Integration

TRIGONOMETRIC IDENTITIES

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A self-contained Tutorial Module for practising integration of expressions involving products of trigonometric functions such as $\sin nx \sin mx$

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1. Theory

Integrals of the form

$$\int \sin nx \sin mx \ ,$$

and similar ones with products like $\sin nx \cos mx$ and $\cos nx \cos mx$, can be solved by making use of the following trigonometric identities:

$$\begin{aligned} \sin A \sin B &= -\frac{1}{2}\left[\cos(A+B) - \cos(A-B)\right] \\ \sin A \cos B &= \frac{1}{2}\left[\sin(A+B) + \sin(A-B)\right] \\ \cos A \cos B &= \frac{1}{2}\left[\cos(A+B) + \cos(A-B)\right] \end{aligned}$$

Using these identities, such **products are expressed as a sum** of trigonometric functions

This sum is generally more straightforward to integrate

2. Exercises

Click on EXERCISE links for full worked solutions (9 exercises in total).

Perform the following integrations:

Exercise 1.

$$\int \cos 3x \cos 2x \, dx$$

Exercise 2.

$$\int \sin 5x \cos 3x \, dx$$

Exercise 3.

$$\int \sin 6x \sin 4x \, dx$$

Exercise 4.

$$\int \cos 2\omega t \sin \omega t \, dt, \text{ where } \omega \text{ is a constant}$$

Exercise 5.

$$\int \cos 4\omega t \cos 2\omega t \, dt \ , \text{ where } \omega \text{ is a constant}$$

Exercise 6.

$$\int \sin^2 x \, dx$$

Exercise 7.

$$\int \sin^2 \omega t \, dt \ , \text{ where } \omega \text{ is a constant}$$

Exercise 8.

$$\int \cos^2 t \, dt$$

Exercise 9.

$\int \cos^2 kx \, dx$, where k is a constant

3. Answers

1. $\frac{1}{10}\sin 5x + \frac{1}{2}\sin x + C$,
2. $-\frac{1}{16}\cos 8x - \frac{1}{4}\cos 2x + C$,
3. $-\frac{1}{20}\sin 10x + \frac{1}{4}\sin 2x + C$,
4. $-\frac{1}{6\omega}\cos 3\omega t + \frac{1}{2\omega}\cos \omega t + C$,
5. $\frac{1}{12\omega}\sin 6\omega t + \frac{1}{4\omega}\sin 2\omega t + C$,
6. $-\frac{1}{4}\sin 2x + \frac{1}{2}\,x + C$,
7. $-\frac{1}{4\omega}\sin 2\omega t + \frac{1}{2}\,t + C$,
8. $\frac{1}{4}\sin 2t + \frac{1}{2}\,t + C$,
9. $\frac{1}{4k}\sin 2kx + \frac{1}{2}\,x + C$.

4. Standard integrals

$f(x)$	$\int f(x)dx$	$f(x)$	$\int f(x)dx$
x^n	$\frac{x^{n+1}}{n+1} \quad (n \neq -1)$	$[g(x)]^n g'(x)$	$\frac{[g(x)]^{n+1}}{n+1} \quad (n \neq -1)$
$\frac{1}{x}$	$\ln\|x\|$	$\frac{g'(x)}{g(x)}$	$\ln\|g(x)\|$
e^x	e^x	a^x	$\frac{a^x}{\ln a} \quad (a > 0)$
$\sin x$	$-\cos x$	$\sinh x$	$\cosh x$
$\cos x$	$\sin x$	$\cosh x$	$\sinh x$
$\tan x$	$-\ln\|\cos x\|$	$\tanh x$	$\ln\cosh x$
$\operatorname{cosec} x$	$\ln\left\|\tan\frac{x}{2}\right\|$	$\operatorname{cosech} x$	$\ln\left\|\tanh\frac{x}{2}\right\|$
$\sec x$	$\ln\|\sec x + \tan x\|$	$\operatorname{sech} x$	$2\tan^{-1} e^x$
$\sec^2 x$	$\tan x$	$\operatorname{sech}^2 x$	$\tanh x$
$\cot x$	$\ln\|\sin x\|$	$\coth x$	$\ln\|\sinh x\|$
$\sin^2 x$	$\frac{x}{2} - \frac{\sin 2x}{4}$	$\sinh^2 x$	$\frac{\sinh 2x}{4} - \frac{x}{2}$
$\cos^2 x$	$\frac{x}{2} + \frac{\sin 2x}{4}$	$\cosh^2 x$	$\frac{\sinh 2x}{4} + \frac{x}{2}$

