Vectors

VECTOR PRODUCT

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A Tutorial Module for learning about the vector product of two vectors

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1. Theory

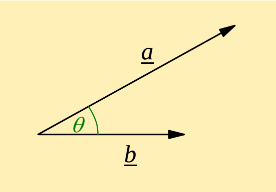
The purpose of this tutorial is to practice working out the vector product of two vectors.

It is called the 'vector product' because the result is a 'vector', i.e. a quantity with both (i) **magnitude**, and (ii) **direction**.

(i) The **MAGNITUDE** of the **vector product** of $\underline{a}$ and $\underline{b}$ is

$$|\underline{a} \times \underline{b}| = |\underline{a}||\underline{b}| \sin\theta$$

where θ is the angle between $\underline{a}$ and $\underline{b}$.

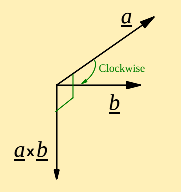


(ii) The **DIRECTION** of the **vector product** of $\underline{a}$ and $\underline{b}$ is **perpendicular** to both $\underline{a}$ and $\underline{b}$,

such that if we look along $\underline{a} \times \underline{b}$

then $\underline{a}$ rotates towards $\underline{b}$

in a clockwise manner.



It can be shown that the above definitions of magnitude and direction of a vector product allow us to calculate the x, y and z **components** of $\underline{a} \times \underline{b}$ from the individual components of the vectors $\underline{a}$ and $\underline{b}$

The **components of the vector** $\underline{a} \times \underline{b}$ are given by the 'determinant' of a matrix with 3 rows.

The components of $\underline{a} = a_x\underline{i} + a_y\underline{j} + a_z\underline{k}$ and $\underline{b} = b_x\underline{i} + b_y\underline{j} + b_z\underline{k}$ appear in the 2nd and 3rd rows.

This 3-row determinant is evaluated by expansion into 2-row determinants, which are themselves then expanded. Matrix theory is itself very useful, and the scheme is shown below

$$\begin{aligned}
\underline{a} \times \underline{b} &= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} \\
&= \underline{i}\begin{vmatrix} a_y & a_z \\ b_y & b_z \end{vmatrix} - \underline{j}\begin{vmatrix} a_x & a_z \\ b_x & b_z \end{vmatrix} + \underline{k}\begin{vmatrix} a_x & a_y \\ b_x & b_y \end{vmatrix} \\
&= \underline{i}\,(a_yb_z - a_zb_y) - \underline{j}\,(a_xb_z - a_zb_x) + \underline{k}\,(a_xb_y - a_yb_x)
\end{aligned}$$

2. Exercises

Click on EXERCISE links for full worked solutions (there are 14 exercises in total).

EXERCISE 1. Calculate $|\underline{a} \times \underline{b}|$ when $|\underline{a}| = 2$, $|\underline{b}| = 4$ and the angle between $\underline{a}$ and $\underline{b}$ is $\theta = 45°$

EXERCISE 2. Calculate $|\underline{a} \times \underline{b}|$ when $|\underline{a}| = 3$, $|\underline{b}| = 5$ and the angle between $\underline{a}$ and $\underline{b}$ is $\theta = 60°$

EXERCISE 3. Calculate $|\underline{a} \times \underline{b}|$ when $|\underline{a}| = 1$, $|\underline{b}| = 3$ and the angle between $\underline{a}$ and $\underline{b}$ is $\theta = 30°$

EXERCISE 4. Calculate $|\underline{a} \times \underline{b}|$ when $|\underline{a}| = 2$, $|\underline{b}| = 5$ and the angle between $\underline{a}$ and $\underline{b}$ is $\theta = 35°$

Exercise 5. Calculate the magnitude of the torque $\underline{\tau} = \underline{s} \times \underline{F}$ when $|\underline{s}| = 2\,\text{m}$, $|\underline{F}| = 4\,\text{N}$ and $\theta = 30^\circ$ (θ is the angle between the position vector $\underline{s}$ and the force $\underline{F}$)

Exercise 6. Calculate the magnitude of the velocity $\underline{v} = \underline{\omega} \times \underline{s}$ when $|\underline{\omega}| = 3\,\text{s}^{-1}$, $|\underline{s}| = 2\,\text{m}$ and $\theta = 45^\circ$ (θ is the angle between the angular velocity $\underline{\omega}$ and the position vector $\underline{s}$)

Exercise 7. If $\underline{a} = 4\underline{i} + 2\underline{j} - \underline{k}$ and $\underline{b} = 2\underline{i} - 6\underline{i} - 3\underline{k}$ then calculate a vector that is perpendicular to both $\underline{a}$ and $\underline{b}$

