Differential Equations

HOMOGENEOUS FUNCTIONS

Graham S McDonald

A Tutorial Module for learning to solve differential equations that involve homogeneous functions

- **Table of contents**
- **Begin Tutorial**

 g.s.mcdonald@salford.ac.uk

Table of contents

1. Theory
2. Exercises
3. Answers
4. Standard integrals
5. Tips on using solutions

Full worked solutions

1. Theory

$M(x, y) = 3x^2 + xy$ is a **homogeneous function** since the sum of the powers of x and y in each term is the same (i.e. x^2 is x to power 2 and $xy = x^1y^1$ giving total power of $1 + 1 = 2$).

The **degree** of this homogeneous function is 2.

Here, we consider differential equations with the following standard form:

$$\frac{dy}{dx} = \frac{M(x, y)}{N(x, y)}$$

where M and N are **homogeneous functions** of the **same degree**.

To find the solution, change the dependent variable from y to v, where

$$y = vx\ .$$

The LHS of the equation becomes:

$$\frac{dy}{dx} = x\frac{dv}{dx} + v$$

using the product rule for differentiation.

Solve the resulting equation by separating the variables v and x. Finally, re-express the solution in terms of x and y.

Note. This method also works for equations of the form:

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right)\ .$$

2. Exercises

Click on Exercise links for full worked solutions (there are 11 exercises in total)

Exercise 1.

Find the general solution of $\frac{dy}{dx} = \frac{xy + y^2}{x^2}$

Exercise 2.

Solve $2xy\frac{dy}{dx} = x^2 + y^2$ given that $y = 0$ at $x = 1$

Exercise 3.

Solve $\frac{dy}{dx} = \frac{x + y}{x}$ and find the particular solution when $y(1) = 1$

Exercise 4.

Solve $x\dfrac{dy}{dx} = x - y$ and find the particular solution when $y(2) = \dfrac{1}{2}$

Exercise 5.

Solve $\dfrac{dy}{dx} = \dfrac{x-2y}{x}$ and find the particular solution when $y(1) = -1$

Exercise 6.

Given that $\dfrac{dy}{dx} = \dfrac{x+y}{x-y}$, prove that $\tan^{-1}\left(\dfrac{y}{x}\right) = \dfrac{1}{2}\ln\left(x^2+y^2\right) + A$, where A is an arbitrary constant

Exercise 7.

Find the general solution of $2x^2\dfrac{dy}{dx} = x^2 + y^2$

Exercise 8.

Find the general solution of $(2x-y)\,\dfrac{dy}{dx} = 2y - x$

<u>Note</u>. The key to solving the next three equations is to recognise that each equation can be written in the form $\dfrac{dy}{dx} = f\left(\dfrac{y}{x}\right) \equiv f(v)$

Exercise 9.

Find the general solution of $\dfrac{dy}{dx} = \dfrac{y}{x} + \tan\left(\dfrac{y}{x}\right)$

Exercise 10.

Find the general solution of $x\,\dfrac{dy}{dx} = y + xe^{\frac{y}{x}}$

Exercise 11.