$f(x)$	$\int f(x)\,dx$	$f(x)$	$\int f(x)\,dx$
$\frac{1}{a^2+x^2}$	$\frac{1}{a}\tan^{-1}\frac{x}{a}$	$\frac{1}{a^2-x^2}$	$\frac{1}{2a}\ln\left\lvert\frac{a+x}{a-x}\right\rvert \quad (0<\lvert x\rvert<a)$
	$(a>0)$	$\frac{1}{x^2-a^2}$	$\frac{1}{2a}\ln\left\lvert\frac{x-a}{x+a}\right\rvert \quad (\lvert x\rvert>a>0)$
$\frac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1}\frac{x}{a}$	$\frac{1}{\sqrt{a^2+x^2}}$	$\ln\left\lvert\frac{x+\sqrt{a^2+x^2}}{a}\right\rvert \quad (a>0)$
	$(-a<x<a)$	$\frac{1}{\sqrt{x^2-a^2}}$	$\ln\left\lvert\frac{x+\sqrt{x^2-a^2}}{a}\right\rvert \quad (x>a>0)$
$\sqrt{a^2-x^2}$	$\frac{a^2}{2}\left[\sin^{-1}\left(\frac{x}{a}\right)\right.$	$\sqrt{a^2+x^2}$	$\frac{a^2}{2}\left[\sinh^{-1}\left(\frac{x}{a}\right)+\frac{x\sqrt{a^2+x^2}}{a^2}\right]$
	$\left.+\frac{x\sqrt{a^2-x^2}}{a^2}\right]$	$\sqrt{x^2-a^2}$	$\frac{a^2}{2}\left[-\cosh^{-1}\left(\frac{x}{a}\right)+\frac{x\sqrt{x^2-a^2}}{a^2}\right]$

5. Tips

- STANDARD INTEGRALS are provided. Do not forget to use these tables when you need to

- When looking at the THEORY, STANDARD INTEGRALS, ANSWERS or TIPS pages, use the Back button (at the bottom of the page) to return to the exercises

- Use the solutions intelligently. For example, they can help you get started on an exercise, or they can allow you to check whether your intermediate results are correct

- Try to make less use of the full solutions as you work your way through the Tutorial

Full worked solutions

Exercise 1.

$$\int \cos 3x \cos 2x \, dx: \qquad \text{Use } \cos A \cos B = \tfrac{1}{2}\left[\cos(A+B) + \cos(A-B)\right]$$

i.e. taking $A = 3x$ and $B = 2x$:

$$\begin{aligned} \int \cos 3x \cos 2x \, dx &= \tfrac{1}{2} \int \left[\cos(3x+2x) + \cos(3x-2x)\right] \\ &= \tfrac{1}{2} \int [\cos 5x + \cos x] \, dx \end{aligned}$$

Each term under the integration sign is a function of a linear function of x, i.e.

$\int f(ax+b) \, dx = \frac{1}{a} \int f(u) du$, where $u = ax+b$, $du = a \, dx$, i.e. $dx = \frac{du}{a}$.

i.e. $\int \cos 3x \cos 2x \, dx = \frac{1}{2}\frac{1}{5} \sin 5x + \frac{1}{2} \sin x + C = \frac{1}{10} \sin 5x + \frac{1}{2} \sin x + C$.

Return to Exercise 1

Exercise 2.

$$\int \sin 5x \cos 3x \, dx: \qquad \text{Use } \sin A \cos B = \tfrac{1}{2}\left[\sin(A+B) + \sin(A-B)\right]$$

i.e. taking $A = 5x$ and $B = 3x$:

$$\begin{aligned} \int \sin 5x \cos 3x \, dx &= \frac{1}{2}\int \left[\sin(5x+3x) + \sin(5x-3x)\right] \\ &= \frac{1}{2}\int \left[\sin 8x + \sin 2x\right] dx \end{aligned}$$

i.e.

$$\begin{aligned} \int \sin 5x \cos 3x \, dx &= -\frac{1}{2}\frac{1}{8}\cos 8x - \frac{1}{2}\frac{1}{2}\cos 2x + C \\ &= -\frac{1}{16}\cos 8x - \frac{1}{4}\cos 2x + C. \end{aligned}$$

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Exercise 3.

$\int \sin 6x \sin 4x \, dx$: Use $\sin A \sin B = -\frac{1}{2}\left[\cos(A+B) - \cos(A-B)\right]$

i.e. taking $A = 6x$ and $B = 4x$:

$$\begin{aligned}\int \sin 6x \sin 4x \, dx &= -\frac{1}{2}\int \left[\cos(6x+4x) - \cos(6x-4x)\right] \\ &= -\frac{1}{2}\int \left[\cos 10x - \cos 2x\right] dx\end{aligned}$$

i.e.

$$\begin{aligned}\int \sin 6x \sin 4x \, dx &= -\frac{1}{2}\frac{1}{10}\sin 10x + \frac{1}{2}\frac{1}{2}\sin 2x + C \\ &= -\frac{1}{20}\sin 10x + \frac{1}{4}\sin 2x + C.\end{aligned}$$

Return to Exercise 3

Exercise 4.

