Differential Equations

PPLATO PROMOTING PHYSICS LEARNING & TEACHING OPPORTUNITIES

SECOND ORDER (inhomogeneous)

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A Tutorial Module for learning to solve 2nd order (inhomogeneous) differential equations

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Table of contents

1 Theory

This Tutorial deals with the solution of second order linear o.d.e.'s with constant coefficients (a, b and c), i.e. of the form:

$$\boxed{a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x) \quad (*)}$$

The first step is to find the general solution of the homogeneous equation [i.e. as $(*)$, except that $f(x) = 0$]. This gives us the "complementary function" y_{CF} .

The second step is to find a particular solution y_{PS} of the full equation $(*)$. Assume that y_{PS} is a more general form of $f(x)$, having undetermined coefficients, as shown in the following table:

$f(x)$	Form of y_{PS}
k (a constant)	C
linear in x	$Cx + D$
quadratic in x	$Cx^2 + Dx + E$
$k \sin px$ or $k \cos px$	$C \cos px + D \sin px$
ke^{px}	Ce^{px}
sum of the above	sum of the above
product of the above	product of the above

(where p is a constant)

Note: If the suggested form of y_{PS} already appears in the complementary function then multiply this suggested form by x.

Substitution of y_{PS} into $(*)$ yields values for the undetermined coefficients (C, D, etc). Then,

$$\boxed{\text{General solution of } (*) = y_{CF} + y_{PS}} \quad .$$

2 Exercises

Find the general solution of the following equations. Where boundary conditions are also given, derive the appropriate particular solution.

Click on Exercise links for full worked solutions (there are 13 exercises in total)

$$\left[\text{Notation:}\quad y'' = \frac{d^2y}{dx^2}\,,\quad y' = \frac{dy}{dx}\right]$$

Exercise 1. $y'' - 2y' - 3y = 6$

Exercise 2. $y'' + 5y' + 6y = 2x$

Exercise 3. (a) $y'' + 5y' - 9y = x^2$
(b) $y'' + 5y' - 9y = \cos 2x$
(c) $y'' + 5y' - 9y = e^{4x}$
(d) $y'' + 5y' - 9y = e^{-2x} + 2 - x$

Exercise 4. $y'' - \lambda^2 y = \sin 2x$

Exercise 5. $y'' - y = e^x$

Exercise 6. $y'' + y' - 2y = e^{-2x}$

Exercise 7. $y'' - 2y' + y = e^x$

Exercise 8. $y'' + 8y' + 17y = 2e^{-3x}$; $\quad y(0) = 2$ and $\quad y\left(\dfrac{\pi}{2}\right) = 0$

Exercise 9. $y'' + y' - 12y = 4e^{2x}$; $\quad y(0) = 7$ and $\quad y'(0) = 0$

Exercise 10. $y'' + 3y' + 2y = 10\cos 2x$; $\quad y(0) = 1$ and $\quad y'(0) = 0$

Exercise 11. $y'' + 4y' + 5y = 2e^{-2x}$; $\quad y(0) = 1$ and $\quad y'(0) = -2$

Exercise 12. $\dfrac{d^2x}{d\tau^2} + 4\dfrac{dx}{d\tau} + 3x = e^{-3\tau}$; $\quad x = \dfrac{1}{2}$ and $\dfrac{dx}{d\tau} = -2$ at $\tau = 0$

Exercise 13. $\dfrac{d^2y}{d\tau^2} + 4\dfrac{dy}{d\tau} + 5y = 6\sin\tau$

3 Answers

1. $y = Ae^{-x} + Be^{3x} - 2\,,$

2. $y = Ae^{-2x} + Be^{-3x} + \frac{x}{3} - \frac{5}{18}\,,$

3. General solutions are $y = y_{CF} + y_{PS}$
   where $y_{CF} = Ae^{m_1 x} + Be^{m_2 x} \quad \left(m_{1,2} = -\frac{5}{2} \pm \frac{1}{2}\sqrt{61}\right)$

   and (a) $y_{PS} = -\frac{1}{9}x^2 - \frac{10}{81}x - \frac{68}{729}$

   (b) $y_{PS} = -\frac{13}{269}\cos 2x + \frac{10}{269}\sin 2x$

   (c) $y_{PS} = \frac{1}{27}e^{4x}$

   (d) $y_{PS} = -\frac{1}{15}e^{-2x} + \frac{1}{9}x - \frac{13}{81}\,,$

4. $y = Ae^{+\lambda x} + Be^{-\lambda x} - \frac{\sin 2x}{4+\lambda^2}\,,$

5. $y = Ae^{x} + Be^{-x} + \frac{1}{2}xe^{x}\,,$

6. $y = Ae^{x} + Be^{-2x} - \frac{1}{3}xe^{-2x}$,

7. $y = (A + Bx)e^{x} + \frac{1}{2}x^{2}e^{x}$,

8. $y = e^{-4x}(A\cos x + B\sin x) + e^{-3x}$;
$y = e^{-4x}(\cos x - e^{\frac{\pi}{2}}\sin x) + e^{-3x}$,