Exercise 8. Calculate the vector $\underline{a} \times \underline{b}$ when $\underline{a} = 2\underline{i} + \underline{j} - \underline{k}$ and $\underline{b} = 3\underline{i} - 6\underline{j} + 2\underline{k}$

Exercise 9. Calculate the vector $\underline{a} \times \underline{b}$ when $\underline{a} = 3\underline{i} + 4\underline{j} - 3\underline{k}$ and $\underline{b} = \underline{i} + 3\underline{j} + 2\underline{k}$

Exercise 10. Calculate the vector $\underline{a} \times \underline{b}$ when $\underline{a} = \underline{i} + 2\underline{j} - \underline{k}$ and $\underline{b} = 3\underline{i} + 3\underline{j} + \underline{k}$

Exercise 11. Calculate the vector $\underline{a} \times \underline{b}$ when $\underline{a} = 2\underline{i} + 4\underline{j} + 2\underline{k}$ and $\underline{b} = \underline{i} + 5\underline{j} - 2\underline{k}$

Exercise 12. Calculate the vector $\underline{a} \times \underline{b}$ when $\underline{a} = 3\underline{i} - 4\underline{j} + \underline{k}$ and $\underline{b} = 2\underline{i} + 5\underline{j} - \underline{k}$

Exercise 13. Calculate the vector $\underline{a} \times \underline{b}$ when $\underline{a} = 2\underline{i} - 3\underline{j} + \underline{k}$ and $\underline{b} = 2\underline{i} + \underline{j} + 4\underline{k}$

Exercise 14. Calculate the vector $\underline{a} \times \underline{b}$ when $\underline{a} = 2\underline{i} + 3\underline{k}$ and $\underline{b} = \underline{i} + 2\underline{j} + 4\underline{k}$

3. Answers

1. $4\sqrt{2}$,
2. $\frac{15}{2}\sqrt{3}$,
3. $\frac{3}{2}$,
4. 5.766,
5. 4 J,
6. $3\sqrt{2}\,\mathrm{m\,s^{-1}}$,
7. $-12\underline{i} + 10\underline{j} - 28\underline{k}$,
8. $-4\underline{i} - 7\underline{j} - 15\underline{k}$,
9. $17\underline{i} - 9\underline{j} + 5\underline{k}$,
10. $5\underline{i} - 4\underline{j} - 4\underline{k}$,
11. $-18\underline{i} + 6\underline{j} + 6\underline{k}$,
12. $-\underline{i} + 5\underline{j} + 23\underline{k}$,
13. $-13\underline{i} - 6\underline{j} + 8\underline{k}$,
14. $-6\underline{i} - 5\underline{j} + 4\underline{k}$.

4. Tips on using solutions

● When looking at the THEORY, ANSWERS, TIPS or NOTATION pages, use the Back button (at the bottom of the page) to return to the exercises

● Use the solutions intelligently. For example, they can help you get started on an exercise, or they can allow you to check whether your intermediate results are correct

● Try to make less use of the full solutions as you work your way through the Tutorial

5. Alternative notation

● Here, we use symbols like $\underline{a}$ to denote a vector.
In some texts, symbols for vectors are **in bold** (eg **a** instead of $\underline{a}$)

● In this Tutorial, vectors are given in terms of the unit Cartesian vectors $\underline{i}$, $\underline{j}$ and $\underline{k}$. A common alternative notation involves quoting the **Cartesian components** within brackets. For example, the vector $\underline{a} = 2\underline{i} + \underline{j} + 5\underline{k}$ can be written as $\underline{a} = (2, 1, 5)$

● The scalar product $\underline{a} \cdot \underline{b}$ is also called a 'dot product' (reflecting the symbol used to denote this type of multiplication). Likewise, the vector product $\underline{a} \times \underline{b}$ is also called a '**cross product**'

● An alternative notation for the vector product is $\underline{a} \wedge \underline{b}$

Full worked solutions

Exercise 1.

Here, we have $|\underline{a}| = 2$, $|\underline{b}| = 4$ and $\theta = 45^\circ$.

$$
\begin{aligned}
|\underline{a} \times \underline{b}| &= |\underline{a}||\underline{b}| \sin\theta \\
&= (2)(4) \sin 45^\circ \\
&= (2)(4) \frac{1}{\sqrt{2}} \\
&= \frac{8}{\sqrt{2}} \\
&= \frac{8}{\sqrt{2}} \frac{\sqrt{2}}{\sqrt{2}} \\
&= \frac{8\sqrt{2}}{2} \\
&= 4\sqrt{2}.
\end{aligned}
$$

Return to Exercise 1

Exercise 2.

