

SEPARATION OF VARIABLES

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A Tutorial Module for learning the technique
of separation of variables

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1. Theory

If one can re-arrange an ordinary differential equation into the following standard form:

$$\frac{dy}{dx} = f(x)g(y),$$

then the solution may be found by the technique of **SEPARATION OF VARIABLES**:

$$\int \frac{dy}{g(y)} = \int f(x) dx .$$

This result is obtained by dividing the standard form by $g(y)$, and then integrating both sides with respect to x .

2. Exercises

Click on [EXERCISE](#) links for full worked solutions (there are 16 exercises in total)

EXERCISE 1.

Find the general solution of $\frac{dy}{dx} = 3x^2e^{-y}$ and the particular solution that satisfies the condition $y(0) = 1$

EXERCISE 2.

Find the general solution of $\frac{dy}{dx} = \frac{y}{x}$

EXERCISE 3.

Solve the equation $\frac{dy}{dx} = \frac{y+1}{x-1}$ given the boundary condition: $y = 1$ at $x = 0$

EXERCISE 4.

Solve $y^2 \frac{dy}{dx} = x$ and find the particular solution when $y(0) = 1$

EXERCISE 5.

Find the solution of $\frac{dy}{dx} = e^{2x+y}$ that has $y = 0$ when $x = 0$

EXERCISE 6.

Find the general solution of $\frac{xy}{x+1} = \frac{dy}{dx}$

EXERCISE 7.

Find the general solution of $x \sin^2 y \cdot \frac{dy}{dx} = (x+1)^2$

EXERCISE 8.

Solve $\frac{dy}{dx} = -2x \tan y$ subject to the condition: $y = \frac{\pi}{2}$ when $x = 0$

EXERCISE 9.

Solve $(1 + x^2) \frac{dy}{dx} + xy = 0$

and find the particular solution when $y(0) = 2$

EXERCISE 10.

Solve $x \frac{dy}{dx} = y^2 + 1$ and find the particular solution when $y(1) = 1$

EXERCISE 11.

Find the general solution of $x \frac{dy}{dx} = y^2 - 1$

EXERCISE 12.

Find the general solution of $\frac{1}{y} \frac{dy}{dx} = \frac{x}{x^2 + 1}$

EXERCISE 13.

Solve $\frac{dy}{dx} = \frac{y}{x(x+1)}$ and find the particular solution when $y(1) = 3$

EXERCISE 14.

Find the general solution of $\sec x \cdot \frac{dy}{dx} = \sec^2 y$

EXERCISE 15.

Find the general solution of $\operatorname{cosec}^3 x \frac{dy}{dx} = \cos^2 y$

EXERCISE 16.

Find the general solution of $(1 - x^2) \frac{dy}{dx} + x(y - a) = 0$, where a is a constant

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3. Answers

1. General solution is $y = \ln(x^3 + A)$, and particular solution is $y = \ln(x^3 + e)$,
2. General solution is $y = kx$,
3. General solution is $y + 1 = k(x - 1)$, and particular solution is $y = -2x + 1$,
4. General solution is $\frac{y^3}{3} = \frac{x^2}{2} + C$, and particular solution is $y^3 = \frac{3x^2}{2} + 1$,
5. General solution is $y = -\ln\left(-\frac{1}{2}e^{2x} - C\right)$, and particular solution is $y = -\ln\left(\frac{3-e^{2x}}{2}\right)$,
6. General solution is $e^x = ky(x + 1)$,

7. General solution is $\frac{y}{2} - \frac{1}{4} \sin 2y = \frac{x^2}{2} + 2x + \ln x + C$,
8. General solution is $\sin y = e^{-x^2+A}$, and particular solution is $\sin y = e^{-x^2}$,
9. General solution is $y(1+x^2)^{\frac{1}{2}} = k$, and particular solution is $y(1+x^2)^{\frac{1}{2}} = 2$,
10. General solution is $\tan^{-1} y = \ln x + C$, and particular solution is $\tan^{-1} y = \ln x + \frac{\pi}{4}$,
11. General solution is $y - 1 = kx^2(y + 1)$,
12. General solution is $y^2 = k(x^2 + 1)$,
13. General solution is $y = \frac{kx}{x+1}$, and particular solution is $y = \frac{6x}{x+1}$,