Find the general solution of $x\frac{dy}{dx} = y + \sqrt{x^2 + y^2}$

3. Answers

1. General solution is $y = -\frac{x}{\ln x + C}$,

2. General solution is $x = C(x^2 - y^2)$, and particular solution is $x = x^2 - y^2$,

3. General solution is $y = x \ln(kx)$, and particular solution is $y = x + x \ln x$,

4. General solution is $1 = Kx(x - 2y)$, and particular solution is $2xy - x^2 = -2$,

5. General solution is $x^2(x - 3y) = K$, and particular solution is $x^2(x - 3y) = 4$,

6. HINT: Try changing the variables from (x, y) to (x, v), where $y = vx$,

7. General solution is $2x = (x - y)(\ln x + C)$,

8. General solution is $y - x = K(x + y)^3$,

9. General solution is $\sin\left(\frac{y}{x}\right) = kx$,

10. General solution is $y = -x \ln(-\ln kx)$,

11. General solution is $\sinh^{-1}\left(\frac{y}{x}\right) = \ln x + C$.

4. Standard integrals

$f(x)$	$\int f(x)dx$	$f(x)$	$\int f(x)dx$
x^n	$\frac{x^{n+1}}{n+1} \quad (n \neq -1)$	$[g(x)]^n g'(x)$	$\frac{[g(x)]^{n+1}}{n+1} \quad (n \neq -1)$
$\frac{1}{x}$	$\ln \lvert x \rvert$	$\frac{g'(x)}{g(x)}$	$\ln \lvert g(x) \rvert$
e^x	e^x	a^x	$\frac{a^x}{\ln a} \quad (a > 0)$
$\sin x$	$-\cos x$	$\sinh x$	$\cosh x$
$\cos x$	$\sin x$	$\cosh x$	$\sinh x$
$\tan x$	$-\ln \lvert \cos x \rvert$	$\tanh x$	$\ln \cosh x$
$\operatorname{cosec} x$	$\ln \left\lvert \tan \frac{x}{2} \right\rvert$	$\operatorname{cosech} x$	$\ln \left\lvert \tanh \frac{x}{2} \right\rvert$
$\sec x$	$\ln \lvert \sec x + \tan x \rvert$	$\operatorname{sech} x$	$2 \tan^{-1} e^x$
$\sec^2 x$	$\tan x$	$\operatorname{sech}^2 x$	$\tanh x$
$\cot x$	$\ln \lvert \sin x \rvert$	$\coth x$	$\ln \lvert \sinh x \rvert$
$\sin^2 x$	$\frac{x}{2} - \frac{\sin 2x}{4}$	$\sinh^2 x$	$\frac{\sinh 2x}{4} - \frac{x}{2}$
$\cos^2 x$	$\frac{x}{2} + \frac{\sin 2x}{4}$	$\cosh^2 x$	$\frac{\sinh 2x}{4} + \frac{x}{2}$

$f(x)$	$\int f(x)\,dx$	$f(x)$	$\int f(x)\,dx$
$\frac{1}{a^2+x^2}$	$\frac{1}{a}\tan^{-1}\frac{x}{a}$	$\frac{1}{a^2-x^2}$	$\frac{1}{2a}\ln\left\lvert\frac{a+x}{a-x}\right\rvert \quad (0<\lvert x\rvert<a)$
	$(a>0)$	$\frac{1}{x^2-a^2}$	$\frac{1}{2a}\ln\left\lvert\frac{x-a}{x+a}\right\rvert \quad (\lvert x\rvert>a>0)$
$\frac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1}\frac{x}{a}$	$\frac{1}{\sqrt{a^2+x^2}}$	$\ln\left\lvert\frac{x+\sqrt{a^2+x^2}}{a}\right\rvert \quad (a>0)$
	$(-a<x<a)$	$\frac{1}{\sqrt{x^2-a^2}}$	$\ln\left\lvert\frac{x+\sqrt{x^2-a^2}}{a}\right\rvert \quad (x>a>0)$
$\sqrt{a^2-x^2}$	$\frac{a^2}{2}\left[\sin^{-1}\left(\frac{x}{a}\right)\right.$	$\sqrt{a^2+x^2}$	$\frac{a^2}{2}\left[\sinh^{-1}\left(\frac{x}{a}\right)+\frac{x\sqrt{a^2+x^2}}{a^2}\right]$
	$\left.+\frac{x\sqrt{a^2-x^2}}{a^2}\right]$	$\sqrt{x^2-a^2}$	$\frac{a^2}{2}\left[-\cosh^{-1}\left(\frac{x}{a}\right)+\frac{x\sqrt{x^2-a^2}}{a^2}\right]$

5. Tips on using solutions

- When looking at the THEORY, ANSWERS, INTEGRALS or TIPS pages, use the Back button (at the bottom of the page) to return to the exercises.

- Use the solutions intelligently. For example, they can help you get started on an exercise, or they can allow you to check whether your intermediate results are correct.

- Try to make less use of the full solutions as you work your way through the Tutorial.

