Vectors

SCALAR PRODUCT

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A Tutorial Module for learning about the scalar product of two vectors

Table of contents

1. Theory

The purpose of this tutorial is to practice using the scalar product of two vectors. It is called the 'scalar product' because the result is a 'scalar', i.e. a quantity with **magnitude** but no associated direction.

The **SCALAR PRODUCT** (or 'dot product') of $\underline{a}$ and $\underline{b}$ is

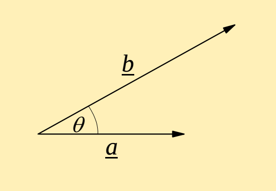
$$\begin{aligned} \underline{a} \cdot \underline{b} &= |\underline{a}|\,|\underline{b}|\cos\theta \\ &= a_x b_x + a_y b_y + a_z b_z \end{aligned}$$

where θ is the angle between $\underline{a}$ and $\underline{b}$

and

$$\underline{a} = a_x\underline{i} + a_y\underline{j} + a_z\underline{k}$$

$$\underline{b} = b_x\underline{i} + b_y\underline{j} + b_z\underline{k}.$$



Note that when

$$\underline{a} = a_x\underline{i} + a_y\underline{j} + a_z\underline{k}$$

and

$$\underline{b} = b_x\underline{i} + b_y\underline{j} + b_z\underline{k}$$

then the magnitudes of $\underline{a}$ and $\underline{b}$ are

$$|\underline{a}| = \sqrt{a_x^2 + a_y^2 + a_z^2}$$

and

$$|\underline{b}| = \sqrt{b_x^2 + b_y^2 + b_z^2}\ ,$$

respectively.

2. Exercises

Click on EXERCISE links for full worked solutions (there are 16 exercises in total)

EXERCISE 1. Calculate $\underline{a} \cdot \underline{b}$ when $\underline{a} = 2\underline{i} - 3\underline{j} + 5\underline{k}, \;\; \underline{b} = \underline{i} + 2\underline{j} + 8\underline{k}$

EXERCISE 2. Calculate $\underline{a} \cdot \underline{b}$ when $\underline{a} = 4\underline{i} - 7\underline{j} + 2\underline{k}, \;\; \underline{b} = 5\underline{i} - \underline{j} - 4\underline{k}$

EXERCISE 3. Calculate $\underline{a} \cdot \underline{b}$ when $\underline{a} = 2\underline{i} + 3\underline{j} + 3\underline{k}, \;\; \underline{b} = 3\underline{i} - 2\underline{j} + 5\underline{k}$

EXERCISE 4. Calculate $\underline{a} \cdot \underline{b}$ when $\underline{a} = 3\underline{i} + 6\underline{j} - \underline{k}, \;\; \underline{b} = 8\underline{i} - 3\underline{j} - \underline{k}$

Exercise 5. Show that $\underline{a}$ is perpendicular to $\underline{b}$ when

$$\underline{a} = \underline{i} + \underline{j} + 3\underline{k}, \quad \underline{b} = \underline{i} - 7\underline{j} + 2\underline{k}$$

Exercise 6. Show that $\underline{a}$ is perpendicular to $\underline{b}$ when

$$\underline{a} = \underline{i} + 23\underline{j} + 7\underline{k}, \quad \underline{b} = 26\underline{i} + \underline{j} - 7\underline{k}$$

Exercise 7. Show that $\underline{a}$ is perpendicular to $\underline{b}$ when

$$\underline{a} = \underline{i} + \underline{j} + 3\underline{k}, \quad \underline{b} = 2\underline{i} + 7\underline{j} - 3\underline{k}$$

Exercise 8. Show that $\underline{a}$ is perpendicular to $\underline{b}$ when

$$\underline{a} = 39\underline{i} + 2\underline{j} + \underline{k}, \quad \underline{b} = \underline{i} - 23\underline{j} + 7\underline{k}$$