9. $y = Ae^{3x} + Be^{-4x} - \frac{2}{3}e^{2x}$;
$y = \frac{32}{7}e^{3x} + \frac{65}{21}e^{-4x} - \frac{2}{3}e^{2x}$,

10. $y = Ae^{-2x} + Be^{-x} - \frac{1}{2}\cos 2x + \frac{3}{2}\sin 2x$;
$y = \frac{3}{2}e^{-2x} - \frac{1}{2}\cos 2x + \frac{3}{2}\sin 2x$,

11. $y = e^{-2x}(A\cos x + B\sin x) + 2e^{-2x}$;
$y = e^{-2x}(2 - \cos x)$,

12. $x = Ae^{-3\tau} + Be^{-\tau} - \frac{1}{2}\tau e^{-3\tau}$;
$x = \frac{1}{2}(1 - \tau)e^{-3\tau}$,

13. $y = e^{-2\tau}(A\cos\tau + B\sin\tau) - \frac{3}{4}(\cos\tau - \sin\tau)$.

4 Standard derivatives

$f(x)$	$f'(x)$	$f(x)$	$f'(x)$
x^n	$n\,x^{n-1}$	$[g(x)]^n$	$n[g(x)]^{n-1}g'(x)$
$\ln x$	$\frac{1}{x} \quad (x > 0)$	$\ln g(x)$	$\frac{1}{g(x)}g'(x) \quad (g(x) > 0)$
e^x	e^x	a^x	$a^x \ln a \quad (a > 0)$
$\sin x$	$\cos x$	$\sinh x$	$\cosh x$
$\cos x$	$-\sin x$	$\cosh x$	$\sinh x$
$\tan x$	$\sec^2 x$	$\tanh x$	$\mathrm{sech}^2 x$
$\mathrm{cosec}\, x$	$-\mathrm{cosec}\, x \cot x$	$\mathrm{cosech}\, x$	$-\mathrm{cosech}\, x \coth\, x$
$\sec x$	$\sec x \tan x$	$\mathrm{sech}\, x$	$-\mathrm{sech}\, x \tanh x$
$\cot x$	$-\mathrm{cosec}^2 x$	$\coth x$	$-\mathrm{cosech}^2 x$

$f(x)$	$f'(x)$	$f(x)$	$f'(x)$
$\tan^{-1} x$	$\frac{1}{1+x^2}$	$\tanh^{-1} x$	$\frac{1}{1-x^2}$ $(-1 < x < 1)$
$\cos^{-1} x$	$-\frac{1}{\sqrt{1-x^2}}$ $(-1 < x < 1)$	$\cosh^{-1} x$	$\frac{1}{\sqrt{x^2-1}}$ $(x > 1)$
$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}$ $(-1 < x < 1)$	$\sinh^{-1} x$	$\frac{1}{\sqrt{x^2+1}}$

where

$$\boxed{\begin{array}{lll} \mathrm{cosec}\, x = 1/\sin x\,, & \sec x = 1/\cos x\,, & \cot x = 1/\tan x \\ \mathrm{cosech}\, x = 1/\sinh x\,, & \mathrm{sech}\, x = 1/\cosh x\,, & \coth x = 1/\tanh x \end{array}}$$

5 Finding y_{CF}

One considers the differential equation with RHS $= 0$. Substituting a trial solution of the form $y = Ae^{mx}$ yields an "auxiliary equation":

$$am^2 + bm + c = 0.$$

This will have two roots (m_1 and m_2).

The general solution y_{CF}, when RHS $= 0$, is then constructed from the possible forms (y_1 and y_2) of the trial solution. The auxiliary equation may have:

i) real different roots,
m_1 and $m_2 \rightarrow y_{CF} = y_1 + y_2 = Ae^{m_1 x} + Be^{m_2 x}$

or ii) real equal roots,
$m_1 = m_2 \rightarrow y_{CF} = y_1 + xy_2 = (A + Bx)e^{m_1 x}$

or iii) complex roots,
$p \pm iq \rightarrow y_{CF} = y_1 + y_2 \equiv e^{px}(A \cos qx + B \sin qx)$

6 Tips on using solutions

- When looking at the THEORY, ANSWERS, DERIVATIVES, FINDING y_{CF} or TIPS pages, use the Back button (at the bottom of the page) to return to the exercises

- Use the solutions intelligently. For example, they can help you get started on an exercise, or they can allow you to check whether your intermediate results are correct

- Try to make less use of the full solutions as you work your way through the Tutorial

Full worked solutions

Exercise 1. $\boxed{y'' - 2y' - 3y = 6}$

Auxiliary equation (A.E.) from the homogeneous equation

$$y'' - 2y' - 3y = 0 \ ,$$

is $\quad m^2 - 2m - 3 = 0$

i.e. $\quad (m-3)(m+1) = 0 \quad$ i.e. $\quad m_1 = 3, m_2 = -1.$

Real different roots : homogeneous equation has general solution

$$y = Ae^{m_1 x} + Be^{m_2 x}$$

i.e. $\underline{y_{CF} = Ae^{-x} + Be^{3x}}$ (complementary function).

$$
\begin{array}{ll}
f(x) = 6 & \text{suggests form of particular solution} \\
y_{PS} = C & (C = \text{undetermined constant}) \\
y''_{PS} = y'_{PS} = 0 & \text{when } y_{PS} = \text{constant} \\
\text{substitute} & y_{PS} = C : 0 + 0 - 3C = 6 \quad \text{i.e. } C = -2.
\end{array}
$$

General solution, $y = y_{CF} + y_{PS} = Ae^{-x} + Be^{3x} - 2$.

