displaystyle\int \frac{x^2+2x+5}{x}\,dx$

Exercise 2. $\displaystyle\int \frac{x^3+4x^2+3x+1}{x^2}\,dx$

Exercise 3. $\displaystyle\int \frac{x^2+3x+4}{x+1}\,dx$

Exercise 4. $\displaystyle\int \frac{2x^2+5x+3}{x+2}\,dx$

Exercise 5. $\displaystyle\int \frac{4x^3+2}{x^4+2x+3}\,dx$

Exercise 6. $\displaystyle\int \frac{x}{x^2-5}\,dx$

Exercise 7. $\displaystyle\int \frac{17-x}{(x-3)(x+4)}\,dx$

Exercise 8. $\displaystyle\int \frac{11x+18}{(2x+5)(x-7)}\,dx$

Exercise 9. $\displaystyle\int \frac{7x+1}{(x+1)(x-2)(x+3)}\,dx$

Exercise 10. $\displaystyle\int \frac{2x+9}{(x+5)^2}\,dx$

Exercise 11. $\int \frac{13x - 4}{(3x-2)(2x+1)}\, dx$

Exercise 12. $\int \frac{27x}{(x-2)^2(x+1)}\, dx$

Exercise 13. $\int \frac{3x^2}{(x-1)(x^2+x+1)}\, dx$

3. Answers

1. $\frac{1}{2}x^2 + 2x + 5\ \ln|x| + C,$
2. $\frac{1}{2}x^2 + 4x + 3\ \ln|x| - \frac{1}{x} + C,$
3. $\frac{1}{2}x^2 + 2x + 2\ \ln|x+1| + C,$
4. $x^2 + x + \ln|x+2| + C,$
5. $\ln|x^4 + 2x + 3| + C,$
6. $\frac{1}{2}\ln|x^2 - 5| + C,$
7. $2\ln|x-3| - 3\ln|x+4| + C,$
8. $\frac{1}{2}\ln|2x+5| + 5\ln|x-7| + C,$
9. $\ln|x+1| + \ln|x-2| - 2\ln|x+3| + C,$
10. $2\ln|x+5| + \frac{1}{x+5} + D,$
11. $\frac{2}{3}\ln|3x-2| + \frac{3}{2}\ln|2x+1| + C,$
12. $3\ln|x-2| - \frac{18}{x-2} - 3\ln|x+1| + D,$
13. $\ln|x-1| + \ln|x^2+x+1| + D.$

4. Standard integrals

$f(x)$	$\int f(x)dx$	$f(x)$	$\int f(x)dx$
x^n	$\frac{x^{n+1}}{n+1} \quad (n \neq -1)$	$[g(x)]^n g'(x)$	$\frac{[g(x)]^{n+1}}{n+1} \quad (n \neq -1)$
$\frac{1}{x}$	$\ln\lvert x\rvert$	$\frac{g'(x)}{g(x)}$	$\ln\lvert g(x)\rvert$
e^x	e^x	a^x	$\frac{a^x}{\ln a} \quad (a > 0)$
$\sin x$	$-\cos x$	$\sinh x$	$\cosh x$
$\cos x$	$\sin x$	$\cosh x$	$\sinh x$
$\tan x$	$-\ln\lvert\cos x\rvert$	$\tanh x$	$\ln\cosh x$
$\operatorname{cosec} x$	$\ln\left\lvert\tan\frac{x}{2}\right\rvert$	$\operatorname{cosech} x$	$\ln\left\lvert\tanh\frac{x}{2}\right\rvert$
$\sec x$	$\ln\lvert\sec x + \tan x\rvert$	$\operatorname{sech} x$	$2\tan^{-1} e^x$
$\sec^2 x$	$\tan x$	$\operatorname{sech}^2 x$	$\tanh x$
$\cot x$	$\ln\lvert\sin x\rvert$	$\coth x$	$\ln\lvert\sinh x\rvert$
$\sin^2 x$	$\frac{x}{2} - \frac{\sin 2x}{4}$	$\sinh^2 x$	$\frac{\sinh 2x}{4} - \frac{x}{2}$
$\cos^2 x$	$\frac{x}{2} + \frac{\sin 2x}{4}$	$\cosh^2 x$	$\frac{\sinh 2x}{4} + \frac{x}{2}$