Exercise 9. Calculate the work done $\underline{F} \cdot \underline{s}$ given $|\underline{F}|$, $|\underline{s}|$ and θ (the angle between the force $\underline{F}$ and the displacement $\underline{s}$) when
$|\underline{F}| = 7\,\mathrm{N}, \quad |\underline{s}| = 3\,\mathrm{m}, \quad \theta = 0^\circ$

Exercise 10. Calculate the work done $\underline{F} \cdot \underline{s}$ given $|\underline{F}|$, $|\underline{s}|$ and θ (the angle between the force $\underline{F}$ and the displacement $\underline{s}$) when
$|\underline{F}| = 4\,\mathrm{N}, \quad |\underline{s}| = 2\,\mathrm{m}, \quad \theta = 27^\circ$

Exercise 11. Calculate the work done $\underline{F} \cdot \underline{s}$ given $|\underline{F}|$, $|\underline{s}|$ and θ (the angle between the force $\underline{F}$ and the displacement $\underline{s}$) when
$|\underline{F}| = 5\,\mathrm{N}, \quad |\underline{s}| = 4\,\mathrm{m}, \quad \theta = 48^\circ$

Exercise 12. Calculate the work done $\underline{F} \cdot \underline{s}$ given $|\underline{F}|$, $|\underline{s}|$ and θ (the angle between the force $\underline{F}$ and the displacement $\underline{s}$) when
$|\underline{F}| = 2\,\mathrm{N}, \quad |\underline{s}| = 3\,\mathrm{m}, \quad \theta = 56^\circ$

Exercise 13. Calculate the angle θ between vectors $\underline{a}$ and $\underline{b}$ when

$$\underline{a} = 2\underline{i} - \underline{j} + 2\underline{k}\,, \quad \underline{b} = \underline{i} + \underline{j} + \underline{k}$$

Exercise 14. Calculate the angle θ between vectors $\underline{a}$ and $\underline{b}$ when

$$\underline{a} = \underline{i} + \underline{j} + \underline{k}\,, \quad \underline{b} = 2\underline{i} - 3\underline{j} + \underline{k}$$

Exercise 15. Calculate the angle θ between vectors $\underline{a}$ and $\underline{b}$ when

$$\underline{a} = \underline{i} - 2\underline{j} + 2\underline{k}\,, \quad \underline{b} = 2\underline{i} + 3\underline{j} + \underline{k}$$

Exercise 16. Calculate the angle θ between vectors $\underline{a}$ and $\underline{b}$ when

$$\underline{a} = 5\underline{i} + 4\underline{j} + 3\underline{k}\,, \quad \underline{b} = 4\underline{i} - 5\underline{j} + 3\underline{k}$$

3. Answers

1. 36,
2. 19,
3. 15,
4. 7,
5. Hint: If $\theta = 90^\circ$ then what will $\underline{a} \cdot \underline{b} = a_x b_x + a_y b_y + a_z b_z$ be?
6. Hint: If $\theta = 90^\circ$ then what will $\underline{a} \cdot \underline{b} = a_x b_x + a_y b_y + a_z b_z$ be?
7. Hint: If $\theta = 90^\circ$ then what will $\underline{a} \cdot \underline{b} = a_x b_x + a_y b_y + a_z b_z$ be?
8. Hint: If $\theta = 90^\circ$ then what will $\underline{a} \cdot \underline{b} = a_x b_x + a_y b_y + a_z b_z$ be?
9. 21 J,
10. 7.128 J,
11. 13.38 J,
12. 3.355 J,
13. 54.7°,

14. 90°,

15. 100.3°,

16. 79.6°.

4. Tips on using solutions

● When looking at the THEORY, ANSWERS, TIPS or NOTATION pages, use the Back button (at the bottom of the page) to return to the exercises

● Use the solutions intelligently. For example, they can help you get started on an exercise, or they can allow you to check whether your intermediate results are correct

● Try to make less use of the full solutions as you work your way through the Tutorial

5. Alternative notation

● Here, we use symbols like $\underline{a}$ to denote a vector.
In some texts, symbols for vectors are **in bold** (eg **a** instead of $\underline{a}$)

● In this Tutorial, vectors are given in terms of the unit Cartesian vectors $\underline{i}$, $\underline{j}$ and $\underline{k}$.