Return to Exercise 1

Exercise 2. $\boxed{y'' + 5y' + 6y = 2x}$

$$\text{A.E. is} \quad m^2 + 5m + 6 = 0 \qquad \text{i.e.} \quad (m+2)(m+3) = 0$$

$$\text{i.e.} \quad m_1 = -2, m_2 = -3, \qquad \underline{y_{CF} = Ae^{-2x} + Be^{-3x}}\,.$$

$f(x) = 2x$ suggests substitution of $y_{PS} = Cx + D$ (C, D are undetermined constants)

$y'_{PS} = C$, $y''_{PS} = 0$, substitution gives

$$0 + 5C + 6(Cx + D) = 2x \;\; \text{i.e.} \;\; 5C + 6D + 6Cx = 2x$$

$$\begin{aligned} \text{Constant term :} \qquad & 5C + 6D = 0 \\ \text{Coefficient of } x\text{:} \qquad & 6C = 2 \end{aligned}$$

$$\begin{aligned} \text{i.e.} \qquad & C = \tfrac{1}{3} \\ \text{and} \qquad & D = -\tfrac{5C}{6} = -\tfrac{5}{18} \end{aligned}$$

General solution, $\quad y = y_{CF} + y_{PS}$

$$\text{i.e.} \quad y = Ae^{-2x} + Be^{-3x} + \tfrac{1}{3}x - \tfrac{5}{18}\ .$$

Return to Exercise 2

Exercise 3. Parts (a)-(d) have same homogeneous equation

i.e. $y'' + 5y' - 9y = 0$ with A.E. $m^2 + 5m - 9 = 0$

$$\text{i.e.} \quad m = \tfrac{1}{2}\left(-5 \pm \sqrt{25-(-36)}\right)$$

$$\text{i.e.} \quad m = -\tfrac{5}{2} \pm \tfrac{1}{2}\sqrt{61}$$

Real different roots: $y_{CF} = Ae^{m_1 x} + Be^{m_2 x}\,,$

where $m_1 = -\frac{5}{2} + \frac{1}{2}\sqrt{61}$

$m_2 = -\frac{5}{2} - \frac{1}{2}\sqrt{61}\,.$

$\boxed{\text{(a)} \quad y'' + 5y' - 9y = x^2}$

Try $y_{PS} = Cx^2 + Dx + E$, $\frac{dy_{PS}}{dx} = 2Cx + D\,,$ $\frac{d^2y_{PS}}{dx^2} = 2C$

Substitution: $2C + 5(2Cx + D) - 9(Cx^2 + Dx + E) = x^2$

$$\begin{aligned} \text{i.e.} \quad 2C + 5D - 9E &= 0 \quad \text{(constant term)} \\ 10Cx - 9Dx &= 0 \quad \text{(terms in } x) \\ -9Cx^2 &= x^2 \quad \text{(terms in } x^2) \end{aligned}$$

$$\Rightarrow \quad -9C = 1 \quad \text{i.e.} \quad C = -\tfrac{1}{9}$$

$$10C - 9D = 0 \quad \text{gives} \quad D = \tfrac{10C}{9} = -\tfrac{10}{81}$$

$$\begin{aligned} 2C + 5D - 9E = 0 \quad \text{gives} \quad E &= \tfrac{1}{9}(2C + 5D) \\ &= \tfrac{1}{9}\left(-\tfrac{2}{9} - \tfrac{50}{81}\right) = \tfrac{1}{9}\left(-\tfrac{68}{81}\right) \\ &= -\tfrac{68}{729} \end{aligned}$$

General solution is:

$$y = y_{CF} + y_{PS} = Ae^{m_1 x} + Be^{m_2 x} - \tfrac{1}{9}x^2 - \tfrac{10}{81}x - \tfrac{68}{729}\,.$$

$$\boxed{\text{(b)} \quad y'' + 5y' - 9y = \cos 2x}$$

$$\begin{aligned}
\text{Try} \qquad y_{PS} &= C\cos 2x + D\sin 2x \\
y'_{PS} &= -2C\sin 2x + 2D\cos 2x \\
y''_{PS} &= -4C\cos 2x - 4D\sin 2x
\end{aligned}$$

$$\begin{aligned}
\text{Substitute:} \quad & -4C\cos 2x - 4D\sin 2x + 5\cdot(-2C\sin 2x + 2D\cos 2x) \\
& -9(C\cos 2x + D\sin 2x) = \cos 2x
\end{aligned}$$

$$\begin{aligned}
\text{i.e.} \quad & (-4C+10D-9C)\cos 2x-(4D+10C+9D)\sin 2x = \cos 2x \\
\text{i.e.} \quad & (-13C+10D)\cos 2x - (13D+10C)\sin 2x = \cos 2x
\end{aligned}$$