14. General solution is $2y + \sin 2y = 4 \sin x + C$,

15. General solution is $\tan y = -\cos x + \frac{1}{3} \cos^3 x + C$,

16. General solution is $y - a = k(1 - x^2)^{\frac{1}{2}}$.

4. Standard integrals

$f(x)$	$\int f(x)dx$	$f(x)$	$\int f(x)dx$
x^n	$\frac{x^{n+1}}{n+1} \quad (n \neq -1)$	$[g(x)]^n g'(x)$	$\frac{[g(x)]^{n+1}}{n+1} \quad (n \neq -1)$
$\frac{1}{x}$	$\ln x $	$\frac{g'(x)}{g(x)}$	$\ln g(x) $
e^x	e^x	a^x	$\frac{a^x}{\ln a} \quad (a > 0)$
$\sin x$	$-\cos x$	$\sinh x$	$\cosh x$
$\cos x$	$\sin x$	$\cosh x$	$\sinh x$
$\tan x$	$-\ln \cos x $	$\tanh x$	$\ln \cosh x$
$\operatorname{cosec} x$	$\ln\left \tan \frac{x}{2}\right $	$\operatorname{cosech} x$	$\ln\left \tanh \frac{x}{2}\right $
$\sec x$	$\ln \sec x + \tan x $	$\operatorname{sech} x$	$2 \tan^{-1} e^x$
$\sec^2 x$	$\tan x$	$\operatorname{sech}^2 x$	$\tanh x$
$\cot x$	$\ln \sin x $	$\operatorname{coth} x$	$\ln \sinh x $
$\sin^2 x$	$\frac{x}{2} - \frac{\sin 2x}{4}$	$\sinh^2 x$	$\frac{\sinh 2x}{4} - \frac{x}{2}$
$\cos^2 x$	$\frac{x}{2} + \frac{\sin 2x}{4}$	$\cosh^2 x$	$\frac{\sinh 2x}{4} + \frac{x}{2}$

$f(x)$	$\int f(x) dx$	$f(x)$	$\int f(x) dx$
$\frac{1}{a^2+x^2}$	$\frac{1}{a} \tan^{-1} \frac{x}{a}$ $(a > 0)$	$\frac{1}{a^2-x^2}$	$\frac{1}{2a} \ln \left \frac{a+x}{a-x} \right $ ($0 < x < a$) $\frac{1}{2a} \ln \left \frac{x-a}{x+a} \right $ ($ x > a > 0$)
$\frac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1} \frac{x}{a}$ $(-a < x < a)$	$\frac{1}{\sqrt{a^2+x^2}}$	$\ln \left \frac{x+\sqrt{a^2+x^2}}{a} \right $ ($a > 0$) $\ln \left \frac{x+\sqrt{x^2-a^2}}{a} \right $ ($x > a > 0$)
$\sqrt{a^2-x^2}$	$\frac{a^2}{2} \left[\sin^{-1} \left(\frac{x}{a} \right) + \frac{x\sqrt{a^2-x^2}}{a^2} \right]$	$\sqrt{a^2+x^2}$	$\frac{a^2}{2} \left[\sinh^{-1} \left(\frac{x}{a} \right) + \frac{x\sqrt{a^2+x^2}}{a^2} \right]$ $\frac{a^2}{2} \left[-\cosh^{-1} \left(\frac{x}{a} \right) + \frac{x\sqrt{x^2-a^2}}{a^2} \right]$

5. Tips on using solutions

- When looking at the THEORY, ANSWERS, INTEGRALS, or TIPS pages, use the [Back](#) button (at the bottom of the page) to return to the exercises.