For example, $\underline{a} = \underline{i} + 2\underline{j} + 3\underline{k}$ implies that $\underline{a}$ can be **decomposed** into the sum of the following three vectors:

	$\underline{i}$	(*one* step along the x-axis)
PLUS	$2\,\underline{j}$	(*two* steps along the y-axis)
PLUS	$3\,\underline{k}$	(*three* steps along the z-axis)

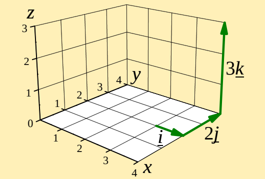
See the figures on the next page . . .

$$\underline{a} = \underline{i} + 2\underline{j} + 3\underline{k}$$

one step along the x-axis

two steps along the y-axis

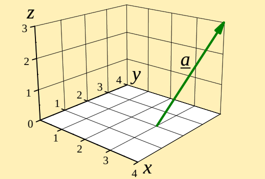
three steps along the z-axis



$\underline{a}$ is the (vector) sum

of $\underline{i}$ and $2\underline{j}$

and $3\underline{k}$



- A common alternative notation for expressing $\underline{a}$ in terms of these **Cartesian components** is given by $\underline{a} = (1, 2, 3)$

Full worked solutions

Exercise 1.

$\underline{a} \cdot \underline{b} = a_x b_x + a_y b_y + a_z b_z$, where

$$\underline{a} = a_x \underline{i} + a_y \underline{j} + a_z \underline{k}$$

$$\underline{b} = b_x \underline{i} + b_y \underline{j} + b_z \underline{k}$$

$\underline{a} = 2\underline{i} - 3\underline{j} + 5\underline{k}$, $\underline{b} = \underline{i} + 2\underline{j} + 8\underline{k}$ gives

$$\begin{aligned} \underline{a} \cdot \underline{b} &= (2)(1) + (-3)(2) + (5)(8) \\ &= 2 - 6 + 40 \\ &= 36. \end{aligned}$$

Return to Exercise 1

Exercise 2.

$\underline{a} \cdot \underline{b} = a_x b_x + a_y b_y + a_z b_z$, where

$$\underline{a} = a_x \underline{i} + a_y \underline{j} + a_z \underline{k}$$

$$\underline{b} = b_x \underline{i} + b_y \underline{j} + b_z \underline{k}$$

$\underline{a} = 4\underline{i} - 7\underline{j} + 2\underline{k}, \quad \underline{b} = 5\underline{i} - \underline{j} - 4\underline{k}$ gives

$$\begin{aligned} \underline{a} \cdot \underline{b} &= (4)(5) + (-7)(-1) + (2)(-4) \\ &= 20 + 7 - 8 \\ &= 19. \end{aligned}$$

Return to Exercise 2

Exercise 3.

$\underline{a} \cdot \underline{b} = a_x b_x + a_y b_y + a_z b_z$, where

$$\underline{a} = a_x \underline{i} + a_y \underline{j} + a_z \underline{k}$$
$$\underline{b} = b_x \underline{i} + b_y \underline{j} + b_z \underline{k}$$

$\underline{a} = 2\underline{i} + 3\underline{j} + 3\underline{k}$, $\underline{b} = 3\underline{i} - 2\underline{j} + 5\underline{k}$ gives

$$\begin{aligned} \underline{a} \cdot \underline{b} &= (2)(3) + (3)(-2) + (3)(5) \\ &= 6 - 6 + 15 \\ &= 15. \end{aligned}$$

Return to Exercise 3

Exercise 4.

$\underline{a} \cdot \underline{b} = a_x b_x + a_y b_y + a_z b_z$, where

$$\underline{a} = a_x \underline{i} + a_y \underline{j} + a_z \underline{k}$$
$$\underline{b} = b_x \underline{i} + b_y \underline{j} + b_z \underline{k}$$

$\underline{a} = 3\underline{i} + 6\underline{j} - \underline{k}, \quad \underline{b} = 8\underline{i} - 3\underline{j} - \underline{k}$ gives

$$\begin{aligned} \underline{a} \cdot \underline{b} &= (3)(8) + (6)(-3) + (-1)(-1) \\ &= 24 - 18 + 1 \\ &= 7. \end{aligned}$$

Return to Exercise 4

Exercise 5.

