

DIRECT INTEGRATION

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A Tutorial Module introducing ordinary differential equations and the method of direct integration

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1. Introduction

$\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3 = x^7$ is an example of an **ordinary** differential equation (o.d.e.) since it contains only **ordinary** derivatives such as $\frac{dy}{dx}$ and not **partial** derivatives such as $\frac{\partial y}{\partial x}$.

The dependent variable is y while the independent variable is x (an o.d.e. has only one independent variable while a partial differential equation has more than one independent variable).

The above example is a **second order** equation since the highest order of derivative involved is **two** (note the presence of the $\frac{d^2y}{dx^2}$ term).

An o.d.e. is **linear** when each term has y and its derivatives only appearing to the power one. The appearance of a term involving the product of y and $\frac{dy}{dx}$ would also make an o.d.e. **nonlinear**.

In the above example, the term $\left(\frac{dy}{dx}\right)^3$ makes the equation **nonlinear**.

The **general solution** of an n^{th} order o.d.e. has n arbitrary constants that can take any values.

In an **initial value problem**, one solves an n^{th} order o.d.e. to find the general solution and then applies n **boundary conditions** (“initial values/conditions”) to find a **particular solution** that does not have any arbitrary constants.

2. Theory

An ordinary differential equation of the following form:

$$\frac{dy}{dx} = f(x)$$

can be solved by integrating both sides with respect to x :

$$y = \int f(x) dx .$$

This technique, called **DIRECT INTEGRATION**, can also be applied when the left hand side is a higher order derivative.

In this case, one integrates the equation a sufficient number of times until y is found.

3. Exercises

Click on [EXERCISE](#) links for full worked solutions (there are 8 exercises in total)

EXERCISE 1.

Show that $y = 2e^{2x}$ is a particular solution of the ordinary differential equation: $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 0$

EXERCISE 2.

Show that $y = 7 \cos 3x - 2 \sin 2x$ is a particular solution of $\frac{d^2y}{dx^2} + 2y = -49 \cos 3x + 4 \sin 2x$

EXERCISE 3.

Show that $y = A \sin x + B \cos x$, where A and B are arbitrary constants, is the general solution of $\frac{d^2y}{dx^2} + y = 0$

EXERCISE 4.

Derive the general solution of $\frac{dy}{dx} = 2x + 3$

EXERCISE 5.

Derive the general solution of $\frac{d^2y}{dx^2} = -\sin x$

EXERCISE 6.

Derive the general solution of $\frac{d^2y}{dt^2} = a$, where $a = \text{constant}$

EXERCISE 7.

Derive the general solution of $\frac{d^3y}{dx^3} = 3x^2$

EXERCISE 8.

Derive the general solution of $e^{-x} \frac{d^2y}{dx^2} = 3$

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4. Answers

1. HINT: Work out $\frac{d^2y}{dx^2}$ and $\frac{dy}{dx}$ and then substitute your results, along with the given form of y , into the differential equation ,
2. HINT: Show that $\frac{d^2y}{dx^2} = -63 \cos 3x + 8 \sin 2x$ and substitute this, along with the given form of y , into the differential equation ,
3. HINT: Show that $\frac{d^2y}{dx^2} = -A \sin x - B \cos x$,
4. $y = x^2 + 3x + C$,
5. $y = \sin x + Ax + B$,
6. $y = \frac{1}{2}at^2 + Ct + D$,
7. $y = \frac{1}{20}x^5 + Cx^2 + Dx + E$,
8. $y = 3e^x + Cx + D$.