- Use the solutions intelligently. For example, they can help you get started on an exercise, or they can allow you to check whether your intermediate results are correct.

- Try to make less use of the full solutions as you work your way through the Tutorial.

Full worked solutions

Exercise 1.

This is of the form $\boxed{\frac{dy}{dx} = f(x)g(y)}$, where $f(x) = 3x^2$ and $g(y) = e^{-y}$, so we can separate the variables and then integrate,

$$\text{i.e. } \int e^y dy = \int 3x^2 dx \quad \text{i.e. } e^y = x^3 + A$$

(where $A =$ arbitrary constant).

$$\text{i.e. } y = \ln(x^3 + A) : \underline{\text{General solution}}$$

$$\underline{\text{Particular solution:}} \quad y(x) = 1 \text{ when } x = 0 \quad \text{i.e. } e^1 = 0^3 + A$$

$$\text{i.e. } A = e \quad \text{and} \quad y = \ln(x^3 + e).$$

[Return to Exercise 1](#)

Exercise 2.

This is of the form $\frac{dy}{dx} = f(x)g(y)$, where $f(x) = \frac{1}{x}$ and $g(y) = y$, so we can separate the variables and then integrate,

$$\int \frac{dy}{y} = \int \frac{dx}{x}$$

$$\begin{aligned} \text{i.e. } \ln y &= \ln x + C \\ &= \ln x + \ln k \quad (\ln k = C = \text{constant}) \end{aligned}$$

$$\text{i.e. } \ln y - \ln x = \ln k$$

$$\text{i.e. } \ln(y/x) = \ln k$$

$$\text{i.e. } y = kx.$$

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Exercise 3.

Find the general solution first. Then apply the boundary condition to get the particular solution.

Equation is of the form: $\frac{dy}{dx} = f(x)g(y)$, where $f(x) = \frac{1}{x-1}$
 $g(y) = y + 1$

so separate variables and integrate.

$$\text{i.e. } \int \frac{dy}{y+1} = \int \frac{dx}{x-1}$$

$$\begin{aligned} \text{i.e. } \ln(y+1) &= \ln(x-1) + C \\ &= \ln(x-1) + \ln k \quad (k = \text{arbitrary constant}) \end{aligned}$$

$$\text{i.e. } \ln(y+1) - \ln(x-1) = \ln k$$

$$\text{i.e. } \ln\left(\frac{y+1}{x-1}\right) = \ln k$$

$$\text{i.e. } \frac{y+1}{x-1} = k$$

$$\text{i.e. } y+1 = k(x-1) \quad (\text{general solution})$$

Now determine k for particular solution with $y(0) = 1$.

$$\begin{array}{l} x = 0 \\ y = 1 \end{array} \quad \text{gives} \quad 1 + 1 = k(0 - 1)$$

$$\text{i.e.} \quad 2 = -k$$

$$\text{i.e.} \quad k = -2$$

Particular solution: $y + 1 = -2(x - 1)$ i.e. $y = -2x + 1$.

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Exercise 4.

Use separation of variables to find the general solution first.

$$\int y^2 dy = \int x dx \quad \text{i.e.} \quad \frac{y^3}{3} = \frac{x^2}{2} + C$$

(general solution)

Particular solution with $y = 1, x = 0$: $\frac{1}{3} = 0 + C$ i.e. $C = \frac{1}{3}$

$$\text{i.e. } y^3 = \frac{3x^2}{2} + 1.$$

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Exercise 5.

General solution first then find particular solution.

Write equation as: $\frac{dy}{dx} = e^{2x}e^y$ ($\equiv f(x)g(y)$)

Separate variables
and integrate:

$$\int \frac{dy}{e^y} = \int e^{2x} dx$$

i.e. $-e^{-y} = \frac{1}{2}e^{2x} + C$

i.e. $e^{-y} = -\frac{1}{2}e^{2x} - C$

i.e. $-y = \ln\left(-\frac{1}{2}e^{2x} - C\right)$

i.e. $y = -\ln\left(-\frac{1}{2}e^{2x} - C\right)$.

