Differential Equations

EXACT EQUATIONS

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A Tutorial Module for learning the technique of solving exact differential equations

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1. Theory

We consider here the following standard form of ordinary differential equation (o.d.e.):

$$\boxed{P(x,y)dx + Q(x,y)dy = 0}$$

If $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ then the o.de. is said to be **exact**.

This means that a function $u(x,y)$ exists such that:

$$\begin{aligned} du &= \frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy \\ &= P\,dx + Q\,dy = 0 \ . \end{aligned}$$

One solves $\frac{\partial u}{\partial x} = P$ and $\frac{\partial u}{\partial y} = Q$ to find $u(x,y)$.

Then $du = 0$ gives $u(x,y) = C$, where C is a constant.
This last equation gives the general solution of $P\,dx + Q\,dy = 0$.

2. Exercises

Click on Exercise links for full worked solutions (there are 11 exercises in total)

Show that each of the following differential equations is exact and use that property to find the general solution:

Exercise 1.

$$\frac{1}{x}dy - \frac{y}{x^2}\,dx = 0$$

Exercise 2.

$$2xy\frac{dy}{dx} + y^2 - 2x = 0$$

Exercise 3.

$$2(y+1)e^x dx + 2(e^x - 2y)dy = 0$$

Exercise 4.

$(2xy + 6x)dx + (x^2 + 4y^3)dy = 0$

Exercise 5.

$(8y - x^2y)\dfrac{dy}{dx} + x - xy^2 = 0$

Exercise 6.

$(e^{4x} + 2xy^2)dx + (\cos y + 2x^2y)dy = 0$

Exercise 7.

$(3x^2 + y\cos x)dx + (\sin x - 4y^3)dy = 0$

Exercise 8.

$$x \tan^{-1} y \cdot dx + \frac{x^2}{2(1+y^2)} \cdot dy = 0$$

Exercise 9.

$$(2x + x^2y^3)dx + (x^3y^2 + 4y^3)dy = 0$$

Exercise 10.

$$(2x^3 - 3x^2y + y^3)\,\frac{dy}{dx} = 2x^3 - 6x^2y + 3xy^2$$

Exercise 11.

$$(y^2 \cos x - \sin x)dx + (2y \sin x + 2)dy = 0$$

3. Answers

1. $y = Ax$,
2. $y^2 x - x^2 = A$,
3. $(y+1)e^x - y^2 = A$,
4. $x^2 y + 3x^2 + y^4 = A$,
5. $\frac{1}{2}x^2(1-y^2) + 4y^2 = A$,
6. $\frac{1}{4}e^{4x} + x^2 y^2 + \sin y = A$,
7. $x^3 + y \sin x - y^4 = A$,
8. $\frac{x^2}{2} \tan^{-1} y = A$,
9. $x^2 + \frac{x^3 y^3}{3} + y^4 = A$,

10. $\frac{x^4}{2} - 2x^3y + \frac{3}{2}\,x^2y^2 - \frac{y^4}{4} = A\,,$

11. $y^2 \sin x + \cos x + 2y = A.$

4. Standard integrals

$f(x)$	$\int f(x)dx$	$f(x)$	$\int f(x)dx$
x^n	$\frac{x^{n+1}}{n+1} \quad (n \neq -1)$	$[g(x)]^n g'(x)$	$\frac{[g(x)]^{n+1}}{n+1} \quad (n \neq -1)$
$\frac{1}{x}$	$\ln \lvert x \rvert$	$\frac{g'(x)}{g(x)}$	$\ln \lvert g(x) \rvert$
e^x	e^x	a^x	$\frac{a^x}{\ln a} \quad (a > 0)$
$\sin x$	$-\cos x$	$\sinh x$	$\cosh x$
$\cos x$	$\sin x$	$\cosh x$	$\sinh x$
$\tan x$	$-\ln \lvert \cos x \rvert$	$\tanh x$	$\ln \cosh x$
$\operatorname{cosec} x$	$\ln \left\lvert \tan \frac{x}{2} \right\rvert$	$\operatorname{cosech} x$	$\ln \left\lvert \tanh \frac{x}{2} \right\rvert$
$\sec x$	$\ln \lvert \sec x + \tan x \rvert$	$\operatorname{sech} x$	$2 \tan^{-1} e^x$
$\sec^2 x$	$\tan x$	$\operatorname{sech}^2 x$	$\tanh x$
$\cot x$	$\ln \lvert \sin x \rvert$	$\coth x$	$\ln \lvert \sinh x \rvert$
$\sin^2 x$	$\frac{x}{2} - \frac{\sin 2x}{4}$	$\sinh^2 x$	$\frac{\sinh 2x}{4} - \frac{x}{2}$
$\cos^2 x$	$\frac{x}{2} + \frac{\sin 2x}{4}$	$\cosh^2 x$	$\frac{\sinh 2x}{4} + \frac{x}{2}$