5. Standard integrals

$f(x)$	$\int f(x)dx$	$f(x)$	$\int f(x)dx$
x^n	$\frac{x^{n+1}}{n+1} \quad (n \neq -1)$	$[g(x)]^n g'(x)$	$\frac{[g(x)]^{n+1}}{n+1} \quad (n \neq -1)$
$\frac{1}{x}$	$\ln x $	$\frac{g'(x)}{g(x)}$	$\ln g(x) $
e^x	e^x	a^x	$\frac{a^x}{\ln a} \quad (a > 0)$
$\sin x$	$-\cos x$	$\sinh x$	$\cosh x$
$\cos x$	$\sin x$	$\cosh x$	$\sinh x$
$\tan x$	$-\ln \cos x $	$\tanh x$	$\ln \cosh x$
$\operatorname{cosec} x$	$\ln\left \tan \frac{x}{2}\right $	$\operatorname{cosech} x$	$\ln\left \tanh \frac{x}{2}\right $
$\sec x$	$\ln \sec x + \tan x $	$\operatorname{sech} x$	$2 \tan^{-1} e^x$
$\sec^2 x$	$\tan x$	$\operatorname{sech}^2 x$	$\tanh x$
$\cot x$	$\ln \sin x $	$\operatorname{coth} x$	$\ln \sinh x $
$\sin^2 x$	$\frac{x}{2} - \frac{\sin 2x}{4}$	$\sinh^2 x$	$\frac{\sinh 2x}{4} - \frac{x}{2}$
$\cos^2 x$	$\frac{x}{2} + \frac{\sin 2x}{4}$	$\cosh^2 x$	$\frac{\sinh 2x}{4} + \frac{x}{2}$

$f(x)$	$\int f(x) dx$	$f(x)$	$\int f(x) dx$
$\frac{1}{a^2+x^2}$	$\frac{1}{a} \tan^{-1} \frac{x}{a}$ $(a > 0)$	$\frac{1}{a^2-x^2}$	$\frac{1}{2a} \ln \left \frac{a+x}{a-x} \right $ ($0 < x < a$) $\frac{1}{2a} \ln \left \frac{x-a}{x+a} \right $ ($ x > a > 0$)
$\frac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1} \frac{x}{a}$ $(-a < x < a)$	$\frac{1}{\sqrt{a^2+x^2}}$	$\ln \left \frac{x+\sqrt{a^2+x^2}}{a} \right $ ($a > 0$) $\ln \left \frac{x+\sqrt{x^2-a^2}}{a} \right $ ($x > a > 0$)
$\sqrt{a^2-x^2}$	$\frac{a^2}{2} \left[\sin^{-1} \left(\frac{x}{a} \right) + \frac{x\sqrt{a^2-x^2}}{a^2} \right]$	$\sqrt{a^2+x^2}$	$\frac{a^2}{2} \left[\sinh^{-1} \left(\frac{x}{a} \right) + \frac{x\sqrt{a^2+x^2}}{a^2} \right]$ $\frac{a^2}{2} \left[-\cosh^{-1} \left(\frac{x}{a} \right) + \frac{x\sqrt{x^2-a^2}}{a^2} \right]$

6. Tips on using solutions

- When looking at the THEORY, ANSWERS, INTEGRALS or TIPS pages, use the [Back](#) button (at the bottom of the page) to return to the exercises.
- Use the solutions intelligently. For example, they can help you get started on an exercise, or they can allow you to check whether your intermediate results are correct.
- Try to make less use of the full solutions as you work your way through the Tutorial.

Full worked solutions

Exercise 1.

We need:

$$\frac{dy}{dx} = 2 \cdot 2e^{2x} = 4e^{2x}$$

$$\frac{d^2y}{dx^2} = 2 \cdot 4e^{2x} = 8e^{2x}.$$

$$\begin{aligned} \therefore \frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y &= 8e^{2x} - 4e^{2x} - 2 \cdot e^{2x} \\ &= (8 - 8)e^{2x} \\ &= 0 \\ &= \text{RHS} \end{aligned}$$

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Exercise 2.

To show that $y = 7 \cos 3x - 2 \sin 2x$ is a particular solution of $\frac{d^2 y}{dx^2} + 2y = -49 \cos 3x + 4 \sin 2x$, work out the following:

$$\frac{dy}{dx} = -21 \sin 3x - 4 \cos 2x$$

$$\frac{d^2 y}{dx^2} = -63 \cos 3x + 8 \sin 2x$$

$$\begin{aligned} \therefore \frac{d^2 y}{dx^2} + 2y &= -63 \cos 3x + 8 \sin 2x + 2(7 \cos 3x - 2 \sin 2x) \\ &= (-63 + 14) \cos 3x + (8 - 4) \sin 2x \\ &= -49 \cos 3x + 4 \sin 2x \\ &= \text{RHS} \end{aligned}$$

