Differential Equations

PPLATO PROMOTING PHYSICS LEARNING & TEACHING OPPORTUNITIES

INTEGRATING FACTOR METHOD

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A Tutorial Module for learning to solve 1st order linear differential equations

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1. Theory

Consider an ordinary differential equation (o.d.e.) that we wish to solve to find out how the variable y depends on the variable x.

If the equation is **first order** then the highest derivative involved is a first derivative.

If it is also a **linear** equation then this means that each term can involve y either as the derivative $\frac{dy}{dx}$ OR through a single factor of y.

Any such linear first order o.d.e. can be re-arranged to give the following standard form:

$$\boxed{\frac{dy}{dx} + P(x)y = Q(x)}$$

where $P(x)$ and $Q(x)$ are functions of x, and in some cases may be constants.

A linear first order o.d.e. can be solved using the **integrating factor method**.

After writing the equation in standard form, $P(x)$ can be identified. One then multiplies the equation by the following "integrating factor":

$$\mathrm{IF}= e^{\int P(x)dx}$$

This factor is defined so that the equation becomes equivalent to:

$$\tfrac{d}{dx}(\mathrm{IF}\, y) = \mathrm{IF}\, Q(x),$$

whereby integrating both sides with respect to x, gives:

$$\mathrm{IF}\, y = \int \mathrm{IF}\, Q(x)\, dx$$

Finally, division by the integrating factor (IF) gives y explicitly in terms of x, i.e. gives the solution to the equation.

2. Exercises

In each case, derive the general solution. When a boundary condition is also given, derive the particular solution.

Click on Exercise links for full worked solutions (there are 10 exercises in total)

Exercise 1.

$$\frac{dy}{dx} + y = x \;\; ; \; y(0) = 2$$

Exercise 2.

$$\frac{dy}{dx} + y = e^{-x} \;\; ; \;\; y(0) = 1$$

Exercise 3.

$$x\,\frac{dy}{dx} + 2y = 10x^2 \;\; ; \;\; y(1) = 3$$

Exercise 4.

$$x\frac{dy}{dx} - y = x^2 \quad ; \quad y(1) = 3$$

Exercise 5.

$$x\,\frac{dy}{dx} - 2y = x^4 \sin x$$

Exercise 6.

$$x\frac{dy}{dx} - 2y = x^2$$

Exercise 7.

$$\frac{dy}{dx} + y \cot x = \operatorname{cosec} x$$

Exercise 8.

$$\frac{dy}{dx} + y \cdot \cot x = \cos x$$

Exercise 9.

$$(x^2 - 1)\frac{dy}{dx} + 2xy = x$$

Exercise 10.

$$\frac{dy}{dx} = y \tan x - \sec x \quad ; \quad y(0) = 1$$

3. Answers

1. General solution is $y = (x-1) + Ce^{-x}$, and particular solution is $y = (x-1) + 3e^{-x}$,

2. General solution is $y = e^{-x}(x + C)$, and particular solution is $y = e^{-x}(x+1)$,

3. General solution is $y = \frac{5}{2}x^2 + \frac{C}{x^2}$, and particular solution is $y = \frac{1}{2}(5x^2 + \frac{1}{x^2})$,