Full worked solutions

Exercise 1.

RHS = quotient of homogeneous functions of same degree (= 2)

$$\underline{\text{Set } y = vx}: \quad \text{i.e.} \quad \frac{d}{dx}(vx) = \frac{xvx + v^2x^2}{x^2}$$

$$\text{i.e.} \quad x\frac{dv}{dx} + v = v + v^2$$

$$\underline{\text{Separate variables}} \quad x\frac{dv}{dx} = v^2 \quad \text{(subtract } v \text{ from both sides)}$$

$$\underline{\text{and integrate}}: \quad \int \frac{dv}{v^2} = \int \frac{dx}{x}$$

$$\text{i.e.} \quad -\frac{1}{v} = \ln x + C$$

$$\underline{\text{Re-express in terms of x,y}}: -\frac{x}{y} = \ln x + C$$

$$\text{i.e.} \quad y = \frac{-x}{\ln x + C}.$$

Return to Exercise 1

Exercise 2.

Standard form: $\dfrac{dy}{dx} = \dfrac{x^2 + y^2}{2xy}$

i.e. quotient of homogeneous functions that have the same degree

Set $y = xv$: $\dfrac{d}{dx}(xv) = \dfrac{x^2 + x^2v^2}{2x \cdot xv}$

i.e. $x\dfrac{dv}{dx} + \dfrac{dx}{dx}v = \dfrac{x^2(1+v^2)}{2x^2v}$

i.e. $x\dfrac{dv}{dx} + v = \dfrac{1+v^2}{2v}$

Separate variables (x, v) and integrate: $x\dfrac{dv}{dx} = \dfrac{1+v^2}{2v} - \dfrac{v(2v)}{(2v)}$

$$\text{i.e.} \quad x\frac{dv}{dx} = \frac{1-v^2}{2v}$$

$$\text{i.e.} \quad \int \frac{2v}{1-v^2}\, dv = \int \frac{dx}{x}$$

$$\left\{\text{Note: } \frac{d}{dv}(1-v^2) = -2v\right\} \quad \text{i.e.} \quad -\int \frac{-2v}{1-v^2}\, dv = \int \frac{dx}{x}$$

$$\text{i.e.} \quad -\ln(1-v^2) = \ln x + \ln C$$

$$\text{i.e.} \quad \ln[(1-v^2)^{-1}] = \ln(Cx)$$

$$\text{i.e.} \quad \frac{1}{1-v^2} = Cx$$

Re-express in terms of x and y: i.e. $\frac{1}{1-\frac{y^2}{x^2}} = Cx$

i.e. $\frac{x^2}{x^2-y^2} = Cx$

i.e. $\frac{x}{C} = x^2 - y^2$.

Particular solution: $x = 1$, $y = 0$ gives $-\frac{1}{C} = 1 - 0$

i.e. $C = 1$

gives $x^2 - y^2 = x$.

Return to Exercise 2

Exercise 3.

Set $y = xv$:

$$\begin{aligned} x\,\frac{dv}{dx} + v &= \frac{x + xv}{x} \\ &= \frac{x}{x}\,(1+v) = 1 + v \\ \text{i.e. } x\,\frac{dv}{dx} &= 1 \end{aligned}$$

Separate variables and integrate:

$$\begin{aligned} \int dv &= \int \frac{dx}{x} \\ \text{i.e.} \qquad v &= \ln\, x + \ln\, k \quad (\ln\, k = \text{constant}) \\ \text{i.e.} \qquad v &= \ln\,(kx) \end{aligned}$$

Re-express in terms of x and y:

$$\begin{aligned} \frac{y}{x} &= \ln(kx) \\ \text{i.e.} \quad y &= x \ln(kx) \quad . \end{aligned}$$

Particular solution with $y = 1$ when $x = 1$:

$$\begin{aligned} 1 &= \ln(k) \\ \text{i.e.} \quad k &= e^1 = e \\ \text{i.e.} \quad y &= x \ln(ex) \\ &= x[\ln e + \ln x] \\ &= x[1 + \ln x] \\ \text{i.e.} \quad y &= x + x \ln x \quad . \end{aligned}$$

Return to Exercise 3

Exercise 4.