- Notes
- The equation is second order, so the general solution would have two arbitrary (undetermined) constants.
 - Notice how similar the particular solution is to the Right-Hand-Side of the equation. It involves the same functions but they have different coefficients i.e.

y is of the form

$$"a \cos 3x + b \sin 2x" \quad \left(\begin{array}{l} a = 7 \\ b = -2 \end{array} \right)$$

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Exercise 3.

$$\begin{aligned}\text{We need: } \quad \frac{dy}{dx} &= A \cos x + B \cdot (-\sin x) \\ \frac{d^2y}{dx^2} &= -A \sin x - B \cos x \\ \therefore \frac{d^2y}{dx^2} + y &= (-A \sin x - B \cos x) + (A \sin x + B \cos x) \\ &= 0 \\ &= \text{RHS}\end{aligned}$$

Since the differential equation is second order and the solution has two arbitrary constants, this solution is the general solution.

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Exercise 4.

This is an equation of the form $\frac{dy}{dx} = f(x)$, and it can be solved by direct integration.

Integrate both sides with respect to x :

$$\int \frac{dy}{dx} dx = \int (2x + 3) dx$$

$$\text{i.e.} \quad \int dy = \int (2x + 3) dx$$

$$\text{i.e.} \quad y = 2 \cdot \frac{1}{2} x^2 + 3x + C$$

$$\text{i.e.} \quad y = x^2 + 3x + C,$$

where C is the (combined) arbitrary constant that results from integrating both sides of the equation. The general solution must have one arbitrary constant since the differential equation is first order.

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Exercise 5.

This is of the form $\boxed{\frac{d^2y}{dx^2} = f(x)}$, so we can solve for y by direct integration.

Integrate both sides with respect to x :

$$\begin{aligned}\frac{dy}{dx} &= - \int \sin x dx \\ &= -(-\cos x) + A\end{aligned}$$

Integrate again:

$$y = \sin x + Ax + B$$

where A, B are the two arbitrary constants of the general solution (the equation is second order).

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Exercise 6.

Integrate both sides with respect to t :

$$\frac{dy}{dt} = \int a dt$$

i.e. $\frac{dy}{dt} = at + C$

Integrate again:

$$y = \int (at + C) dt$$

i.e. $y = \frac{1}{2}at^2 + Ct + D,$

where C and D are the two arbitrary constants required for the general solution of the second order differential equation.

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Exercise 7.

Integrate both sides with respect to x :

$$\frac{d^2y}{dx^2} = \int 3x^2 dx$$

i.e. $\frac{d^2y}{dx^2} = 3 \cdot \frac{1}{3}x^3 + C$

i.e. $\frac{d^2y}{dx^2} = x^3 + C$

Integrate again:

$$\frac{dy}{dx} = \int (x^3 + C) dx$$

i.e. $\frac{dy}{dx} = \frac{x^4}{4} + Cx + D$

Integrate again: $y = \int \left(\frac{x^4}{4} + Cx + D \right) dx$

i.e. $y = \frac{1}{4} \cdot \frac{1}{5} x^5 + C \cdot \frac{1}{2} x^2 + Dx + E$

i.e. $y = \frac{1}{20} x^5 + C' x^2 + Dx + E$

where $C' = \frac{C}{2}$, D and E are the required three arbitrary constants of the general solution of the third order differential equation.

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Exercise 8.

Multiplying both sides of the equation by e^x gives:

$$e^x \cdot e^{-x} \frac{d^2y}{dx^2} = e^x \cdot 3$$

$$\text{i.e. } \frac{d^2y}{dx^2} = 3e^x$$

This is now of the form $\frac{d^2y}{dx^2} = f(x)$, where $f(x) = 3e^x$, and the solution y can be found by direct integration.

Integrating both sides with respect to x :

$$\frac{dy}{dx} = \int 3e^x dx$$

$$\text{i.e. } \frac{dy}{dx} = 3e^x + C.$$

Integrate again:

$$y = \int (3e^x + C) dx$$

$$\text{i.e. } y = 3e^x + Cx + D,$$

where C and D are the two arbitrary constants of the general solution of the original second order differential equation.

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