Particular solution: $x = 0$ gives $0 = -\ln\left(-\frac{1}{2} - C\right)$
 $y = 0$

i.e. $-\frac{1}{2} - C = 1$

i.e. $C = -\frac{3}{2}$

$\therefore y = -\ln\left(\frac{3-e^{2x}}{2}\right)$.

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Exercise 6.

Separate variables and integrate:

$$\int \frac{x}{x+1} dx = \int \frac{dy}{y}$$

↗

Numerator and denominator of same degree in x : reduce degree of numerator using long division.

i.e. $\frac{x}{x+1} = \frac{x+1-1}{x+1} = \frac{x+1}{x+1} - \frac{1}{x+1} = 1 - \frac{1}{x+1}$

i.e. $\int \left(1 - \frac{1}{x+1}\right) dx = \int \frac{dy}{y}$

i.e. $x - \ln(x+1) = \ln y + \ln k$ ($\ln k = \text{constant of integration}$)

i.e. $x = \ln(x+1) + \ln y + \ln k$

$$= \ln[ky(x+1)]$$

i.e. $e^x = ky(x+1)$. General solution.

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Exercise 7.

Separate variables and integrate:

$$\text{i.e.} \quad \int \sin^2 y dy = \int \frac{(x+1)^2}{x} dx$$

$$\text{i.e.} \quad \int \frac{1}{2}(1 - \cos 2y) dy = \int \frac{x^2 + 2x + 1}{x} dx$$

$$\text{i.e.} \quad \frac{1}{2} \int dy - \frac{1}{2} \int \cos 2y dy = \int \left(x + 2 + \frac{1}{x} \right) dx$$

$$\text{i.e.} \quad \frac{1}{2} y - \frac{1}{2} \cdot \frac{1}{2} \sin 2y = \frac{1}{2} x^2 + 2x + \ln x + C .$$

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Exercise 8.

General solution first.

Separate variables: i.e. $\frac{dy}{\tan y} = -2x dx$

Integrate: i.e. $\int \cot y dy = -2 \int x dx$

i.e. $\ln(\sin y) = -2 \cdot \frac{x^2}{2} + A$

i.e. $\ln(\sin y) = -x^2 + A$

i.e. $\sin y = e^{-x^2+A}$

$\left\{ \text{Note: } \int \frac{\cos y}{\sin y} dy \text{ is of form } \int \frac{f'(y)}{f(y)} dy = \ln[f(y)] + C \right\}$

Particular solution: $y = \frac{\pi}{2}$ when $x = 0$

gives $\sin \frac{\pi}{2} = e^A$

i.e. $1 = e^A$

i.e. $A = 0$

\therefore Required solution is $\sin y = e^{-x^2}$.

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Exercise 9.

Separate variables and integrate:

$$(1 + x^2) \frac{dy}{dx} = -xy$$

$$\text{i.e.} \quad \int \frac{dy}{y} = - \int \frac{x}{1 + x^2} dx$$

$$\text{i.e.} \quad \int \frac{dy}{y} = -\frac{1}{2} \int \frac{2x}{1 + x^2} dx$$

$$[\text{compare with } \int \frac{f'(x)}{f(x)} dx]$$

$$\text{i.e.} \quad \ln y = -\frac{1}{2} \ln(1 + x^2) + \ln k \quad (\ln k = \text{constant})$$

$$\text{i.e.} \quad \ln y + \ln(1 + x^2)^{\frac{1}{2}} = \ln k$$

$$\text{i.e.} \quad \ln \left[y(1 + x^2)^{\frac{1}{2}} \right] = \ln k$$

$$\text{i.e.} \quad y(1 + x^2)^{\frac{1}{2}} = k, \quad (\underline{\text{general solution}}).$$