$f(x)$	$\int f(x)\,dx$	$f(x)$	$\int f(x)\,dx$
$\frac{1}{a^2+x^2}$	$\frac{1}{a}\tan^{-1}\frac{x}{a}$	$\frac{1}{a^2-x^2}$	$\frac{1}{2a}\ln\left\lvert\frac{a+x}{a-x}\right\rvert \quad (0<\lvert x\rvert<a)$
	$(a>0)$	$\frac{1}{x^2-a^2}$	$\frac{1}{2a}\ln\left\lvert\frac{x-a}{x+a}\right\rvert \quad (\lvert x\rvert>a>0)$
$\frac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1}\frac{x}{a}$	$\frac{1}{\sqrt{a^2+x^2}}$	$\ln\left\lvert\frac{x+\sqrt{a^2+x^2}}{a}\right\rvert \quad (a>0)$
	$(-a<x<a)$	$\frac{1}{\sqrt{x^2-a^2}}$	$\ln\left\lvert\frac{x+\sqrt{x^2-a^2}}{a}\right\rvert \quad (x>a>0)$
$\sqrt{a^2-x^2}$	$\frac{a^2}{2}\left[\sin^{-1}\left(\frac{x}{a}\right)\right.$	$\sqrt{a^2+x^2}$	$\frac{a^2}{2}\left[\sinh^{-1}\left(\frac{x}{a}\right)+\frac{x\sqrt{a^2+x^2}}{a^2}\right]$
	$\left.+\frac{x\sqrt{a^2-x^2}}{a^2}\right]$	$\sqrt{x^2-a^2}$	$\frac{a^2}{2}\left[-\cosh^{-1}\left(\frac{x}{a}\right)+\frac{x\sqrt{x^2-a^2}}{a^2}\right]$

5. Polynomial division

You can use formal long division to simplify an improper algebraic fraction. In this Tutorial, we us another technique (that is sometimes called **'algebraic juggling'**)

● In each step of the technique, we re-write the top line in a way that the algebraic fraction can be broken into two separate fractions, where a simplifying cancellation is forced to appear in the first of these two fractions

● The technique involves re-writing the top-line term with the highest power of x using the expression from the bottom line

The detail of how the method works is best illustrated with a long example

One such example follows on the next page ...

$$\begin{aligned}
\frac{x^3+3x^2-2x-1}{x+1} &= \frac{x^2(x+1)-x^2 \qquad +3x^2-2x-1}{x+1} \\
&\quad \{ \text{ the bottom line has been used to write } x^3 \text{ as } x^2(x+1)-x^2 \ \} \\
&= \frac{x^2(x+1) \qquad +2x^2-2x-1}{x+1} \\
&= \frac{x^2(x+1)}{x+1} + \frac{2x^2-2x-1}{x+1} \\
&= x^2 + \frac{2x^2-2x-1}{x+1} \\
&= x^2 + \frac{2x(x+1)-2x \qquad -2x-1}{x+1} \\
&\quad \{ \text{ writing } 2x^2 \text{ as } 2x(x+1)-2x \ \}
\end{aligned}$$

$$
\begin{aligned}
\text{i.e. } \frac{x^3+3x^2-2x-1}{x+1} &= x^2 + \frac{2x(x+1) \qquad -4x-1}{x+1} \\
&= x^2 + \frac{2x(x+1)}{x+1} + \frac{-4x-1}{x+1} \\
&= x^2 + 2x + \frac{-4x-1}{x+1} \\
&= x^2 + 2x + \frac{-4(x+1)+4 \qquad -1}{x+1} \\
&\qquad \{ \text{ writing } -4x \text{ as } -4(x+1)+4 \ \} \\
&= x^2 + 2x + \frac{-4(x+1) \qquad +3}{x+1} \\
&= x^2 + 2x + \frac{-4(x+1)}{x+1} + \frac{3}{x+1}
\end{aligned}
$$

$$\text{i.e.} \quad \frac{x^3+3x^2-2x-1}{x+1} = x^2 + 2x + \frac{-4(x+1)}{x+1} + \frac{3}{x+1}$$

$$= x^2 + 2x - 4 + \frac{3}{x+1}$$

We have now written the original improper algebraic fraction as a sum of terms that do not involve any further improper fractions, and our task is complete!