Coefficients of $\cos 2x$: $\quad -13C + 10D = 1 \quad$ (i)

Coefficients of $\sin 2x$: $\quad 13D + 10C = 0 \quad$ (ii)

$$\begin{aligned} \text{(ii) gives } D = -\tfrac{10C}{13} \text{ then (i) gives} \quad & -13C - \tfrac{100C}{13} = 1 \\ \text{i.e.} \quad & -\tfrac{13\cdot 13 - 100}{13} = \tfrac{1}{C} \\ \text{i.e.} \quad & -\tfrac{13}{269} = C \end{aligned}$$

$$\text{then } D = -\tfrac{10}{13} \cdot \left(-\tfrac{13}{269}\right) = +\tfrac{10}{269}$$

$$\text{i.e. } y_{PS} = -\tfrac{13}{269} \cos 2x + \tfrac{10}{269} \sin 2x$$

General solution is:

$$y = y_{CF} + y_{PS} = Ae^{m_1 x} + Be^{m_2 x} + \frac{1}{269}(10 \sin 2x - 13 \cos 2x)\,.$$

$$\boxed{\text{(c)} \quad y'' + 5y' - 9y = e^{4x}}$$

$$\begin{aligned}
&\text{Try} && y_{PS} = Ce^{4x} \\
&\text{i.e.} && y'_{PS} = 4Ce^{4x} \\
& && y''_{PS} = 16Ce^{4x}
\end{aligned}$$

$$\begin{aligned}
\text{Substitute:} \quad 16Ce^{4x} + 20Ce^{4x} - 9Ce^{4x} &= e^{4x} \\
\text{i.e.} \qquad (16C + 20C - 9C)e^{4x} &= e^{4x} \\
\text{i.e.} \qquad 27C &= 1 \\
\text{i.e.} \qquad C &= \tfrac{1}{27}
\end{aligned}$$

(by comparing coefficients of e^{4x})

i.e. $y_{PS} = \frac{1}{27}e^{4x}$

General solution is:

$$y = y_{CF} + y_{PS} = Ae^{m_1x} + Be^{m_2x} + \frac{1}{27}e^{4x} \, .$$

$$\boxed{\text{(d)} \quad y'' + 5y' - 9y = e^{-2x} + 2 - x}$$

Try $\quad y_{PS} = Ce^{-2x} + Dx + E$ (i.e. sum of the forms "ke^{px}" and "linear in x")

i.e. $\quad y'_{PS} = -2Ce^{-2x} + D$

$$y''_{PS} = +4Ce^{-2x}$$

Substitute: $4Ce^{-2x} - 10Ce^{-2x} + 5D - 9Ce^{-2x} - 9Dx - 9E = e^{-2x} + 2 - x$

Coeff. e^{-2x} : $\quad 4C - 10C - 9C = 1 \quad \rightarrow \quad C = -\frac{1}{15}$

Coeff. x : $\quad -9D = -1 \quad \rightarrow \quad D = \frac{1}{9}$

Coeff. x^0 : $\quad 5D - 9E = 2 \quad \rightarrow \quad 5\left(\frac{1}{9}\right) - 9E = 2$

i.e. $E = -\frac{13}{81}$

i.e. $y_{PS} = -\frac{1}{15}e^{-2x} + \frac{1}{9}x - \frac{13}{81}$

General solution is:

$$y = y_{CF} + y_{PS} = Ae^{m_1 x} + Be^{m_2 x} - \frac{1}{15}e^{-2x} + \frac{1}{9}x - \frac{13}{81}\ .$$

Return to Exercise 3

Exercise 4. $\boxed{y'' - \lambda^2 y = \sin 2x}$

A.E. is $\quad m^2 - \lambda^2 = 0 \quad$ i.e. $\quad m = \pm\lambda \quad$ (real different roots)

$$\underline{y_{CF} = Ae^{\lambda x} + Be^{-\lambda x}}$$

$f(x) = \sin 2x$ suggests trying $\quad y_{PS} = A\sin 2x + B\cos 2x$

i.e. $\quad y'_{PS} = 2A\cos 2x - 2B\sin 2x$

$$y''_{PS} = -4A\sin 2x - 4B\cos 2x$$

Substitute:

$$-4A\sin 2x - 4B\cos 2x - \lambda^2 A\sin 2x - \lambda^2 B\cos 2x = \sin 2x$$

$$\text{i.e. } (-4A - \lambda^2 A)\sin 2x + (-4B - \lambda^2 B)\cos 2x = \sin 2x$$

Coeff. $\cos 2x$: $\quad -4B - \lambda^2 B = 0 \quad$ i.e. $B = 0$

Coeff. $\sin 2x$: $\quad -4A - \lambda^2 A = 1 \quad$ i.e. $A = -\frac{1}{4+\lambda^2}$

i.e. $\quad y_{PS} = -\frac{\sin 2x}{4+\lambda^2}$;

general solution $\quad y = y_{CF} + y_{PS} = Ae^{\lambda x} + Be^{-\lambda x} - \frac{\sin 2x}{4+\lambda^2}$.