Particular solution

$$y(0) = 2, \quad \text{i.e. } y(x) = 2 \text{ when } x = 0$$

$$\text{i.e. } 2(1 + 0)^{\frac{1}{2}} = k$$

$$\text{i.e. } k = 2$$

$$\text{i.e. } y(1 + x^2)^{\frac{1}{2}} = 2.$$

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Exercise 10.

$$\int \frac{dy}{y^2 + 1} = \int \frac{dx}{x}$$

$$\left\{ \text{Standard integral: } \int \frac{dy}{1 + y^2} = \tan^{-1} y + C \right\}$$

i.e. $\tan^{-1} y = \ln x + C$. General solution.

Particular solution with $y = 1$ when $x = 1$:

$$\tan \frac{\pi}{4} = 1 \quad \therefore \tan^{-1}(1) = \frac{\pi}{4} \quad , \quad \text{while} \quad \ln 1 = 0 \quad (\text{i.e. } 1 = e^0)$$

$$\therefore \frac{\pi}{4} = 0 + C \quad \text{i.e. } C = \frac{\pi}{4}$$

Particular solution is: $\tan^{-1} y = \ln x + \frac{\pi}{4}$.

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Exercise 11.

$$\begin{aligned} \text{Partial fractions : } \int \frac{dy}{y^2 - 1} &= \int \frac{dx}{x} \\ \frac{1}{y^2 - 1} &= \frac{A}{y - 1} + \frac{B}{y + 1} = \frac{A(y + 1) + B(y - 1)}{(y - 1)(y + 1)} \\ &= \frac{(A + B)y + (A - B)}{y^2 - 1} \end{aligned}$$

Compare numerators: $1 = (A + B)y + (A - B)$ [true for all y]

\therefore

$$\begin{array}{r} A + B = 0 \\ A - B = 1 \\ \hline 2A = 1 \end{array}$$

$$\therefore A = \frac{1}{2}, B = -\frac{1}{2}.$$

$$\text{i.e.} \quad \int \frac{A}{y-1} + \frac{B}{y+1} dy = \int \frac{dx}{x}$$

$$\text{i.e.} \quad \frac{1}{2} \int \frac{1}{y-1} - \frac{1}{y+1} dy = \int \frac{dx}{x}$$

$$\text{i.e.} \quad \frac{1}{2} [\ln(y-1) - \ln(y+1)] = \ln x + \ln k$$

$$\text{i.e.} \quad \ln(y-1) - \ln(y+1) - 2 \ln x = 2 \ln k$$

$$\text{i.e.} \quad \ln \left[\frac{y-1}{(y+1)x^2} \right] = 2 \ln k$$

$$\text{i.e.} \quad y-1 = k'x^2(y+1), \quad (k' = k^2 = \text{constant}).$$

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Exercise 12.

$$\int \frac{dy}{y} = \int \frac{x}{x^2 + 1} dx = \frac{1}{2} \int \frac{2x}{x^2 + 1} dx$$

$$\left\{ \underline{\text{Note}} : \int \frac{f'(x)}{f(x)} dx = \ln[f(x)] + A \right\}$$

$$\text{i.e. } \ln y = \frac{1}{2} \ln(x^2 + 1) + C$$

$$\text{i.e. } \frac{1}{2} \ln y^2 = \frac{1}{2} \ln(x^2 + 1) + C \quad \{\text{get same coefficients to allow log manipulations}\}$$

$$\text{i.e. } \frac{1}{2} \ln \left[\frac{y^2}{x^2 + 1} \right] = C$$

$$\text{i.e. } \frac{y^2}{x^2 + 1} = e^{2C}$$

$$\text{i.e. } y^2 = k(x^2 + 1), \quad (\text{where } k = e^{2C} = \text{constant}).$$

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Exercise 13.