$\underline{a} \cdot \underline{b} = |\underline{a}||\underline{b}| \cos\theta$, where θ is the angle between $\underline{a}$ and $\underline{b}$

$$
\begin{aligned}
\underline{a} \text{ perpendicular to } \underline{b} \text{ gives} \quad & \theta = 90^\circ \\
\text{i.e.} \quad & \cos\theta = 0 \\
\text{i.e.} \quad & \underline{a} \cdot \underline{b} = 0
\end{aligned}
$$

To show that this is true, use $\underline{a} \cdot \underline{b} = a_x b_x + a_y b_y + a_z b_z$

$$
\begin{aligned}
\text{i.e.} \quad \underline{a} \cdot \underline{b} &= (1)(1) + (1)(-7) + (3)(2) \\
&= 1 - 7 + 6 \\
&= 0.
\end{aligned}
$$

Return to Exercise 5

Exercise 6.

$\underline{a} \cdot \underline{b} = |\underline{a}||\underline{b}| \cos\theta$, where θ is the angle between $\underline{a}$ and $\underline{b}$

$$\begin{aligned} \underline{a} \text{ perpendicular to } \underline{b} \text{ gives} \quad & \theta = 90^\circ \\ \text{i.e.} \quad & \cos\theta = 0 \\ \text{i.e.} \quad & \underline{a} \cdot \underline{b} = 0 \end{aligned}$$

To show that this is true, use $\underline{a} \cdot \underline{b} = a_x b_x + a_y b_y + a_z b_z$

$$\begin{aligned} \text{i.e.} \qquad \underline{a} \cdot \underline{b} &= (1)(26) + (23)(1) + (7)(-7) \\ &= 26 + 23 - 49 \\ &= 0. \end{aligned}$$

Return to Exercise 6

Exercise 7.

$\underline{a} \cdot \underline{b} = |\underline{a}||\underline{b}| \cos\theta$, where θ is the angle between $\underline{a}$ and $\underline{b}$

$$\begin{aligned} \underline{a} \text{ perpendicular to } \underline{b} \text{ gives} \quad & \theta = 90^\circ \\ \text{i.e.} \quad & \cos\theta = 0 \\ \text{i.e.} \quad & \underline{a} \cdot \underline{b} = 0 \end{aligned}$$

To show that this is true, use $\underline{a} \cdot \underline{b} = a_x b_x + a_y b_y + a_z b_z$

$$\begin{aligned} \text{i.e.} \qquad \underline{a} \cdot \underline{b} &= (1)(2) + (1)(7) + (3)(-3) \\ &= 2 + 7 - 9 \\ &= 0. \end{aligned}$$

Return to Exercise 7

Exercise 8.

$\underline{a} \cdot \underline{b} = |\underline{a}||\underline{b}| \cos\theta$, where θ is the angle between $\underline{a}$ and $\underline{b}$

$$\begin{aligned} \underline{a} \text{ perpendicular to } \underline{b} \text{ gives} \quad \theta &= 90^\circ \\ \text{i.e.} \quad \cos\theta &= 0 \\ \text{i.e.} \quad \underline{a} \cdot \underline{b} &= 0 \end{aligned}$$

To show that this is true, use $\underline{a} \cdot \underline{b} = a_x b_x + a_y b_y + a_z b_z$

$$\begin{aligned} \text{i.e.} \qquad \underline{a} \cdot \underline{b} &= (39)(1) + (2)(-23) + (1)(7) \\ &= 39 - 46 + 7 \\ &= 0. \end{aligned}$$

Return to Exercise 8

Exercise 9.