6. Tips on using solutions

● When looking at the THEORY, ANSWERS, INTEGRALS, P-DIVISION or TIPS pages, use the Back button (at the bottom of the page) to return to the exercises

● Use the solutions intelligently. For example, they can help you get started on an exercise, or they can allow you to check whether your intermediate results are correct

● Try to make less use of the full solutions as you work your way through the Tutorial

Full worked solutions

Exercise 1.

$$\int \frac{x^2+2x+5}{x}\, dx$$ top line is quadratic in x

bottom line is linear in x

$\Rightarrow$ we have an <u>improper</u> algebraic fraction

$\rightarrow$ we need simple polynomial division ...

$$\begin{aligned}
\text{i.e. } \int \frac{x^2+2x+5}{x}\, dx &= \int \left(\frac{x^2}{x}+\frac{2x}{x}+\frac{5}{x}\right) dx \\
&= \int \left(x+2+\frac{5}{x}\right) dx \\
&= \int x\, dx + \int 2\, dx + 5\int \frac{1}{x}\, dx
\end{aligned}$$

$$\text{i.e.} \int \frac{x^2+2x+5}{x}\,dx \quad = \quad \frac{1}{2}x^2+2x+5\ \ln|x|+C,$$

where C is a constant of integration.

Return to Exercise 1

Exercise 2.

$$\int \frac{x^3 + 4x^2 + 3x + 1}{x^2}\, dx$$

top line is cubic in x

bottom line is quadratic in x

$\Rightarrow$ an <u>improper</u> algebraic fraction

$\rightarrow$ simple polynomial division ...

$$\begin{aligned}
\int \frac{x^3 + 4x^2 + 3x + 1}{x^2}\, dx &= \int \left(\frac{x^3}{x^2} + \frac{4x^2}{x^2} + \frac{3x}{x^2} + \frac{1}{x^2} \right) dx \\
&= \int \left(x + 4 + \frac{3}{x} + \frac{1}{x^2} \right) dx \\
&= \int x\, dx + \int 4\, dx + 3\int \frac{1}{x}\, dx + \int x^{-2}\, dx
\end{aligned}$$

$$
\begin{aligned}
\text{i.e. } \int \frac{x^3+4x^2+3x+1}{x^2}\,dx &= \frac{1}{2}x^2+4x+3\ \ln|x|+\frac{x^{-1}}{(-1)}+C \\
&= \frac{1}{2}x^2+4x+3\ \ln|x|-\frac{1}{x}+C,
\end{aligned}
$$

where C is a constant of integration.

Return to Exercise 2

Exercise 3.

$$\int \frac{x^2+3x+4}{x+1}\,dx$$

top line is quadratic in x

bottom line is linear in x

$\Rightarrow$ an <u>improper</u> algebraic fraction

$\rightarrow$ polynomial division ...

Now we have more than just a single term in the bottom line and we need to do full polynomial division

If you are unfamiliar with this technique, there is some extra help within the P-Division section

Here, we will go through the polynomial division first, and we will leave the integration until later ...

$$\begin{aligned}
\frac{x^2+3x+4}{x+1} &= \frac{x(x+1)-x \qquad +3x+4}{x+1} \\
&\quad \{ \text{ the bottom line has been used} \\
&\quad \text{ to write } \ x^2 \ \text{ as } \ x(x+1)-x \ \} \\
&= \frac{x(x+1) \qquad +2x+4}{x+1} \\
&= \frac{x(x+1)}{x+1} \ + \ \frac{2x+4}{x+1} \\
&= x \ + \ \frac{2x+4}{x+1} \\
&= x \ + \ \frac{2(x+1)-2 \qquad +4}{x+1} \\
&\quad \{ \text{ writing } \ 2x \ \text{ as } \ 2x(x+1)-2 \ \}
\end{aligned}$$

$$
\begin{aligned}
\text{i.e. } \frac{x^2+3x+4}{x+1} &= x + \frac{2(x+1) \qquad +2}{x+1} \\
&= x + \frac{2(x+1)}{x+1} + \frac{2}{x+1} \\
&= x + 2 + \frac{2}{x+1}
\end{aligned}
$$

{ polynomial division is complete,
since we no longer have any
improper algebraic fractions }

$$
\begin{aligned}
\therefore \int \frac{x^2+3x+4}{x+1}\,dx &= \int \left(x+2+\frac{2}{x+1} \right) dx \\
&= \frac{1}{2}x^2 + 2x + 2\ \ln|x+1| + C.
\end{aligned}
$$

Return to Exercise 3

Exercise 4.

$$\int \frac{2x^2+5x+3}{x+2}\,dx \quad \text{top line is quadratic in } x$$

bottom line is linear in x
$\Rightarrow$ an <u>improper</u> algebraic fraction
$\rightarrow$ polynomial division ...