$$\int \frac{dy}{y} = \int \frac{dx}{x(x+1)}$$

Use partial fractions:

$$\begin{aligned} \frac{1}{x(x+1)} &= \frac{A}{x} + \frac{B}{x+1} = \frac{A(x+1) + Bx}{x(x+1)} \\ &= \frac{(A+B)x + A}{x(x+1)} \end{aligned}$$

Compare numerators: $1 = (A+B)x + A$ (true for all x)

i.e. $A+B=0$ and $A=1$, $\therefore B=-1$

$$\text{i.e. } \int \frac{dy}{y} = \int \left(\frac{1}{x} - \frac{1}{x+1} \right) dx$$

i.e. $\ln y = \ln x - \ln(x+1) + C$

$$\text{i.e. } \ln y - \ln x + \ln(x+1) = \ln k \quad (\ln k = C = \text{constant})$$

$$\text{i.e. } \ln \left[\frac{y(x+1)}{x} \right] = \ln k$$

$$\text{i.e. } \frac{y(x+1)}{x} = k$$

$$\text{i.e. } y = \frac{kx}{x+1}. \quad \text{General solution.}$$

Particular solution with $y(1) = 3$:

$$x = 1, \quad y = 3 \quad \text{gives} \quad 3 = \frac{k}{1+1}$$

$$\text{i.e. } k = 6$$

$$\text{i.e. } y = \frac{6x}{x+1} .$$

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Exercise 14.

$$\int \frac{dy}{\sec^2 y} = \int \frac{dx}{\sec x}$$

$$\text{i.e.} \quad \int \cos^2 y \, dy = \int \cos x \, dx$$

$$\text{i.e.} \quad \int \frac{1 + \cos 2y}{2} \, dy = \int \cos x \, dx$$

$$\text{i.e.} \quad \frac{y}{2} + \frac{1}{2} \cdot \frac{1}{2} \sin 2y = \sin x + C$$

$$\text{i.e.} \quad 2y + \sin 2y = 4 \sin x + C'$$

(where $C' = 4C = \text{constant}$).

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Exercise 15.

$$\begin{aligned} \text{i.e. } \int \frac{dy}{\cos^2 y} &= \int \frac{dx}{\operatorname{cosec}^3 x} \\ &= \int \sin^3 x \, dx \\ &= \int \sin^2 x \cdot \sin x \, dx \\ &= \int (1 - \cos^2 x) \cdot \sin x \, dx \\ &= \int \sin x \, dx - \underbrace{\int \cos^2 x \cdot \sin x \, dx}_{\substack{\text{set } u = \cos x, \text{ so } \frac{du}{dx} = -\sin x \\ \text{and } \cos^2 x \cdot \sin x \, dx = -u^2 du}} \end{aligned}$$

LHS is standard integral

$$\int \sec^2 y \, dy = \tan y + A.$$

This gives, $\tan y = -\cos x - \left(-\frac{\cos^3 x}{3}\right) + C$

i.e. $\tan y = -\cos x + \frac{\cos^3 x}{3} + C.$

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Exercise 16.

i.e. $(1 - x^2) \frac{dy}{dx} = -x(y - a)$

i.e. $\int \frac{dy}{y-a} = - \int \frac{x}{1-x^2} dx$

i.e. $\int \frac{dy}{y-a} = +\frac{1}{2} \int \frac{-2x}{1-x^2} dx$ [compare RHS integral with $\int \frac{f'(x)}{f(x)} dx$]

i.e. $\ln(y - a) = \frac{1}{2} \ln(1 - x^2) + \ln k$

i.e. $\ln(y - a) - \ln(1 - x^2)^{\frac{1}{2}} = \ln k$

i.e. $\ln \left[\frac{y-a}{(1-x^2)^{\frac{1}{2}}} \right] = \ln k$

$\therefore y - a = k(1 - x^2)^{\frac{1}{2}}.$

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