$f(x)$	$\int f(x)\,dx$	$f(x)$	$\int f(x)\,dx$
$\frac{1}{a^2+x^2}$	$\frac{1}{a}\tan^{-1}\frac{x}{a}$	$\frac{1}{a^2-x^2}$	$\frac{1}{2a}\ln\left\lvert\frac{a+x}{a-x}\right\rvert$ $(0<\lvert x\rvert<a)$
	$(a>0)$	$\frac{1}{x^2-a^2}$	$\frac{1}{2a}\ln\left\lvert\frac{x-a}{x+a}\right\rvert$ $(\lvert x\rvert>a>0)$
$\frac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1}\frac{x}{a}$	$\frac{1}{\sqrt{a^2+x^2}}$	$\ln\left\lvert\frac{x+\sqrt{a^2+x^2}}{a}\right\rvert$ $(a>0)$
	$(-a<x<a)$	$\frac{1}{\sqrt{x^2-a^2}}$	$\ln\left\lvert\frac{x+\sqrt{x^2-a^2}}{a}\right\rvert$ $(x>a>0)$
$\sqrt{a^2-x^2}$	$\frac{a^2}{2}\left[\sin^{-1}\left(\frac{x}{a}\right)+\frac{x\sqrt{a^2-x^2}}{a^2}\right]$	$\sqrt{a^2+x^2}$	$\frac{a^2}{2}\left[\sinh^{-1}\left(\frac{x}{a}\right)+\frac{x\sqrt{a^2+x^2}}{a^2}\right]$
		$\sqrt{x^2-a^2}$	$\frac{a^2}{2}\left[-\cosh^{-1}\left(\frac{x}{a}\right)+\frac{x\sqrt{x^2-a^2}}{a^2}\right]$

5. Tips on using solutions

- When looking at the THEORY, ANSWERS, INTEGRALS or TIPS pages, use the Back button (at the bottom of the page) to return to the exercises.

- Use the solutions intelligently. For example, they can help you get started on an exercise, or they can allow you to check whether your intermediate results are correct.

- Try to make less use of the full solutions as you work your way through the Tutorial.

Full worked solutions

Exercise 1.

Standard form: $P(x,y)dx + Q(x,y)dy = 0$

$$\text{i.e. } P(x,y) = -\tfrac{y}{x^2} \quad \text{and} \quad Q(x,y) = \tfrac{1}{x}$$

Equation is exact if $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$

Check: $\frac{\partial P}{\partial y} = -\frac{1}{x^2} = \frac{\partial Q}{\partial x}$ $\therefore$ o.d.e. is exact.

Since equation exact, $u(x,y)$ exists such that

$$\begin{aligned} du &= \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy \\ &= P\,dx + Q\,dy = 0 \end{aligned}$$

and equation has solution $u = C, \ C =$ constant.

$\frac{\partial u}{\partial x} = P$ gives i) $\frac{\partial u}{\partial x} = -\frac{y}{x^2}$

$\frac{\partial u}{\partial y} = Q$ gives ii) $\frac{\partial u}{\partial y} = \frac{1}{x}$

Integrate i) partially with respect to x,

$$\boxed{u = \frac{y}{x} + \phi(y),}$$

where $\phi(y)$ is an arbitrary function of y.