$\underline{F} \cdot \underline{s} = |\underline{F}||\underline{s}| \cos\theta$,

$$|\underline{F}| = 7\text{ N}$$

$$|\underline{s}| = 3\text{ m}$$

$$\theta = 0^\circ \text{ gives } \cos\theta = 1$$

$\therefore\ \underline{F} \cdot \underline{s} = |\underline{F}||\underline{s}| \cos\theta = (7\text{ N})(3\text{ m})(1) = 21\text{ J}.$

<u>Note:</u> When the angle θ is zero then

$$\underline{F} \cdot \underline{s} = |\underline{F}||\underline{s}|$$

and one simply multiplies the magnitudes of $\underline{F}$ and $\underline{s}$.

Return to Exercise 9

Exercise 10.

$$\underline{F} \cdot \underline{s} = |\underline{F}||\underline{s}| \cos\theta, \qquad |\underline{F}| = 4\ \mathrm{N}$$

$$|\underline{s}| = 2\ \mathrm{m}$$

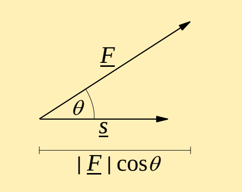
$$\theta = 27^\circ \text{ gives } \cos\theta \simeq 0.8910$$

$$\therefore\ \underline{F} \cdot \underline{s} \simeq (4\ \mathrm{N})(2\ \mathrm{m})(0.8910) \simeq 7.128\ \mathrm{J}.$$

$$\underline{F} \cdot \underline{s} = |\underline{F}||\underline{s}| \cos\theta$$

and

$$\underline{F} \cdot \underline{s} = (|\underline{F}| \cos\theta)\ |\underline{s}|$$



Note: $\underline{F} \cdot \underline{s}$ is the product of $|\underline{s}|$ and the **projected component** of force $|\underline{F}| \cos\theta$ along the direction of $\underline{s}$.

Return to Exercise 10

Exercise 11.

$\underline{F} \cdot \underline{s} = |\underline{F}||\underline{s}| \cos\theta$,

$$|\underline{F}| = 5 \text{ N}$$

$$|\underline{s}| = 4 \text{ m}$$

$$\theta = 48^\circ \text{ gives } \cos\theta \simeq 0.6691$$

$\therefore\ \underline{F} \cdot \underline{s} \simeq (5\text{ N})(4\text{ m})(0.6691) \simeq 13.38\text{ J}.$

Return to Exercise 11

Exercise 12.

$\underline{F} \cdot \underline{s} = |\underline{F}||\underline{s}| \cos\theta$,

$$|\underline{F}| = 2\ \mathrm{N}$$

$$|\underline{s}| = 3\ \mathrm{m}$$

$$\theta = 56^\circ \text{ gives } \cos\theta \simeq 0.5592$$

$\therefore\ \underline{F} \cdot \underline{s} \simeq (2\ \mathrm{N})(3\ \mathrm{m})(0.5592) \simeq 3.355\ \mathrm{J}.$

Return to Exercise 12

Exercise 13.

$\underline{a} \cdot \underline{b} = |\underline{a}||\underline{b}| \cos\theta$, where θ is the angle between vectors $\underline{a}$ and $\underline{b}$

$$\therefore \quad \cos\theta = \frac{\underline{a}\cdot\underline{b}}{|\underline{a}||\underline{b}|}$$

$$\begin{aligned} \text{If} \quad \underline{a} &= a_x\underline{i} + a_y\underline{j} + a_z\underline{k} \\ \underline{b} &= b_x\underline{i} + b_y\underline{j} + b_z\underline{k} \end{aligned}$$

then $\cos\theta = \frac{\underline{a}\cdot\underline{b}}{|\underline{a}||\underline{b}|}$ where

$$\begin{aligned} \underline{a}\cdot\underline{b} &= a_xb_x + a_yb_y + a_zb_z \\ |\underline{a}| &= \sqrt{a_x^2 + a_y^2 + a_z^2} \\ |\underline{b}| &= \sqrt{b_x^2 + b_y^2 + b_z^2} \end{aligned}$$

$$\underline{a} = 2\underline{i} - \underline{j} + 2\underline{k}\ , \quad \underline{b} = \underline{i} + \underline{j} + \underline{k}$$