$$\frac{2x^2+5x+3}{x+2} = \frac{2x(x+2)-4x \qquad +5x+3}{x+2}$$

{ the bottom line has been used
to write $2x^2$ as $2x(x+2)-4x$ }

$$= \frac{2x(x+2) \qquad +x+3}{x+2}$$

$$= \frac{2x(x+2)}{x+2} + \frac{x+3}{x+2}$$

$$
\begin{aligned}
\text{i.e. } \frac{2x^2+5x+3}{x+2} &= 2x + \frac{x+3}{x+2} \\
&= 2x + \frac{(x+2)-2 \qquad +3}{x+2} \\
&\qquad \{ \text{ writing } x \text{ as } (x+2)-2 \ \} \\
&= 2x + \frac{(x+2) \qquad +1}{x+2} \\
&= 2x + \frac{(x+2)}{x+2} + \frac{1}{x+2} \\
&= 2x + 1 + \frac{1}{x+2} \\
&\qquad \{ \text{ no improper algebraic fractions } \} \\
\therefore \int \frac{2x^2+5x+3}{x+2}\,dx &= \int \left(2x+1+\frac{1}{x+2}\right) dx \\
&= x^2 + x + \ln|x+2| + C.
\end{aligned}
$$

Return to Exercise 4

Exercise 5.

$$\int \frac{4x^3+2}{x^4+2x+3}\,dx \quad \text{top line is degree 3 in } x$$

bottom line is degree 4 in x

$\Rightarrow$ we have a proper algebraic fraction

$\rightarrow$ factorise bottom line for partial fractions?

No! First, check if this is of the form $\int \frac{k\,g'(x)}{g(x)}\,dx$, where k =constant

If $g(x) = x^4 + 2x + 3$ (the bottom line), $g'(x) = \frac{dg}{dx} = 4x^3 + 2$ (which exactly equals the top line). So we can use the standard integral

$$\int \frac{k\,g'(x)}{g(x)}\,dx = k\ \ln|g(x)| + C, \quad \text{with } \ k = 1$$

(or employ substitution techniques by setting $u = x^4 + 2x + 3$)

$$\therefore \int \frac{4x^3+2}{x^4+2x+3}\,dx \quad = \quad \ln|x^4 + 2x + 3| + C.$$

Return to Exercise 5

Exercise 6.

$$\int \frac{x}{x^2-5}\,dx \quad \text{top line is degree 1 in } x$$

bottom line is degree 2 in x
$\Rightarrow$ we have a proper algebraic fraction
$\rightarrow$ consider for partial fractions?

No! First, check if this is of the form $\int \frac{k\,g'(x)}{g(x)}\,dx$, where k =constant

If $g(x) = x^2 - 5$ (the bottom line), $g'(x) = \frac{dg}{dx} = 2x$ (which is proportional to the top line). So we can use the standard integral

$$\int \frac{k\,g'(x)}{g(x)}\,dx = k\,\ln|g(x)| + C, \quad \text{with } k = \frac{1}{2}$$

(or employ substitution techniques by setting $u = x^2 - 5$)

$$\text{i.e.} \int \frac{x}{x^2-5}\,dx = \int \frac{\frac{1}{2}\cdot 2x}{x^2-5}\,dx = \frac{1}{2}\ln|x^2-5| + C.$$

Return to Exercise 6

Exercise 7.

$$\int \frac{17-x}{(x-3)(x+4)}\,dx \quad \text{is a proper algebraic fraction,}$$

and the top line is <u>not</u> a multiple of the derivative of bottom line

Try partial fractions

$$\begin{aligned}\frac{17-x}{(x-3)(x+4)} &= \frac{A}{x-3}+\frac{B}{x+4}\\ &= \frac{A(x+4)+B(x-3)}{(x-3)(x+4)}\end{aligned}$$

$\therefore \quad 17-x = A(x+4)+B(x-3) \qquad$ [if true then true for all x]

$\underline{x=-4}$ gives $17+4=0+(-4-3)B$ i.e. $21=-7B$, $B=-3$

$\underline{x=3}$ gives $17-3=(3+4)A+0$ i.e. $14=7A$, $A=2$

$$\therefore \int \frac{17-x}{(x-3)(x+4)}\,dx = \int \frac{2}{x-3} + \frac{(-3)}{x+4}\,dx$$
$$= 2\int \frac{dx}{x-3} - 3\int \frac{dx}{x+4}$$
$$= 2\ln|x-3| - 3\ln|x+4| + C.$$

Note.