4. General solution is $y = x^2 + Cx$, and particular solution is $y = x^2 + 2x$,

5. General solution is $y = -x^3 \cos x + x^2 \sin x + Cx^2$,

6. General solution is $y = x^2 \ln\, x + C\, x^2$,

7. General solution is $y \sin x = x + C$,

8. General solution is $4y \sin x + \cos 2x = C$,

9. General solution is $(x^2 - 1)y = \frac{x^2}{2} + C$,

10. General solution is $y \cos x = C - x$, and particular solution is $y \cos x = 1 - x$.

4. Standard integrals

$f(x)$	$\int f(x)dx$	$f(x)$	$\int f(x)dx$
x^n	$\frac{x^{n+1}}{n+1} \quad (n \neq -1)$	$[g(x)]^n g'(x)$	$\frac{[g(x)]^{n+1}}{n+1} \quad (n \neq -1)$
$\frac{1}{x}$	$\ln\lvert x\rvert$	$\frac{g'(x)}{g(x)}$	$\ln\lvert g(x)\rvert$
e^x	e^x	a^x	$\frac{a^x}{\ln a} \quad (a > 0)$
$\sin x$	$-\cos x$	$\sinh x$	$\cosh x$
$\cos x$	$\sin x$	$\cosh x$	$\sinh x$
$\tan x$	$-\ln\lvert\cos x\rvert$	$\tanh x$	$\ln\cosh x$
$\operatorname{cosec} x$	$\ln\left\lvert\tan\frac{x}{2}\right\rvert$	$\operatorname{cosech} x$	$\ln\left\lvert\tanh\frac{x}{2}\right\rvert$
$\sec x$	$\ln\lvert\sec x + \tan x\rvert$	$\operatorname{sech} x$	$2\tan^{-1} e^x$
$\sec^2 x$	$\tan x$	$\operatorname{sech}^2 x$	$\tanh x$
$\cot x$	$\ln\lvert\sin x\rvert$	$\coth x$	$\ln\lvert\sinh x\rvert$
$\sin^2 x$	$\frac{x}{2} - \frac{\sin 2x}{4}$	$\sinh^2 x$	$\frac{\sinh 2x}{4} - \frac{x}{2}$
$\cos^2 x$	$\frac{x}{2} + \frac{\sin 2x}{4}$	$\cosh^2 x$	$\frac{\sinh 2x}{4} + \frac{x}{2}$

$f(x)$	$\int f(x)\,dx$	$f(x)$	$\int f(x)\,dx$
$\frac{1}{a^2+x^2}$	$\frac{1}{a}\tan^{-1}\frac{x}{a}$	$\frac{1}{a^2-x^2}$	$\frac{1}{2a}\ln\left\|\frac{a+x}{a-x}\right\|$ $(0<\|x\|<a)$
	$(a>0)$	$\frac{1}{x^2-a^2}$	$\frac{1}{2a}\ln\left\|\frac{x-a}{x+a}\right\|$ $(\|x\|>a>0)$
$\frac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1}\frac{x}{a}$	$\frac{1}{\sqrt{a^2+x^2}}$	$\ln\left\|\frac{x+\sqrt{a^2+x^2}}{a}\right\|$ $(a>0)$
	$(-a<x<a)$	$\frac{1}{\sqrt{x^2-a^2}}$	$\ln\left\|\frac{x+\sqrt{x^2-a^2}}{a}\right\|$ $(x>a>0)$
$\sqrt{a^2-x^2}$	$\frac{a^2}{2}\left[\sin^{-1}\left(\frac{x}{a}\right)\right.$	$\sqrt{a^2+x^2}$	$\frac{a^2}{2}\left[\sinh^{-1}\left(\frac{x}{a}\right)+\frac{x\sqrt{a^2+x^2}}{a^2}\right]$
	$\left.+\frac{x\sqrt{a^2-x^2}}{a^2}\right]$	$\sqrt{x^2-a^2}$	$\frac{a^2}{2}\left[-\cosh^{-1}\left(\frac{x}{a}\right)+\frac{x\sqrt{x^2-a^2}}{a^2}\right]$

5. Tips on using solutions

- When looking at the THEORY, ANSWERS, INTEGRALS, TIPS or NOTATION pages, use the Back button (at the bottom of the page) to return to the exercises.

- Use the solutions intelligently. For example, they can help you get started on an exercise, or they can allow you to check whether your intermediate results are correct.

- Try to make less use of the full solutions as you work your way through the Tutorial.

6. Alternative notation

The linear first order differential equation:

$$\frac{dy}{dx} + P(x)\,y = Q(x)$$

has the integrating factor IF=$e^{\int P(x)\,dx}$.