$\int \cos 2\omega t \sin \omega t \, dt$: Use $\sin A \cos B = \frac{1}{2}\left[\sin(A+B) + \sin(A-B)\right]$

i.e. taking $A = \omega t$ and $B = 2\omega t$:

$$\begin{aligned}
\int \cos 2\omega t \sin \omega t \, dt &= \frac{1}{2}\int \left[\sin(\omega t + 2\omega t) + \sin(\omega t - 2\omega t)\right] \\
&= \frac{1}{2}\int \left[\sin 3\omega t + \sin(-\omega t)\right] dt \\
&= \frac{1}{2}\int \left[\sin 3\omega t - \sin \omega t\right] dt \\
&= -\frac{1}{2}\frac{1}{3\omega}\cos 3\omega t + \frac{1}{2}\frac{1}{\omega}\cos \omega t + C \\
&= -\frac{1}{6\omega}\cos 3\omega t + \frac{1}{2\omega}\cos \omega t + C.
\end{aligned}$$

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Exercise 5.

$\int \cos 4\omega t \cos 2\omega t \, dt$:   Use $\cos A \cos B = \frac{1}{2}\left[\cos(A+B) + \cos(A-B)\right]$

i.e. taking $A = 4\omega t$ and $B = 2\omega t$:

$$\begin{aligned}
\int \cos 4\omega t \cos 2\omega t \, dt &= \frac{1}{2}\int \left[\cos(4\omega t + 2\omega t) + \cos(4\omega t - 2\omega t)\right] \\
&= \frac{1}{2}\int \left[\cos 6\omega t + \cos 2\omega t\right] dt \\
&= \frac{1}{2}\frac{1}{6\omega}\sin 6\omega t + \frac{1}{2}\frac{1}{2\omega}\sin 2\omega t + C \\
&= \frac{1}{12\omega}\sin 6\omega t + \frac{1}{4\omega}\sin 2\omega t + C.
\end{aligned}$$

Return to Exercise 5

Exercise 6.

$\int \sin^2 x\, dx$: For the particular case: $A{=}B{=}x$,

the formula: $\sin A \sin B = -\frac{1}{2}\left[\cos(A+B) - \cos(A-B)\right]$,

reduces to: $\sin^2 x = -\frac{1}{2}(\cos 2x - 1)$ (a half-angle formula)

i.e.

$$\begin{aligned} \int \sin^2 x\, dx &= -\frac{1}{2}\int (\cos 2x - 1)\, dx \\ &= -\frac{1}{2}\frac{1}{2}\sin 2x + \frac{1}{2}x + C \\ &= -\frac{1}{4}\sin 2x + \frac{1}{2}x + C. \end{aligned}$$

Return to Exercise 6

Exercise 7.

$\int \sin^2 \omega t \, dt$: For the particular case: $A{=}B{=}\omega t$,

the formula: $\sin A \sin B = -\frac{1}{2}\left[\cos\left(A+B\right) - \cos\left(A-B\right)\right]$,

reduces to: $\sin^2 \omega t = -\frac{1}{2}(\cos 2\omega t - 1)$

i.e.

$$\begin{aligned}\int \sin^2 \omega t \, dt &= -\frac{1}{2}\int (\cos 2\omega t - 1)\, dt \\ &= -\frac{1}{2}\frac{1}{2\omega}\sin 2\omega t + \frac{1}{2}t + C \\ &= -\frac{1}{4\omega}\sin 2\omega t + \frac{1}{2}\, t + C.\end{aligned}$$

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Exercise 8.

$\int \cos^2 t\, dt$: For the particular case: $A{=}B{=}t$,

the formula: $\cos A \cos B = \frac{1}{2}\left[\cos\left(A+B\right) + \cos\left(A-B\right)\right]$,

reduces to: $\cos^2 t = \frac{1}{2}(\cos 2t + 1)$

i.e.

$$\begin{aligned}
\int \cos^2 t\, dt &= \frac{1}{2}\int (\cos 2t + 1)\, dt \\
&= \frac{1}{2}\frac{1}{2}\sin 2t + \frac{1}{2}\, t + C \\
&= \frac{1}{4}\sin 2t + \frac{1}{2}\, t + C.
\end{aligned}$$

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Exercise 9.

$\int \cos^2 kx\, dx$: For the particular case: $A{=}B{=}kx$,

the formula: $\cos A \cos B = \frac{1}{2}\left[\cos\left(A+B\right) + \cos\left(A-B\right)\right]$,

reduces to: $\cos^2 kx = \frac{1}{2}(\cos 2kx + 1)$

i.e.

$$
\begin{aligned}
\int \cos^2 kx\, dx &= \frac{1}{2}\int (\cos 2kx + 1) dx \\
&= \frac{1}{2}\frac{1}{2k}\sin 2kx + \frac{1}{2}\, x + C \\
&= \frac{1}{4k}\sin 2kx + \frac{1}{2}\, x + C.
\end{aligned}
$$

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