In the above we have used $\displaystyle\int \frac{dx}{ax+b} = \frac{1}{a}\ln|ax+b| + D$

Return to Exercise 7

Exercise 8.

$$\int \frac{11x+18}{(2x+5)(x-7)}\,dx$$ is a proper algebraic fraction,

and the top line is <u>not</u> a multiple of the derivative of bottom line

Try partial fractions

$$\begin{aligned}\frac{11x+18}{(2x+5)(x-7)} &= \frac{A}{2x+5}+\frac{B}{x-7} \\ &= \frac{A(x-7)+B(2x+5)}{(2x+5)(x-7)}\end{aligned}$$

$$\therefore \quad 11x+18 = A(x-7)+B(2x+5)$$

$\underline{x=7}$ gives $77+18=(14+5)B$ i.e. $95=19B$, $B=5$

$\underline{x=-\frac{5}{2}}$ gives $-\frac{55}{2}+18=(-\frac{5}{2}-7)A$ i.e. $\frac{19}{2}=\frac{19}{2}A$, $A=1$

$$\begin{aligned}
\therefore \int \frac{11x+18}{(2x+5)(x-7)}\,dx &= \int \frac{1}{2x+5} + \frac{5}{x-7}\,dx \\
&= \int \frac{dx}{2x+5} + 5\int \frac{dx}{x-7} \\
&= \frac{1}{2}\ln|2x+5| + 5\ln|x-7| + C.
\end{aligned}$$

Note.

In the above we have used $\int \frac{dx}{ax+b} = \frac{1}{a}\ln|ax+b| + D$

Return to Exercise 8

Exercise 9.

$$\int \frac{7x+1}{(x+1)(x-2)(x+3)}\,dx$$

is a proper algebraic fraction,

and the top line is <u>not</u> a multiple of the derivative of bottom line

Try partial fractions

$$\begin{aligned}\frac{7x+1}{(x+1)(x-2)(x+3)} &= \frac{A}{x+1}+\frac{B}{x-2}+\frac{C}{x+3}\\ &= \frac{A(x-2)(x+3)+B(x+1)(x+3)+C(x+1)(x-2)}{(x+1)(x-2)(x+3)}\end{aligned}$$

$$\therefore 7x+1 = A(x-2)(x+3)+B(x+1)(x+3)+C(x+1)(x-2)$$

$$7x + 1 = A(x-2)(x+3) + B(x+1)(x+3) + C(x+1)(x-2)$$

$\underline{x = -1}$ gives $-6 = A(-3)(2)$ i.e. $-6 = -6A$ i.e. $A = 1$

$\underline{x = 2}$ gives $15 = B(3)(5)$ i.e. $15 = 15B$ i.e. $B = 1$

$\underline{x = -3}$ gives $-20 = C(-2)(-5)$ i.e. $-20 = 10C$ i.e. $C = -2$

$$\therefore \int \frac{7x+1}{(x+1)(x-2)(x+3)} dx = \int \frac{1}{x+1} + \frac{1}{x-2} - 2\frac{1}{x+3} dx$$

$$= \ln|x+1| + \ln|x-2| - 2\ln|x+3| + C.$$

Return to Exercise 9

Exercise 10.

Proper algebraic fraction and we can use partial fractions

$$\int \frac{2x+9}{(x+5)^2}\,dx = \int \frac{A}{(x+5)} + \frac{B}{(x+5)^2}\,dx$$

where $\dfrac{2x+9}{(x+5)^2} = \dfrac{A(x+5)+B}{(x+5)^2}$ i.e. $2x+9 = A(x+5)+B$

$\underline{x=-5}$ gives $-10+9=B$ i.e. $B=-1$

$\underline{x=0}$ gives $9=5A+B=5A-1$ i.e. $10=5A$ i.e. $A=2$

$$\therefore \int \frac{2x+9}{(x+5)^2}\,dx = \int \frac{2}{x+5}+\frac{(-1)}{(x+5)^2}\,dx$$

$$= 2\int \frac{dx}{x+5} - \int \frac{dx}{(x+5)^2}$$

$$\begin{aligned}\text{i.e. } \int \frac{2x+9}{(x+5)^2}\, dx &= 2\ln|x+5| - \int (x+5)^{-2} dx + C \\ &= 2\ln|x+5| - \frac{(x+5)^{-1}}{(-1)} + C \\ &= 2\ln|x+5| + \frac{1}{x+5} + C,\end{aligned}$$

where, in the last integral, we have used

$$\int (ax+b)^n = \frac{(ax+b)^{n+1}}{n+1} + C, \quad (n \neq -1).$$

Return to Exercise 10

Exercise 11.