$\frac{dy}{dx} = \frac{x-y}{x}$: $\underline{\text{Set } y = vx}$: i.e. $x\frac{dv}{dx} + v = 1 - v$

i.e. $x\frac{dv}{dx} = 1 - 2v$ i.e. $\int \frac{dv}{1-2v} = \int \frac{dx}{x}$

i.e. - $\frac{1}{2}\ln(1 - 2v) = \ln\, x + \ln\, k$

i.e. $\ln\left[(1 - 2v)^{-\frac{1}{2}}\right] - \ln\, x = \ln\, k$

i.e. $\ln\left[\frac{1}{(1-2v)^{\frac{1}{2}}x}\right] = \ln\, k$

$$\text{i.e.} \quad 1 = kx(1-2v)^{\frac{1}{2}}$$

Re-express in x, y: $$1 = kx\left(1 - \frac{2y}{x}\right)^{\frac{1}{2}}$$

$$\text{i.e.} \quad 1 = kx\left(\frac{x-2y}{x}\right)^{\frac{1}{2}}$$

(square both sides) $$1 = K\,x^2\left(\frac{x-2y}{x}\right) \quad , \quad (k^2 = K)$$

$$\text{i.e.} \quad 1 = K\,x(x-2y)$$

Particular solution: $1 = K \cdot 2 \cdot \left(2 - 2\left(\frac{1}{2}\right)\right) = K \cdot 2 \cdot 1, \quad \text{i.e. } K = \frac{1}{2}$

$y(2) = \frac{1}{2}$ i.e. $\begin{array}{l} x = 2 \\ y = \frac{1}{2} \end{array}$ gives $2 = x^2 - 2xy.$

Return to Exercise 4

Exercise 5.

Set $y = xv$:

$$\begin{aligned} x\,\frac{dv}{dx} + v &= \frac{x - 2xv}{x} \\ &= 1 - 2v \\ \text{i.e. } x\,\frac{dv}{dx} &= 1 - 3v \end{aligned}$$

Separate variables and integrate:

$$\begin{aligned} \int \frac{dv}{1-3v} &= \int \frac{dx}{x} \\ \text{i.e.} \quad \frac{1}{(-3)}\ln(1-3v) &= \ln x + \ln k \qquad (\ln k = \text{ constant}) \\ \text{i.e.} \quad \ln(1-3v) &= -3\ln x - 3\ln k \\ \text{i.e} \quad \ln(1-3v) + \ln x^3 &= -3\ln k \\ \text{i.e} \quad \ln[x^3(1-3v)] &= -3\ln k \\ \text{i.e} \quad x^3(1-3v) &= K \qquad (K = \text{constant}) \end{aligned}$$

Re-express in terms of x and y:

$$\begin{aligned} x^3\left(1-\frac{3y}{x}\right) &= K \\ \text{i.e. } x^3\left(\frac{x-3y}{x}\right) &= K \\ \text{i.e. } x^2\left(x-3y\right) &= K \; . \end{aligned}$$

Particular solution with $y(1)=-1$:

$$\begin{aligned} 1(1+3) &= K \qquad \text{i.e. } \; K=4 \\ \therefore \quad x^2\left(x-3y\right) &= 4 \; . \end{aligned}$$

Return to Exercise 5

Exercise 6.
Already in standard form, with quotient of two first degree homogeneous functions.

Set $y = xv$:

$$x\frac{dv}{dx} + v = \frac{x + vx}{x - vx}$$

i.e.

$$x\frac{dv}{dx} = \frac{x(1+v)}{x(1-v)} - v$$

$$= \frac{1 + v - v(1-v)}{1-v}$$

i.e.

$$x\frac{dv}{dx} = \frac{1+v^2}{1-v}$$

Separate variables and integrate:

$$\begin{aligned}
\int \frac{1-v}{1+v^2}dv &= \int \frac{dx}{x} \\
\text{i.e.} \quad \int \frac{dv}{1+v^2} - \frac{1}{2}\int \frac{2v}{1+v^2} &= \int \frac{dx}{x} \\
\text{i.e.} \quad \tan^{-1} v - \frac{1}{2}\ \ln(1+v^2) &= \ln\ x + A
\end{aligned}$$