The integrating factor method is sometimes explained in terms of simpler forms of differential equation. For example, when constant coefficients a and b are involved, the equation may be written as:

$$a\,\frac{dy}{dx} + b\,y = Q(x)$$

In our standard form this is:

$$\frac{dy}{dx} + \frac{b}{a}\,y = \frac{Q(x)}{a}$$

with an integrating factor of:

$$IF = e^{\int \frac{b}{a}\,dx} = e^{\frac{bx}{a}}$$

Full worked solutions

Exercise 1.

Compare with form: $\frac{dy}{dx} + P(x)y = Q(x)$ (P, Q are functions of x)

Integrating factor: $P(x) = 1$.

$$\begin{aligned} \text{Integrating factor,} \quad \text{IF} &= e^{\int P(x)dx} \\ &= e^{\int dx} \\ &= e^{x} \end{aligned}$$

Multiply equation by IF:

$$\begin{aligned} e^x \frac{dy}{dx} + e^x y &= e^x x \\ \text{i.e.} \quad \frac{d}{dx}\left[e^x y\right] &= e^x x \end{aligned}$$

Integrate both sides with respect to x:

$$e^x y = e^x(x-1) + C$$

$$\begin{aligned}
\{\ \underline{\text{Note}}:\ \int u\,\frac{dv}{dx}dx &= uv - \int v\,\frac{du}{dx}\,dx \quad \text{i.e. integration by parts with} \\
u &\equiv x, \quad \frac{dv}{dx} \equiv e^x \\
&\to xe^x - \int e^x dx \\
&\to xe^x - e^x = e^x(x-1)\ \} \\
\text{i.e.} \quad y &= (x-1) + Ce^{-x}\ .
\end{aligned}$$

Particular solution with $y(0) = 2$:

$$\begin{aligned}
2 &= (0-1) + Ce^0 \\
&= -1 + C \quad \text{i.e. } C = 3 \text{ and } y = (x-1) + 3e^{-x}\ .
\end{aligned}$$

Return to Exercise 1

Exercise 2.

Integrating Factor: $P(x) = 1$, $\mathrm{IF} = e^{\int P dx} = e^{\int dx} = e^x$

Multiply equation:

$$\begin{aligned} e^x \frac{dy}{dx} + e^x y &= e^x e^{-x} \\ \text{i.e.} \quad \frac{d}{dx}\left[e^x y\right] &= 1 \end{aligned}$$

Integrate:

$$\begin{aligned} e^x y &= x + C \\ \text{i.e.} \quad y &= e^{-x}(x + C) \quad . \end{aligned}$$

Particular solution:

$$\begin{aligned} \begin{matrix} y = 1 \\ x = 0 \end{matrix} \quad \text{gives} \quad 1 &= e^0(0 + C) \\ &= 1.C \qquad \text{i.e. } C = 1 \\ \text{and} \quad y &= e^{-x}(x + 1) \quad . \end{aligned}$$

Return to Exercise 2

Exercise 3.

Equation is linear, 1st order i.e. $\frac{dy}{dx} + P(x)y = Q(x)$

i.e. $\frac{dy}{dx} + \frac{2}{x}y = 10x,$ where $P(x) = \frac{2}{x},\ Q(x) = 10x$

Integrating factor : IF $= e^{\int P(x)dx} = e^{2\int \frac{dx}{x}} = e^{2\ln x} = e^{\ln x^2} = x^2$.

$$
\begin{aligned}
\text{Multiply equation:} \quad x^2\frac{dy}{dx} + 2xy &= 10x^3 \\
\text{i.e.} \quad \frac{d}{dx}\left[x^2 \cdot y\right] &= 10x^3 \\
\text{Integrate:} \quad x^2y &= \frac{5}{2}x^4 + C \\
\text{i.e.} \quad y &= \frac{5}{2}x^2 + \frac{C}{x^2}
\end{aligned}
$$

<u>Particular solution</u> $\quad y(1) = 3 \quad$ i.e. $\; y(x) = 3 \quad$ when $x = 1$.

i.e. $\quad 3 \; = \frac{5}{2} \cdot 1 + \frac{C}{1}$

i.e. $\quad \frac{6}{2} \; = \frac{5}{2} + C$

i.e. $\quad C = \frac{1}{2}$

$\therefore \quad y \; = \frac{5}{2}x^2 + \frac{1}{2x^2} = \frac{1}{2}\left(5x^2 + \frac{1}{x^2}\right)$.