Proper algebraic fraction and we need to use partial fractions

$$\int \frac{13x-4}{(3x-2)(2x+1)}\,dx = \int \frac{A}{(3x-2)} + \frac{B}{(2x+1)}\,dx$$

where $\displaystyle \frac{13x-4}{(3x-2)(2x+1)} = \frac{A(2x+1)+B(3x-2)}{(3x-2)(2x+1)}$

$$13x-4 = A(2x+1)+B(3x-2)$$

and

$\underline{x=-\frac{1}{2}}$ gives $-\frac{13}{2}-4 = B\left(-\frac{3}{2}-2\right)$ i.e. $-\frac{21}{2} = -\frac{7}{2}B$, i.e. $B=3$

$\underline{x=\frac{2}{3}}$ gives $\frac{26}{3}-\frac{12}{3} = A\left(\frac{4}{3}+\frac{3}{3}\right)$ i.e. $\frac{14}{3} = \frac{7}{3}A$ i.e. $A=2$

$$
\begin{aligned}
\therefore \int \frac{13x-4}{(3x-2)(2x+1)}\,dx &= \int \frac{2}{3x-2}+\frac{3}{2x+1}\,dx \\
&= 2\int \frac{dx}{3x-2} + 3\int \frac{dx}{2x+1} \\
&= 2\left(\frac{1}{3}\right)\ln|3x-2| + 3\left(\frac{1}{2}\right)\ln|2x+1| + C \\
&= \frac{2}{3}\ln|3x-2| + \frac{3}{2}\ln|2x+1| + C,
\end{aligned}
$$

where $\displaystyle\int \frac{dx}{ax+b} = \frac{1}{a}\ln|ax+b| + C$ has been used.

Return to Exercise 11

Exercise 12.

Use Partial fractions

$$\int \frac{27x}{(x-2)^2(x+1)}\,dx = \int \frac{A}{(x-2)} + \frac{B}{(x-2)^2} + \frac{C}{x+1}\,dx$$

where

$$27x = A(x-2)(x+1) + B(x+1) + C(x-2)^2$$

$\underline{x=2}$	gives	$54 = 3B$	i.e. $B = 18$
$\underline{x=-1}$	gives	$-27 = C(-3)^2$	i.e. $C = -3$
$\underline{x=0}$	gives	$0 = A(-2) + 18 + (-3)(4)$	i.e. $A = 3$

$$\therefore \int \frac{27x}{(x-2)^2(x+1)}\,dx = \int \frac{3}{x-2} + \frac{18}{(x-2)^2} - \frac{3}{x+1}\,dx$$

$$= 3\ln|x-2| + 18\int (x-2)^{-2}dx - 3\ln|x+1| + D$$

$$= 3\ln|x-2| + \frac{18}{(-1)}(x-2)^{-1} - 3\ln|x+1| + D$$

$$= 3\ln|x-2| - \frac{18}{x-2} - 3\ln|x+1| + D.$$

Return to Exercise 12

Exercise 13.

$$\int \frac{3x^2}{(x-1)(x^2+x+1)}dx = \int \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1}dx$$

Note that x^2+x+1 does not give real linear factors
One thus uses the partial fraction $\frac{Bx+C}{x^2+x+1}$

We then have

$$3x^2 = A\left(x^2+x+1\right) + (Bx+C)(x-1)$$

$\underline{x=1}$ gives $3 = 3A$ i.e. $A = 1$

$\underline{x=0}$ gives $0 = A - C$ i.e. $C = A = 1$

$\underline{x=-1}$ gives $3 = A(1-1+1) + (-B+C)(-2)$

i.e. $3 = A + 2B - 2C$

i.e. $3 = 1 + 2B - 2$

i.e. $4 = 2B$ i.e. $B = 2$

$$
\begin{aligned}
\therefore \int \frac{3x^2}{(x-1)(x^2+x+1)}\,dx &= \int \frac{A}{x-1} + \int \frac{Bx+C}{(x^2+x+1)}dx \\
&= \int \frac{dx}{x-1} + \int \frac{2x+1}{x^2+x+1}dx \\
&= \ln|x-1| + \ln|x^2+x+1| + D,
\end{aligned}
$$

and we note that the second integral is of the form

$$\int \frac{g'(x)}{g(x)}dx = \ln|g(x)| + D.$$

Return to Exercise 13