Return to Exercise 4

Exercise 5. $\boxed{y'' - y = e^x}$

A.E. is $\quad m^2 - 1 = 0 \quad$ i.e. $\quad m = \pm 1, \quad \underline{y_{CF} = Ae^x + Be^{-x}}$

$f(x) = e^x$ suggests trying $y_{PS} = Ce^x$ <u>BUT</u> THIS ALREADY APPEARS IN y_{CF}, therefore multiply this trial form by x.

$$
\begin{aligned}
\text{Try} \qquad & y_{PS} = Cxe^x,\ y'_{PS} = Cxe^x + Ce^x = C(1+x)e^x \\
& y''_{PS} = C(1+x)e^x + Ce^x = C(2+x)e^x \\
\text{Substitute:} \qquad & C(2+x)e^x - Cxe^x = e^x \\
\text{Coeff. } e^x\text{:} \qquad & C(2+x) - Cx = 1 \quad \text{i.e.} \quad 2C = 1 \quad \text{i.e.} \quad C = \tfrac{1}{2}
\end{aligned}
$$

$\therefore y_{PS} = \frac{1}{2}xe^x$ and general solution is

$$y = y_{CF} + y_{PS} = Ae^x + Be^{-x} + \frac{1}{2}xe^x\ .$$

Return to Exercise 5

Exercise 6. $\boxed{y'' + y' - 2y = e^{-2x}}$

A.E. is $\quad m^2 + m - 2 = 0 \quad$ i.e. $\quad (m+2)(m-1) = 0$

i.e. $\quad m = 1$ or $m = -2 \quad$ i.e. $\quad \underline{y_{CF} = Ae^x + Be^{-2x}}$

Try $\quad y_{PS} = Ce^{-2x}$? $\quad$ No. This already appears in y_{CF}

Try $\quad y_{PS} = Cxe^{-2x} \quad$ (i.e. multiply trial solution by x, until it does not appear in y_{CF})

$$\text{i.e.} \quad y'_{PS} = -2Cxe^{-2x} + Ce^{-2x} = (1-2x)Ce^{-2x}$$

$$y''_{PS} = -2(1-2x)Ce^{-2x} + (-2)Ce^{-2x} = (-4+4x)Ce^{-2x}$$

Substitute: $(-4+4x)Ce^{-2x} + (1-2x)Ce^{-2x} - 2Cxe^{-2x} = e^{-2x}$

Coeff. e^{-2x}: $-4C+4xC + C-2xC-2Cx = 1$

$$\text{i.e. } C = -\tfrac{1}{3}, \quad \text{i.e.} \quad y_{PS} = -\tfrac{1}{3}xe^{-2x}$$

General solution is $y = y_{CF} + y_{PS} = Ae^{x} + Be^{-2x} - \frac{1}{3}xe^{-2x}$.

Return to Exercise 6

Exercise 7. $\boxed{y'' - 2y' + y = e^x}$

A.E. is $m^2 - 2m + 1 = 0$ i.e. $(m-1)^2 = 0$

i.e. $m = 1$ (twice) i.e. $\underline{y_{CF} = (A + Bx)e^x}$

Try $y_{PS} = Ce^x$? No, it's in y_{CF}

Try $y_{PS} = Cxe^x$? No. Also in y_{CF}!

Try $y_{PS} = Cx^2e^x$.

$$\begin{array}{rcl} y'_{PS} & = & 2xCe^x + Cx^2e^x, \quad y''_{PS} = (2{+}2x)Ce^x + (2x{+}x^2)Ce^x \\ \text{i.e. } y'_{PS} & = & (2x + x^2)Ce^x \qquad\qquad\quad = (2 + 4 + x^2)Ce^x \end{array}$$

Substitute: $(2 + 4x + x^2)Ce^x - 2(2x + x^2)Ce^x + Cx^2e^x = e^x$

Coeff. e^x : $(2 + 4x + x^2 - 4x - 2x^2 + x^2)C = 1$ i.e. $C = \frac{1}{2}$

$\therefore$ $y_{PS} = \frac{1}{2}x^2e^x$ and general solution is

$$y = y_{CF} + y_{PS} = (A + Bx)e^x + \tfrac{1}{2}x^2e^x \ .$$

Return to Exercise 7

Exercise 8. $\boxed{y'' + 8y' + 17y = 2e^{-3x}\ ; \quad y(0) = 2 \quad \text{and} \quad y\left(\frac{\pi}{2}\right) = 0}$