Re-express in terms of x and y:

$$\begin{aligned}
\tan^{-1}\left(\frac{y}{x}\right) &- \frac{1}{2}\ln\left(1+\frac{y^2}{x^2}\right) = \ln\ x + A \\
\text{i.e.} \quad \tan^{-1}\left(\frac{y}{x}\right) &= \frac{1}{2}\ln\left(\frac{x^2+y^2}{x^2}\right) + \frac{1}{2}\ln\ x^2 + A \\
&= \frac{1}{2}\ln\left[\left(\frac{x^2+y^2}{\not x^2}\right)\cdot \not x^2\right] + A
\end{aligned}$$

Return to Exercise 6

Exercise 7.

$$\frac{dy}{dx} = \frac{x^2 + y^2}{2x^2}$$

Set $y = xv$:

$$\begin{aligned} x\,\frac{dv}{dx} + v &= \frac{x^2 + x^2v^2}{2x^2} \\ &= \frac{1 + v^2}{2} \\ \text{i.e.} \quad x\,\frac{dv}{dx} &= \frac{1 + v^2}{2} - \frac{2v}{2} \\ &= \frac{1 + v^2 - 2v}{2} \end{aligned}$$

Separate variables and integrate:

$$\int \frac{dv}{1-2v+v^2} = \frac{1}{2}\int \frac{dx}{x}$$

$$\text{i.e.} \quad \int \frac{dv}{(1-v)^2} = \frac{1}{2}\int \frac{dx}{x}$$

[Note: $1-v$ is a linear function of v, therefore use standard integral and divide by coefficient of v. In other words,

$w = 1 - v$

$\frac{dw}{dv} = -1$ and $\int \frac{dv}{(1-v)^2} = \frac{1}{(-1)}\int \frac{dw}{w^2}$.]

$$\text{i.e.} \quad -\int \frac{dw}{w^2} = \frac{1}{2}\int \frac{dx}{x}$$

$$\text{i.e.} \quad -\left(-\frac{1}{w}\right) = \frac{1}{2}\ln x + C$$

$$\text{i.e.} \quad \frac{1}{1-v} = \frac{1}{2}\ln x + C$$

Re-express in terms of x and y:

$$\begin{aligned} \frac{1}{1-\frac{y}{x}} &= \frac{1}{2}\ \ln x + C \\ \text{i.e.} \quad \frac{x}{x-y} &= \frac{1}{2}\ \ln x + C \\ \text{i.e.} \quad 2x &= (x-y)(\ln x + C'), \qquad (C' = 2C). \end{aligned}$$

Return to Exercise 7

Exercise 8.

$\frac{dy}{dx} = \frac{2y-x}{2x-y}$. Set $y = vx$, $x\frac{dv}{dx} + v = \frac{2v-1}{2-v}$

$$\therefore \; x\frac{dv}{dx} = \frac{2v-1-v(2-v)}{2-v} = \frac{v^2-1}{2-v} \; ; \; \int \frac{2-v}{v^2-1}dv = \int \frac{dx}{x}$$

Partial fractions: $\frac{2-v}{v^2-1} = \frac{A}{v-1} + \frac{B}{v+1} = \frac{A(v+1)+B(v-1)}{v^2-1}$

$$\text{i.e.} \quad \begin{array}{r} A+B=-1 \\ A-B=2 \\ \hline 2A=1 \\ \hline \end{array}$$

$$\text{i.e. } A = \tfrac{1}{2}, \; B = -\tfrac{3}{2}$$

i.e. $\frac{1}{2}\int \frac{1}{v-1} - \frac{3}{v+1}dv = \int \frac{dx}{x}$

i.e. $\frac{1}{2}\ln(v-1) - \frac{3}{2}\ln(v+1) = \ln x + \ln k$

i.e. $\ln\left[\frac{(v-1)^{\frac{1}{2}}}{(v+1)^{\frac{3}{2}}x}\right] = \ln\, k$