Differentiate with respect to y,

$$\frac{\partial u}{\partial y} = \frac{1}{x} + \frac{\partial \phi}{\partial y} = \frac{1}{x} + \frac{d\phi}{dy}$$

(since $\phi = \phi(y)$ only)

Compare with equation ii)

$$\begin{aligned}
\frac{1}{x}+\frac{d\phi}{dy} &= \frac{1}{x} \\
\text{i.e.} \qquad \frac{d\phi}{dy} &= 0 \\
\text{i.e.} \qquad \phi &= C', \quad C' = \text{constant} \\
\text{and} \qquad u &= \frac{y}{x}+C'.
\end{aligned}$$

$\underline{du = 0 \quad \text{implies} \quad u = C}, \quad C = \text{ constant}$

$$\therefore \ \tfrac{y}{x} = A \quad , \quad A = C - C'$$

$$= \text{constant.}$$

Return to Exercise 1

Exercise 2.

Standard form: $(y^2 - 2x)dx + 2xy\,dy = 0$

Exact if $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$, where $P(x,y) = y^2 - 2x$
$Q(x,y) = 2xy$

$$\frac{\partial P}{\partial y} = 2y = \frac{\partial Q}{\partial x} \qquad \text{i.e. o.d.e. is exact.}$$

$\therefore$ $u(x,y)$ exists such that $du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$

$$= P\,dx + Q\,dy = 0,$$

giving i) $\frac{\partial u}{\partial x} = y^2 - 2x$, ii) $\frac{\partial u}{\partial y} = 2xy$.

Integrate i): $\boxed{u = xy^2 - x^2 + \phi(y)}$, ϕ is arbitrary function.

Differentiate and compare with ii):

$$\frac{\partial u}{\partial y} = 2xy + \frac{d\phi}{dy} = 2xy$$

$$\therefore \qquad \frac{d\phi}{dy} = 0 \quad \text{and} \quad \phi = C' \qquad \text{(constant)}$$

$$\therefore \qquad u = xy^2 - x^2 + C'$$

$\underline{du = 0 \text{ implies } u = C}$, $\quad \therefore\ xy^2 - x^2 = A$, where $A = C - C'$.

Return to Exercise 2

Exercise 3.

$$P(x,y)\,dx + Q(x,y)\,dy = 0 \quad \text{where} \quad \begin{aligned} P(x,y) &= 2(y+1)e^x \\ Q(x,y) &= 2(e^x - 2y) \end{aligned}$$

$\frac{\partial P}{\partial y} = 2e^x = \frac{\partial Q}{\partial x}$, $\quad\therefore$ o.d.e. is exact.

$\therefore$ $\underline{u(x,y) \text{ exists such that}}$

$$\begin{aligned} du &= \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy \\ &= P\,dx + Q\,dy = 0, \end{aligned}$$

Giving i) $\frac{\partial u}{\partial x} = 2(y+1)e^x$, ii) $\frac{\partial u}{\partial y} = 2(e^x - 2y)$.

Integrate i): $\boxed{u = 2(y+1)e^x + \phi(y)}$

Differentiate: $\frac{\partial u}{\partial y} = 2e^x + \frac{d\phi}{dy} = 2(e^x - 2y)$, using ii) i.e. $\frac{d\phi}{dy} = -4y$ i.e. $\int d\phi = -4 \int y\, dy$ i.e. $\phi = -2y^2 + C'$

$\therefore\ u = 2(y+1)e^x - 2y^2 + C'$

$\underline{du = 0 \quad \text{gives} \quad u = C}$,

$\therefore\ (y+1)e^x - y^2 = A$, where $A = (C - C')/2$.