A.E. is $\quad m^2 + 8m + 17 = 0 \quad$ i.e. $\ m = \frac{1}{2}\left(-8 \pm \sqrt{64 - 68}\right) = -4 \pm i$

i.e. $\quad \underline{y_{CF} = e^{-4x}(A\cos x + B\sin x).}$

$f(x) = 2e^{-3x}$, so try $\quad y_{PS} = Ce^{-3x}$

i.e. $y'_{PS} = -3Ce^{-3x}, \quad y''_{PS} = 9Ce^{-3x}$

Substitute: $\qquad 9Ce^{-3x} + 8 \cdot (-3)Ce^{-3x} + 17Ce^{-3x} = 2e^{-3x}$

Coeff. e^{-3x} : $\quad 9C - 24C + 17C = 2 \quad$ i.e. $C = 1$

$y_{PS} = e^{-3x}$, $\quad$ general solution

$$\boxed{y = y_{CF} + y_{PS} = e^{-4x}(A\cos x + B\sin x) + e^{-3x}\ .}$$

Now use the boundary conditions.

$y(0)=2$	means $y=2$ when $x=0$	i.e. $2=e^0(A\cos 0+B\sin 0)+e^0$
		i.e. $2 = A+1$ i.e. $A=1$
$y(\frac{\pi}{2})=0$	means $y=0$ when $x=\frac{\pi}{2}$	i.e. $0=e^{-2\pi}(A\cos\frac{\pi}{2}+B\sin\frac{\pi}{2})+e^{-\frac{3\pi}{2}}$
		$\cos\frac{\pi}{2}=0$
		$\sin\frac{\pi}{2}=1$
		i.e. $0=e^{-2\pi}(0+B)+e^{-\frac{3\pi}{2}}$
		i.e. $0=e^{-2\pi}B+e^{-\frac{3\pi}{2}}$
(multiply both sides by $e^{2\pi}$)		i.e. $0=B+e^{-\frac{3\pi}{2}}\cdot e^{2\pi}$
		i.e. $0=B+e^{\frac{\pi}{2}}$ i.e. $B=-e^{\frac{\pi}{2}}$
Required particular solution is		$y=e^{-4x}\left(\cos x-e^{\frac{\pi}{2}}\sin x\right)+e^{-3x}$.

Return to Exercise 8

Exercise 9. $\boxed{y'' + y' - 12y = 4e^{2x}\ ; \quad y(0) = 7 \quad \text{and} \quad y'(0) = 0}$

A.E. is $\quad m^2 + m - 12 = 0 \quad$ i.e. $\quad (m-3)(m+4) = 0$

i.e. $\quad m = 3$ or $m = -4 \quad$ i.e. $\quad \underline{y_{CF} = Ae^{3x} + Be^{-4x}}$

Try $y_{PS} = Ce^{2x}$, $y'_{PS} = 2Ce^{2x}$, $y''_{PS} = 4Ce^{2x}$

Substitute: $\quad 4Ce^{2x} + 2Ce^{2x} - 12Ce^{2x} = 4e^{2x}$

Coeff. e^{2x} : $\quad 4C + 2C - 12C = 4 \quad$ i.e. $C = -\frac{2}{3}$

$y_{PS} = -\frac{2}{3}e^{2x}$, $\quad \boxed{\text{general solution is } y = Ae^{3x} + Be^{-4x} - \tfrac{2}{3}e^{2x}\ .}$

Apply boundary conditions to general solution (as always)

$y(0) = 7$ means $y = 7$ when $x = 0$ i.e. $7 = Ae^0 + Be^0 - \frac{2}{3}e^0$

i.e. $7 = A + B - \frac{2}{3}$

i.e. $\boxed{21 = 3A + 3B - 2}$ (i)

$y'(0) = 0$ means $y' = 0$ when $x = 0$,

where $y' \equiv \frac{dy}{dx} = 3Ae^{3x} - 4Be^{-4x} - \frac{4}{3}e^{2x}$

i.e. $0 = 3Ae^0 - 4Be^0 - \frac{4}{3}e^0$

i.e. $\boxed{\frac{4}{3} = 3A - 4B}$ (ii)

<u>Solve equations (i) and (ii) simultaneously to find A and B</u>

(i)-(ii) gives $21 - \frac{4}{3} = 7B - 2$ i.e. $\left(23 - \frac{4}{3}\right) \cdot \frac{1}{7} = B$

i.e. $\frac{65}{3} \cdot \frac{1}{7} = B$ i.e. $B = \frac{65}{21}$,

then (ii) gives $3A = \frac{4}{3} + 4B$ i.e. $\frac{4}{3} + \frac{260}{21} = \frac{288}{21} = 3A$

i.e. $A = \frac{96}{21} = \frac{32}{7}$.