i.e. $\frac{v-1}{(v+1)^3x^2} = k^2$

<u>Re-express in x, y:</u>

$$\frac{\left(\frac{y}{x}-1\right)}{\left(\frac{y}{x}+1\right)^3 x^2} = k^2$$

$$\text{i.e. } \frac{\left(\frac{y-x}{x}\right)}{\left(\frac{y+x}{x}\right)^3 x^2} = k^2$$

$$\text{i.e. } y - x = K(y+x)^3.$$

Return to Exercise 8

Exercise 9.

RHS is only a function of $v = \frac{y}{x}$, so substitute and separate variables.

Set $y = xv$:

$$\begin{aligned} x\,\frac{dv}{dx} + v &= v + \tan v \\ \text{i.e.} \qquad x\,\frac{dv}{dx} &= \tan v \end{aligned}$$

Separate variables and integrate:

$$\int \frac{dv}{\tan v} = \int \frac{dx}{x}$$

$$\{ \text{ \underline{Note}: } \int \frac{\cos v}{\sin v}\,dx \equiv \int \frac{f'(v)}{f(v)} dv = \ln[f(v)] + C \ \}$$

$$\begin{aligned}
\text{i.e.} \quad \ln[\sin v] &= \ln\, x + \ln\, k \quad (\ln\, k = \text{constant}) \\
\text{i.e.} \quad \ln\left[\frac{\sin v}{x}\right] &= \ln\, k \\
\text{i.e.} \quad \frac{\sin v}{x} &= k \\
\text{i.e.} \quad \sin v &= kx
\end{aligned}$$

Re-express in terms of x and y: $\sin\left(\frac{y}{x}\right) = kx.$

Return to Exercise 9

Exercise 10.

$$\frac{dy}{dx} = \left(\frac{y}{x}\right) + e^{\left(\frac{y}{x}\right)}$$

i.e. RHS is function of $v = \frac{y}{x}$, only.

Set $y = vx$:

$$\begin{aligned} x\frac{dv}{dx} + v &= v + e^v \\ \text{i.e.} \quad x\frac{dv}{dx} &= e^v \\ \text{i.e.} \quad \int e^{-v} dv &= \int \frac{dx}{x} \\ \text{i.e.} \quad -e^{-v} &= \ln x + \ln k \\ &= \ln(kx) \\ \text{i.e.} \quad e^{-v} &= -\ln(kx) \end{aligned}$$

Re-express in terms of x, y:

$$\begin{aligned} e^{-\frac{y}{x}} &= -\ln(kx) \\ \text{i.e.} \quad -\frac{y}{x} &= \ln[-\ln(kx)] \\ \text{i.e.} \quad y &= -x\ \ln[-\ln(kx)]. \end{aligned}$$

Return to Exercise 10

Exercise 11.

$$\begin{aligned}\frac{dy}{dx} &= \frac{y}{x} + \frac{1}{x}\sqrt{x^2+y^2} \\ &= \frac{y}{x} + \sqrt{1+\left(\frac{y}{x}\right)^2}\end{aligned}$$

[Note RHS is a function of only $v = \frac{y}{x}$, so substitute and separate the variables]

i.e. Set $y = xv$:

$$\begin{aligned}x\frac{dv}{dx} + v &= v + \sqrt{1+v^2} \\ \text{i.e.} \qquad x\frac{dv}{dx} &= \sqrt{1+v^2}\end{aligned}$$

Separate variables and integrate:

$$\int \frac{dv}{\sqrt{1+v^2}} = \int \frac{dx}{x}$$

$$\left\{ \text{Standard integral: } \int \frac{dv}{\sqrt{1+v^2}} = \sinh^{-1}(v) + C \right\}$$

$$\text{i.e.} \quad \sinh^{-1}(v) = \ln x + A$$

Re-express in terms of x and y

$$\sinh^{-1}\left(\frac{y}{x}\right) = \ln x + A \quad .$$

Return to Exercise 11