Return to Exercise 3

Exercise 4.

Standard form: $$\frac{dy}{dx} - \left(\frac{1}{x}\right) y = x$$

Compare with $\frac{dy}{dx} + P(x)y = Q(x)$, giving $P(x) = -\frac{1}{x}$

Integrating Factor: $\text{IF} = e^{\int P(x)dx} = e^{-\int \frac{dx}{x}} = e^{-\ln x} = e^{\ln(x^{-1})} = \frac{1}{x}$.

Multiply equation:

$$\begin{aligned} \frac{1}{x}\frac{dy}{dx} - \frac{1}{x^2}\, y &= 1 \\ \text{i.e.} \quad \frac{d}{dx}\left[\frac{1}{x}\, y\right] &= 1 \end{aligned}$$

Integrate:

$$\begin{aligned} \frac{1}{x} y &= x + C \\ \text{i.e.} \quad y &= x^2 + Cx \ . \end{aligned}$$

Particular solution with $y(1) = 3$:

$$\begin{aligned} 3 &= 1 + C \\ \text{i.e.} \quad C &= 2 \end{aligned}$$

$$\text{Particular solution is} \quad y = x^2 + 2x \ .$$

Return to Exercise 4

Exercise 5.

Linear in y : $\frac{dy}{dx} - \frac{2}{x}y = x^3 \sin x$

Integrating factor: IF $= e^{-2\int \frac{dx}{x}} = e^{-2\ln x} = e^{\ln x^{-2}} = \frac{1}{x^2}$

$$\begin{aligned}
\text{Multiply equation:} \quad \frac{1}{x^2}\frac{dy}{dx} - \frac{2}{x^3}y &= x \sin x \\
\text{i.e.} \quad \frac{d}{dx}\left[\frac{1}{x^2}y\right] &= x \sin x \\
\text{Integrate:} \quad \frac{y}{x^2} &= -x\cos x - \int 1\cdot(-\cos x)dx + C'
\end{aligned}$$

[Note: integration by parts,

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx,\ u = x,\ \frac{dv}{dx} = \sin x]$$

$$\begin{aligned}
\text{i.e.} \quad \frac{y}{x^2} &= -x\cos x + \sin x + C \\
\text{i.e.} \quad y &= -x^3\cos x + x^2\sin x + Cx^2.
\end{aligned}$$

Return to Exercise 5

Exercise 6.

Standard form: $\qquad \dfrac{dy}{dx} - \left(\dfrac{2}{x}\right) y = x$

Integrating Factor: $\qquad P(x) = -\dfrac{2}{x}$

$$\mathrm{IF} = e^{\int P dx} = e^{-2\int \frac{dx}{x}} = e^{-2\ln x} = e^{\ln x^{-2}} = \frac{1}{x^2}$$

Multiply equation: $\qquad \dfrac{1}{x^2}\dfrac{dy}{dx} - \dfrac{2}{x^3}\, y = \dfrac{1}{x}$

$$\text{i.e.} \qquad \frac{d}{dx}\left[\frac{1}{x^2}\, y\right] = \frac{1}{x}$$

Integrate:
$$\frac{1}{x^2}\,y = \int \frac{dx}{x}$$

i.e.
$$\frac{1}{x^2}y = \ln\, x + C$$

i.e.
$$y = x^2\,\ln\, x + Cx^2 \quad .$$

Return to Exercise 6

Exercise 7.