Particular solution is

$$y = \frac{32}{7}e^{3x} + \frac{65}{21}e^{-4x} - \frac{2}{3}e^{2x} \ .$$

Return to Exercise 9

Exercise 10. $\boxed{y'' + 3y' + 2y = 10\cos 2x \; ; \quad y(0) = 1 \quad \text{and} \quad y'(0) = 0}$

A.E. is $\quad m^2 + 3m + 2 = 0 \quad$ i.e. $\quad (m+2)(m+1) = 0$

i.e. $\quad m = -2$ or $m - 1 \quad$ i.e. $\quad \underline{y_{CF} = Ae^{-2x} + Be^{-x}}$

Try $\quad y_{PS} = C\cos 2x + D\sin 2x \quad$ i.e. $\quad y'_{PS} = -2C\sin 2x + 2D\cos 2x$

$$y''_{PS} = -4C\cos 2x - 4D\sin 2x$$

Substitute:

$$-4C\cos 2x - 4D\sin 2x \quad + \quad 3(-2C\sin 2x + 2D\cos 2x)$$
$$+ \quad 2(C\cos 2x + D\sin 2x) = 10\cos 2x$$

Coeff. $\cos 2x$: $\quad -4C + 6D + 2C = 10 \quad$ i.e. $\; -2C + 6D = 10 \qquad$ (i)

Coeff. $\sin 2x$: $\quad -4D - 6C + 2D = 0 \quad$ i.e. $\; -2D - 6C = 0 \qquad$ (ii)

Solve (i) and (ii) for C and D

$$\begin{array}{rrcl}
(-3) \text{ times (i) gives} & +6C - 18D &=& -30 \quad \text{(i)} \\
\text{ADD} & -6C - 2D &=& 0 \quad \text{(ii)} \\
\hline
& -20D &=& -30 \quad \text{i.e. } D = \frac{3}{2}\,,
\end{array}$$

(ii) then gives $\quad -2\left(\frac{3}{2}\right) - 6C = 0 \quad$ i.e. $\quad -3 - 6C = 0$

i.e. $\quad C = -\frac{1}{2}\,.$

$\therefore\ y_{PS} = -\frac{1}{2}\cos 2x + \frac{3}{2}\sin 2x\,;$

$$\boxed{\text{general solution is: } y = Ae^{-2x} + Be^{-x} - \tfrac{1}{2}\cos 2x + \tfrac{3}{2}\sin 2x}$$

Boundary condition $y(0) = 1$ gives $\quad 1 = A + B - \frac{1}{2}$ i.e. $\underline{A+B = \frac{3}{2}}$ (iii)

$y' = -2Ae^{-2x} - Be^{-x} + \sin 2x + 3\cos 2x$

$\therefore\ y'(0) = 0$ gives $0 = -2A - B + 3$ i.e. $\underline{2A+B=3}$ (iv)

Solve (iii) and (iv) to find A and B (iv)-(iii) gives $A = 3 - \frac{3}{2} = \frac{3}{2}$.

Then, (iv) gives $2\left(\frac{3}{2}\right) + B = 3$ i.e. $B = 0$

Particular solution is $y = \frac{3}{2}e^{-2x} - \frac{1}{2}\cos 2x + \frac{3}{2}\sin 2x$.

Return to Exercise 10

Exercise 11. $\boxed{y'' + 4y' + 5y = 2e^{-2x}\ ; \quad y(0) = 1 \quad \text{and} \quad y'(0) = -2}$

A.E. is $\quad m^2 + 4m + 5 = 0 \quad$ i.e. $m = \frac{1}{2}\left(-4 \pm \sqrt{16-20}\right) = -2 \pm i$

i.e. $\quad \underline{y_{CF} = e^{-2x}(A\cos x + B\sin x)}$

Try $\quad y_{PS} = Ce^{-2x},\ y'_{PS} = -2Ce^{-2x},\ y''_{PS} = 4Ce^{-2x}$

Substitute: $\quad 4Ce^{-2x} - 8Ce^{-2x} + 5Ce^{-2x} = 2e^{-2x}$

Coeff. e^{-2x}: $\quad 4C - 8C + 5C = 2 \quad$ i.e. $\quad C = 2\,.$

$y_{PS} = 2e^{-2x}$, $\quad$ general solution is

$$\boxed{y = e^{-2x}(A\cos x + B\sin x) + 2e^{-2x}}$$

$$\underline{\text{Boundary condition}} \quad y(0) = 1: \quad 1 = A\cos 0 + B\sin 0 + 2$$
$$\text{i.e.} \quad 1 = A + 2 \quad \text{i.e. } A = -1$$

$$y' = -2e^{-2x}(A\cos x + B\sin x) + e^{-2x}\cdot(-A\sin x + B\cos x) - 4e^{-2x}$$

$y'(0) = -2$ gives

$$\begin{aligned} & -2 = -2\cdot(-1+0) + (0+B) - 4 \\ \text{i.e.} \quad & -2 = 2 + B - 4 \quad \text{i.e. } B = 0\,. \end{aligned}$$