Return to Exercise 3

Exercise 4.

$$P(x,y)dx + Q(x,y)dy = 0 \quad \text{where} \quad \begin{array}{l} P(x,y) = 2xy + 6x\,, \\ Q(x,y) = x^2 + 4y^3 \end{array}$$

$\frac{\partial P}{\partial y} = 2x = \frac{\partial Q}{\partial x}$, $\quad \therefore$ o.d.e. is exact.

$\therefore$ $\underline{u(x,y) \text{ exists such that}}$

$$\begin{aligned} du &= \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy \\ &= P\,dx + Q\,dy = 0, \end{aligned}$$

Giving i) $\frac{\partial u}{\partial x} = 2xy + 6x$, ii) $\frac{\partial u}{\partial y} = x^2 + 4y^3$.

Integrate i): $\boxed{u = x^2y + 3x^2 + \phi(y)}$

Differentiate: $\frac{\partial u}{\partial y} = x^2 + \frac{d\phi}{dy} = x^2 + 4y^3$, using ii)

i.e. $\frac{d\phi}{dy} = 4y^3$ i.e. $\int d\phi = 4\int y^3\,dy$ i.e. $\phi = y^4 + C'$

$\therefore\ u = x^2y + 3x^2 + y^4 + C'$

$\underline{du = 0 \quad \text{gives} \quad u = C}$,

$\therefore\ x^2y + 3x^2 + y^4 = A$, where $A = C - C'$.

Return to Exercise 4

Exercise 5.

$(x - xy^2)dx + (8y - x^2y)dy = 0$

$P(x, y) = x - xy^2$

$Q(x, y) = 8y - x^2y\,. \qquad \frac{\partial P}{\partial y} = -2xy = \frac{\partial Q}{\partial x}, \qquad \therefore$ o.d.e. is exact.

$\therefore$ $\underline{u(x, y) \text{ exists where}}$ $\quad du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$

$$= P\,dx + Q\,dy = 0$$

Giving $\quad$ i) $\frac{\partial u}{\partial x} = x - xy^2; \qquad$ ii) $\frac{\partial u}{\partial y} = 8y - x^2y.$

Integrate i): $\boxed{u = \frac{1}{2}x^2(1 - y^2) + \phi(y)}$

Differentiate: $\frac{\partial u}{\partial y} = -\frac{1}{2}x^2 \cdot 2y + \frac{d\phi}{dy} = 8y - x^2y$, using ii)

$\therefore \quad \frac{d\phi}{dy} = 8y \qquad$ i.e. $\quad \int d\phi = 8\int y dy$

i.e. $\phi(y) = 4y^2 + C'$ and $u = \frac{1}{2}x^2(1-y^2) + 4y^2 + C'$

$\underline{du = 0 \quad \text{gives} \quad u = C}$, $\qquad \therefore \ \frac{1}{2}x^2(1-y^2) + 4y^2 = A \ , \ A = C - C' \ .$

Return to Exercise 5

Exercise 6.

$P(x,y)dx + Q(x,y)dy = 0$ where $P(x,y) = e^{4x} + 2xy^2$, $Q(x,y) = \cos y + 2x^2 y$

$\frac{\partial P}{\partial y} = 4xy = \frac{\partial Q}{\partial x}$, $\therefore$ o.d.e. is exact.

$\therefore$ $\underline{u(x,y) \text{ exists such that}}$

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = P\,dx + Q\,dy = 0,$$

Giving i) $\frac{\partial u}{\partial x} = e^{4x} + 2xy^2$, ii) $\frac{\partial u}{\partial y} = \cos y + 2x^2 y$.

Integrate i): $\boxed{u = \frac{1}{4}e^{4x} + x^2y^2 + \phi(y)}$

Differentiate: $\frac{\partial u}{\partial y} = 2x^2 y + \frac{d\phi}{dy} = \cos y + 2x^2 y$, using ii)

i.e. $\frac{d\phi}{dy} = \cos y$ i.e. $\int d\phi = \int \cos y\, dy$ i.e. $\phi = \sin y + C'$

$\therefore\ u = \frac{1}{4}e^{4x} + x^2y^2 + \sin y + C'$

$\underline{du = 0 \quad \text{gives} \quad u = C}$,

$\therefore\ \frac{1}{4}e^{4x} + x^2y^2 + \sin y = A$, where $A = C - C'$.