Of the form: $\dfrac{dy}{dx} + P(x)y = Q(x)$ (i.e. linear, 1st order o.d.e.)

where $\quad P(x) = \cot x.$

Integrating factor :

$$\begin{aligned} \text{IF} &= e^{\int P(x)dx} = e^{\int \frac{\cos x}{\sin x} dx} \quad \left\{ \equiv e^{\int \frac{f'(x)}{f(x)} dx} \right\} \\ &= e^{\ln(\sin x)} = \sin x \end{aligned}$$

Multiply equation :

$$\sin x \cdot \tfrac{dy}{dx} + \sin x \left(\tfrac{\cos x}{\sin x}\right) y = \tfrac{\sin x}{\sin x}$$

i.e. $\quad \sin x \cdot \tfrac{dy}{dx} + \cos x \cdot y = 1$

i.e. $\quad \tfrac{d}{dx}\left[\sin x \cdot y\right] = 1$

Integrate: $\quad (\sin x)y = x + C\,.$

Return to Exercise 7

Exercise 8.

Integrating factor: $\quad P(x) = \cot x = \dfrac{\cos x}{\sin x}$

$$\mathrm{IF} = e^{\int P dx} = e^{\int \frac{\cos x}{\sin x} dx} = e^{\ln(\sin x)} = \sin x$$

$\left\{ \text{Note:} \quad \dfrac{\cos x}{\sin x} \equiv \dfrac{f'(x)}{f(x)} \right\}$

Multiply equation:

$$\begin{aligned} \sin x \cdot \frac{dy}{dx} + \sin x \cdot y \cdot \frac{\cos x}{\sin x} &= \sin x \cdot \cos x \\ \text{i.e.} \quad \frac{d}{dx}\left[\sin x \cdot y\right] &= \sin x \cdot \cos x \end{aligned}$$

Integrate: $\quad y \sin x = \int \sin x \cdot \cos x \, dx$

$$\left\{ \underline{\text{Note}}: \int \sin x \cos x \, dx \equiv \int f(x) f'(x) dx \equiv \int f \, \frac{df}{dx} \cdot dx \right.$$
$$\left. \equiv \int f df = \frac{1}{2} f^2 + C \right\}$$

$$\begin{aligned} \text{i.e.} \quad y \sin x &= \tfrac{1}{2} \sin^2 x + C \\ &= \tfrac{1}{2} \cdot \tfrac{1}{2}(1 - \cos 2x) + C \end{aligned}$$

$$\text{i.e.} \quad 4y \sin x + \cos 2x = C'$$

$$\left(\begin{aligned} \text{where } C' &= 4C + 1 \\ &= \text{constant} \end{aligned} \right) \quad .$$

Return to Exercise 8

Exercise 9.

Standard form: $$\frac{dy}{dx} + \left(\frac{2x}{x^2-1}\right) y = \frac{x}{x^2-1}$$

Integrating factor: $$P(x) = \frac{2x}{x^2-1}$$

$$\begin{aligned} \text{IF} = e^{\int P dx} = e^{\int \frac{2x}{x^2-1} dx} &= e^{\ln(x^2-1)} \\ &= x^2 - 1 \end{aligned}$$

Multiply equation: $$(x^2-1)\frac{dy}{dx} + 2x\, y = x$$

(the original form of the equation was half-way there!)

$$\text{i.e.} \quad \frac{d}{dx}\left[(x^2-1)y\right] = x$$

Integrate: $$(x^2-1)y = \frac{1}{2}x^2 + C\,.$$

Return to Exercise 9

Exercise 10.

$$P(x) = -\tan x \qquad Q(x) = -\sec x$$

$$\text{IF} = e^{-\int \tan x \, dx} = e^{-\int \frac{\sin x}{\cos x} dx} = e^{+\int \frac{-\sin x}{\cos x} dx} = e^{\ln(\cos x)} = \cos x$$

Multiply by IF: $\cos x \frac{dy}{dx} - \cos x \cdot \frac{\sin x}{\cos x} y = -\cos x \cdot \sec x$

i.e. $\frac{d}{dx}[\cos x \cdot y] = -1$ i.e. $y \cos x = -x + C$.

$y(0) = 1$ i.e. $y = 1$ when $x = 0$ gives

$\cos(0) = 0 + C \;\therefore\; C = 1$

i.e. $y \cos x = -x + 1$.

Return to Exercise 10