Particular solution is

$$\begin{aligned} & y = e^{-2x}\cdot(-\cos x + 0) + 2e^{-2x} \\ \text{i.e.} \quad & y = e^{-2x}(2 - \cos x)\,. \end{aligned}$$

Return to Exercise 11

Exercise 12. $\boxed{\frac{d^2x}{d\tau^2} + 4\frac{dx}{d\tau} + 3x = e^{-3\tau}\ ; \quad x = \frac{1}{2} \text{ and } \frac{dx}{d\tau} = -2 \text{ at } \tau = 0}$

A.E. is $\quad m^2 + 4m + 3 = 0 \qquad$ i.e. $\quad (m+3)(m+1) = 0$

i.e. $\quad m = -3$ or $m = -1 \quad$ i.e. $\quad \underline{x_{CF} = Ae^{-3\tau} + Be^{-\tau}}$

Try $\quad x_{PS} = Ce^{-3\tau}$? $\qquad$ No. Already in x_{CF}

Try $\quad x_{PS} = C\tau e^{-3\tau}\,, \qquad \frac{dx_{PS}}{d\tau} = Ce^{-3\tau} - 3\tau Ce^{-3\tau}$
$= (1-3\tau)Ce^{-3\tau}\,.$

$$\begin{aligned} \frac{d^2x_{PS}}{d\tau^2} &= -3Ce^{-3\tau} - 3(1-3\tau)Ce^{-3\tau} \\ &= (9\tau - 6)Ce^{-3\tau} \end{aligned}$$

Substitute: $\quad (9\tau - 6)Ce^{-3\tau} + 4(1-3\tau)Ce^{-3\tau} + 3C\tau e^{-3\tau} = e^{-3\tau}$

Coeff. $e^{-3\tau}$: $\quad 9\tau C - 6C + 4C - 12\tau C + 3C\tau = 1$

i.e. $C = -\frac{1}{2}$

$x_{PS} = -\frac{1}{2}\tau e^{-3\tau}$, general solution is

$$\boxed{x = x_{CF} + x_{PS} = Ae^{-3\tau} + Be^{-\tau} - \frac{1}{2}\tau e^{-3\tau}}$$

Boundary conditions $x = \frac{1}{2}$ when $\tau = 0$: $\frac{1}{2} = A + B$ (i)

$$\frac{dx}{d\tau} = -3Ae^{-3\tau} - Be^{-\tau} + \frac{3}{2}\tau e^{-3\tau} - \frac{1}{2}e^{-3\tau}$$

$\frac{dx}{d\tau} = -2$ when $\tau = 0$: $-2 = -3A - B - \frac{1}{2}$

i.e. $-\frac{3}{2} = -3A - B$ (ii)

Solve (i) and (ii) for A and B: (i)+(ii) gives $\frac{1}{2} - \frac{3}{2} = A - 3A$ i.e. $A = \frac{1}{2}$

Then, (i) gives $B = 0$.

Particular solution is

$$x = \frac{1}{2}e^{-3\tau} - \frac{1}{2}\tau e^{-3\tau} = \frac{1}{2}(1-\tau)e^{-3\tau}\ .$$

Return to Exercise 12

Exercise 13. $\boxed{\frac{d^2y}{d\tau^2} + 4\frac{dy}{d\tau} + 5y = 6\sin\tau}$

A.E. is $\quad m^2 + 4m + 5 = 0 \qquad$ i.e. $m = \frac{1}{2}\left(-4 \pm \sqrt{16-20}\right) = -2 \pm i$

$$\underline{y_{CF} = e^{-2\tau}(A\cos\tau + B\sin\tau)}$$

[Since $e^{-2\tau}$ multiplies $\sin\tau$ in $y_{CF}, \sin\tau$ is an independent function with respect to the components of y_{CF}].

Try
$$\begin{aligned} y_{PS} &= C\cos\tau + D\sin\tau \\ y'_{PS} &= -C\sin\tau + D\cos\tau \\ y''_{PS} &= -C\cos\tau - D\sin\tau \qquad \text{(each dash denoting } \tfrac{d}{d\tau}\text{, here)} \end{aligned}$$

Substitute:
$$-C\cos\tau - D\sin\tau + 4(-C\sin\tau + D\cos\tau) + 5(C\cos\tau + D\sin\tau) = 6\sin\tau$$

$$\begin{array}{lrlrr}
\text{Coeff. } \cos\tau: & -C+4D+5C=0 & \text{i.e.} & C+D=0 & \text{(i)} \\
\text{Coeff. } \sin\tau: & -D-4C+5D=6 & \text{i.e.} & -C+D=\frac{3}{2} & \text{(ii)} \\
\hline
 & & \text{ADD} & 2D=\frac{3}{2} & \\
\hline
 & & \text{i.e.} & D=\frac{3}{4} \text{ and} & \\
 & & & C=-\frac{3}{4} &
\end{array}$$

General solution: $y = e^{-2\tau}(A\cos\tau + B\sin\tau) - \frac{3}{4}(\cos\tau - \sin\tau)$.

Return to Exercise 13