Return to Exercise 6

Exercise 7.

$$P(x,y)\,dx + Q(x,y)\,dy = 0 \quad \text{where} \quad \begin{array}{l} P(x,y) = 3x^2 + y\cos x \\ Q(x,y) = \sin x - 4y^3 \end{array}$$

$\frac{\partial P}{\partial y} = \cos x = \frac{\partial Q}{\partial x}$, $\quad \therefore$ o.d.e. is exact.

$\therefore$ $\underline{u(x,y) \text{ exists such that}}$

$$\begin{aligned} du &= \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy \\ &= P\,dx + Q\,dy = 0 \end{aligned}$$

Giving i) $\frac{\partial u}{\partial x} = 3x^2 + y\cos x$, ii) $\frac{\partial u}{\partial y} = \sin x - 4y^3$.

Integrate i): $\boxed{u = x^3 + y\sin x + \phi(y)}$

Differentiate: $\frac{\partial u}{\partial y} = \sin x + \frac{d\phi}{dy} = \sin x - 4y^3$, using ii)

$\therefore \quad \frac{d\phi}{dy} = -4y^3 \qquad$ i.e. $\quad \int d\phi = -4\int y^3 dy$

i.e. $\phi = -y^4 + C' \quad$ and $\quad u = x^3 + y\sin x - y^4 + C'$

$\underline{du = 0 \quad \text{gives} \quad u = C}, \qquad \therefore\ x^3 + y\sin x - y^4 = A\ ,\ A = C - C'\ .$

Return to Exercise 7

Exercise 8.

$$P(x,y)\,dx + Q(x,y)\,dy = 0 \quad \text{where} \quad P(x,y) = x\tan^{-1} y, \quad Q(x,y) = \frac{x^2}{2(1+y^2)}$$

$$\frac{\partial P}{\partial y} = \frac{x}{1+y^2} = \frac{\partial Q}{\partial x}, \qquad \therefore \text{ o.d.e. is exact.}$$

$\therefore$ $\underline{u(x,y) \text{ exists such that}}$

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = P\,dx + Q\,dy = 0,$$

Giving i) $\frac{\partial u}{\partial x} = x\tan^{-1} y$, ii) $\frac{\partial u}{\partial y} = \frac{x^2}{2(1+y^2)}$.

Integrate i): $\boxed{u = \frac{x^2}{2}\tan^{-1} y + \phi(y)}$

Differentiate: $\frac{\partial u}{\partial y} = \frac{x^2}{2}\frac{1}{(1+y^2)} + \frac{d\phi}{dy} = \frac{x^2}{2(1+y^2)}$, using ii)

$\therefore \quad \frac{d\phi}{dy} = 0 \qquad$ i.e. $\quad \phi(y) = C'$

and $u = \frac{x^2}{2}\tan^{-1} y + C'$

$\underline{du = 0 \quad \text{implies} \quad u = C\,,\ C = \text{constant}}$

$$\therefore \ \frac{x^2}{2}\tan^{-1} y = A, \quad A = C - C'\,.$$

Return to Exercise 8

Exercise 9.

$$P(x,y)dx + Q(x,y)dy = 0 \quad \text{where} \quad \begin{array}{l} P(x,y) = 2x + x^2y^3\,, \\ Q(x,y) = x^3y^2 + 4y^3 \end{array}$$

$$\frac{\partial P}{\partial y} = 3x^2y^2 = \frac{\partial Q}{\partial x}\,, \qquad \therefore \text{ o.d.e. is exact.}$$

$\therefore$ <u>$u(x,y)$ exists such that</u>

$$\begin{aligned} du &= \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy \\ &= P\,dx + Q\,dy = 0, \end{aligned}$$

Giving i) $\frac{\partial u}{\partial x} = 2x + x^2y^3$, ii) $\frac{\partial u}{\partial y} = x^3y^2 + 4y^3$.