i.e. $a_x = 2,\ a_y = -1,\ a_z = 2$ and $b_x = 1,\ b_y = 1,\ b_z = 1$

then

$$\underline{a} \cdot \underline{b} = (2)(1) + (-1)(1) + (2)(1) = 2 - 1 + 2 = 3$$

$$|\underline{a}| = \sqrt{2^2 + (-1)^2 + 2^2} = \sqrt{4 + 1 + 4} = \sqrt{9} = 3$$

$$|\underline{b}| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$$

$$\therefore \quad \cos\theta = \frac{\underline{a} \cdot \underline{b}}{|\underline{a}||\underline{b}|} = \frac{3}{3\sqrt{3}} \simeq 0.5774$$

so $\theta \simeq \cos^{-1}(0.5774) \simeq 54.7^\circ$. Return to Exercise 13

Exercise 14.

$\underline{a} \cdot \underline{b} = |\underline{a}||\underline{b}| \cos\theta$, where θ is the angle between vectors $\underline{a}$ and $\underline{b}$

$$\therefore \quad \cos\theta = \frac{\underline{a} \cdot \underline{b}}{|\underline{a}||\underline{b}|}$$

$$\begin{aligned} \text{If} \quad \underline{a} &= a_x\underline{i} + a_y\underline{j} + a_z\underline{k} \\ \underline{b} &= b_x\underline{i} + b_y\underline{j} + b_z\underline{k} \end{aligned}$$

then $\cos\theta = \frac{\underline{a} \cdot \underline{b}}{|\underline{a}||\underline{b}|}$ where

$$\begin{aligned} \underline{a} \cdot \underline{b} &= a_x b_x + a_y b_y + a_z b_z \\ |\underline{a}| &= \sqrt{a_x^2 + a_y^2 + a_z^2} \\ |\underline{b}| &= \sqrt{b_x^2 + b_y^2 + b_z^2} \end{aligned}$$

$$\underline{a} = \underline{i} + \underline{j} + \underline{k}\ , \quad \underline{b} = 2\underline{i} - 3\underline{j} + \underline{k}$$

i.e. $a_x = 1,\ a_y = 1,\ a_z = 1$ and $b_x = 2,\ b_y = -3,\ b_z = 1$

then

$$\underline{a} \cdot \underline{b} = (1)(2) + (1)(-3) + (1)(1) = 2 - 3 + 1 = 0$$

$$|\underline{a}| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$$

$$|\underline{b}| = \sqrt{2^2 + (-3)^2 + 1^2} = \sqrt{4 + 9 + 1} = \sqrt{14}$$

$$\therefore \quad \cos\theta = \frac{\underline{a}\cdot\underline{b}}{|\underline{a}||\underline{b}|} = \frac{0}{\sqrt{3}\sqrt{14}} = \frac{0}{\sqrt{42}} = 0$$

so $\theta = \cos^{-1}(0) = 90^\circ.$

Return to Exercise 14

Exercise 15.

$\underline{a} \cdot \underline{b} = |\underline{a}||\underline{b}| \cos\theta$, where θ is the angle between vectors $\underline{a}$ and $\underline{b}$

$$\therefore \quad \cos\theta = \frac{\underline{a}\cdot\underline{b}}{|\underline{a}||\underline{b}|}$$

$$\begin{aligned} \text{If} \quad \underline{a} &= a_x\underline{i} + a_y\underline{j} + a_z\underline{k} \\ \underline{b} &= b_x\underline{i} + b_y\underline{j} + b_z\underline{k} \end{aligned}$$

then $\cos\theta = \frac{\underline{a}\cdot\underline{b}}{|\underline{a}||\underline{b}|}$ where

$$\begin{aligned} \underline{a}\cdot\underline{b} &= a_xb_x + a_yb_y + a_zb_z \\ |\underline{a}| &= \sqrt{a_x^2 + a_y^2 + a_z^2} \\ |\underline{b}| &= \sqrt{b_x^2 + b_y^2 + b_z^2} \end{aligned}$$