<u>Integrate i)</u>: $\boxed{u = x^2 + \frac{x^3y^3}{3} + \phi(y)}$

<u>Differentiate</u>: $\frac{\partial u}{\partial y} = x^3y^2 + \frac{d\phi}{dy} = x^3y^2 + 4y^3$, using ii)

i.e. $\frac{d\phi}{dy} = 4y^3$ i.e. $\int d\phi = 4\int y^3\,dy$ i.e. $\phi = y^4 + C'$

$\therefore\ u = x^2 + \frac{x^3y^3}{3} + y^4 + C'$

$\underline{du = 0 \quad \text{gives} \quad u = C}$,

$\therefore\ x^2 + \frac{x^3y^3}{3} + y^4 = A$, where $A = C - C'$.

Return to Exercise 9

Exercise 10.

$$(2x^3 - 6x^2y + 3xy^2)dx + (-2x^3 + 3x^2y - y^3)dy = 0$$

$$\frac{\partial P}{\partial y} = -6x^2 + 6xy = \frac{\partial Q}{\partial x}, \quad \therefore \text{ o.d.e. is exact.}$$

$\therefore$ $\underline{u(x,y) \text{ exists such that}}$

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = P\,dx + Q\,dy = 0$$

Giving i) $\frac{\partial u}{\partial x} = 2x^3 - 6x^2y + 3xy^2$, ii) $\frac{\partial u}{\partial y} = -2x^3 + 3x^2y - y^3$.

Integrate i): $\boxed{u = \frac{x^4}{2} - 2x^3y + \frac{3}{2}x^2y^2 + \phi(y)}$

Differentiate: $\frac{\partial u}{\partial y} = -2x^3 + 3x^2y + \frac{d\phi}{dy} = -2x^3 + 3x^2y - y^3$, using ii)

$\therefore \quad \frac{d\phi}{dy} = -y^3 \qquad$ i.e. $\quad \int d\phi = -\int y^3 dy$

i.e. $\phi(y) = -\frac{1}{4}y^4 + C'$

and $\quad u(x,y) = \frac{x^4}{2} - 2x^3y + \frac{3}{2}x^2y^2 - \frac{y^4}{4} + C'$

$\underline{du = 0 \quad \text{gives} \quad u = C}$,

$\therefore \ \frac{x^4}{2} - 2x^3y + \frac{3}{2}x^2y^2 - \frac{y^4}{4} = A \ , \ A = C - C' \ .$

Return to Exercise 10

Exercise 11.

$$P(x,y)dx + Q(x,y)dy = 0 \quad \text{where} \quad \begin{aligned} P(x,y) &= y^2\cos x - \sin x\,, \\ Q(x,y) &= 2y\sin x + 2 \end{aligned}$$

$\frac{\partial P}{\partial y} = 2y\cos x = \frac{\partial Q}{\partial x}$, $\therefore$ o.d.e. is exact.

$\therefore$ <u>$u(x,y)$ exists such that</u>

$$\begin{aligned} du &= \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy \\ &= P\,dx + Q\,dy = 0, \end{aligned}$$

Giving i) $\frac{\partial u}{\partial x} = y^2\cos x - \sin x\,,$ ii) $\frac{\partial u}{\partial y} = 2y\sin x + 2\,.$

<u>Integrate i)</u>: $\boxed{u = y^2\sin x + \cos x + \phi(y)}$

<u>Differentiate</u>: $\frac{\partial u}{\partial y} = 2y\sin x + \frac{d\phi}{dy} = 2y\sin x + 2$, using ii)

i.e. $\frac{d\phi}{dy} = 2$ i.e. $\int d\phi = 2\int dy$ i.e. $\phi = 2y + C'$

$\therefore\ u = y^2 \sin x + \cos x + 2y + C'$

$\underline{du = 0 \quad \text{gives} \quad u = C}$,

$\therefore\ y^2 \sin x + \cos x + 2y = A$, where $A = C - C'$.

Return to Exercise 11