$$\underline{a} = \underline{i} - 2\underline{j} + 2\underline{k}\ , \quad \underline{b} = 2\underline{i} + 3\underline{j} + \underline{k}$$

i.e. $a_x = 1,\ a_y = -2,\ a_z = 2$ and $b_x = 2,\ b_y = 3,\ b_z = 1$

then

$$\underline{a} \cdot \underline{b} = (1)(2) + (-2)(3) + (2)(1) = 2 - 6 + 2 = -2$$

$$|\underline{a}| = \sqrt{1^2 + (-2)^2 + 2^2} = \sqrt{1 + 4 + 4} = \sqrt{9} = 3$$

$$|\underline{b}| = \sqrt{2^2 + 3^2 + 1^2} = \sqrt{4 + 9 + 1} = \sqrt{14}$$

$$\therefore \quad \cos\theta = \frac{\underline{a}\cdot\underline{b}}{|\underline{a}||\underline{b}|} = \frac{-2}{3\sqrt{14}} \simeq -0.1782$$

so $\quad \theta \simeq \cos^{-1}(-0.1782) \simeq 100.3^\circ,$

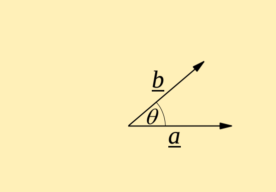
see also the Note on the next page . . .

Note:

When $\underline{a} \cdot \underline{b}$ is positive

$\cos\theta$ is positive

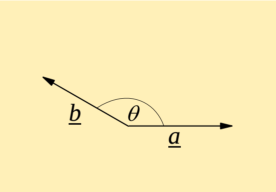
and θ is an acute angle



When $\underline{a} \cdot \underline{b}$ is negative

$\cos\theta$ is negative

and θ is an obtuse angle



End of Note. Return to Exercise 15

Exercise 16.

$\underline{a} \cdot \underline{b} = |\underline{a}||\underline{b}| \cos\theta$, where θ is the angle between vectors $\underline{a}$ and $\underline{b}$

$$\therefore \quad \cos\theta = \frac{\underline{a}\cdot\underline{b}}{|\underline{a}||\underline{b}|}$$

$$\begin{aligned} \text{If} \quad \underline{a} &= a_x\underline{i} + a_y\underline{j} + a_z\underline{k} \\ \underline{b} &= b_x\underline{i} + b_y\underline{j} + b_z\underline{k} \end{aligned}$$

then $\cos\theta = \frac{\underline{a}\cdot\underline{b}}{|\underline{a}||\underline{b}|}$ where

$$\begin{aligned} \underline{a}\cdot\underline{b} &= a_xb_x + a_yb_y + a_zb_z \\ |\underline{a}| &= \sqrt{a_x^2 + a_y^2 + a_z^2} \\ |\underline{b}| &= \sqrt{b_x^2 + b_y^2 + b_z^2} \end{aligned}$$

$$\underline{a} = 5\underline{i} + 4\underline{j} + 3\underline{k}\ , \quad \underline{b} = 4\underline{i} - 5\underline{j} + 3\underline{k}$$

i.e. $a_x = 5,\ a_y = 4,\ a_z = 3$ and $b_x = 4,\ b_y = -5,\ b_z = 3$

then

$$\underline{a} \cdot \underline{b} = (5)(4) + (4)(-5) + (3)(3) = 20 - 20 + 9 = 9$$

$$|\underline{a}| = \sqrt{5^2 + 4^2 + 3^2} = \sqrt{25 + 16 + 9} = \sqrt{50}$$

$$|\underline{b}| = \sqrt{4^2 + (-5)^2 + 3^2} = \sqrt{16 + 25 + 9} = \sqrt{50}$$

$$\therefore \quad \cos\theta = \frac{\underline{a}\cdot\underline{b}}{|\underline{a}||\underline{b}|} = \frac{9}{\sqrt{50}\sqrt{50}} = \frac{9}{50} = 0.18$$

so $\theta = \cos^{-1}(0.18) \simeq 79.6^\circ$.

Return to Exercise 16