Here, we have $|\underline{a}| = 3$, $|\underline{b}| = 5$ and $\theta = 60^\circ$.

$$\begin{aligned} |\underline{a} \times \underline{b}| &= |\underline{a}||\underline{b}| \sin\theta \\ &= (3)(5)\sin 60^\circ \\ &= (3)(5)\frac{\sqrt{3}}{2} \\ &= \frac{15}{2}\sqrt{3}. \end{aligned}$$

Return to Exercise 2

Exercise 3.

Here, we have $|\underline{a}| = 1$, $|\underline{b}| = 3$ and $\theta = 30°$.

$$\begin{aligned} |\underline{a} \times \underline{b}| &= |\underline{a}||\underline{b}| \sin\theta \\ &= (1)(3) \sin 30° \\ &= (1)(3)\frac{1}{2} \\ &= \frac{3}{2}. \end{aligned}$$

Return to Exercise 3

Exercise 4.

Here, we have $|\underline{a}| = 2$, $|\underline{b}| = 5$ and $\theta = 35^\circ$.

$$\begin{aligned} |\underline{a} \times \underline{b}| &= |\underline{a}||\underline{b}| \sin\theta \\ &= (2)(5)\sin 35^\circ \\ &\simeq (2)(5)(0.5736) \\ &= 5.736. \end{aligned}$$

Return to Exercise 4

Exercise 5.

Here, we have $|\underline{s}| = 2\text{ m}$, $|\underline{F}| = 4\text{ N}$ and $\theta = 30^\circ$.

$$\begin{aligned}
|\tau| &= |\underline{s} \times \underline{F}| \\
&= |\underline{s}||\underline{F}| \sin\theta \\
&= (2\text{ m})(4\text{ N}) \sin 30^\circ \\
&= (2\text{ m})(4\text{ N}) \frac{1}{2} \\
&= (8)(\text{N m}) \frac{1}{2} \\
&= (8)(\text{J}) \frac{1}{2} \\
&= 4\text{ J}.
\end{aligned}$$

Return to Exercise 5

Exercise 6.

Here, we have $|\underline{\omega}| = 3\,\mathrm{s}^{-1}$, $|\underline{s}| = 2\,\mathrm{m}$ and $\theta = 45^\circ$.

$$\begin{aligned}
|\tau| &= |\underline{\omega} \times \underline{s}| \\
&= |\underline{\omega}||\underline{s}|\sin\theta \\
&= (3\,\mathrm{s}^{-1})(2\,\mathrm{m})\sin 45^\circ \\
&= (3\,\mathrm{s}^{-1})(2\,\mathrm{m})\frac{1}{\sqrt{2}} \\
&= (6)(\mathrm{m\,s}^{-1})\frac{1}{\sqrt{2}}\frac{\sqrt{2}}{\sqrt{2}} \\
&= (6)(\mathrm{m\,s}^{-1})\frac{\sqrt{2}}{2} \\
&= 3\sqrt{2}\,\mathrm{m\,s}^{-1}.
\end{aligned}$$

Return to Exercise 6

Exercise 7.

$\underline{a} \times \underline{b}$ is a vector that is perpendicular to both $\underline{a}$ and $\underline{b}$.
Here, we have $\underline{a} = 4\underline{i} + 2\underline{j} - \underline{k}$ and $\underline{b} = 2\underline{i} - 6\underline{j} - 3\underline{k}$.

$$\begin{aligned}
\underline{a} \times \underline{b} &= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 4 & 2 & -1 \\ 2 & -6 & -3 \end{vmatrix} \\
&= \underline{i}\begin{vmatrix} 2 & -1 \\ -6 & -3 \end{vmatrix} - \underline{j}\begin{vmatrix} 4 & -1 \\ 2 & -3 \end{vmatrix} + \underline{k}\begin{vmatrix} 4 & 2 \\ 2 & -6 \end{vmatrix} \\
&= \underline{i}[(2)(-3)-(-1)(-6)] - \underline{j}[(4)(-3)-(-1)(2)] + \underline{k}[(4)(-6)-(2)(2)] \\
\underline{a} \times \underline{b} &= \underline{i}[-6-6] - \underline{j}[-12+2] + \underline{k}[-24-4] \\
&= -12\underline{i} + 10\underline{j} - 28\underline{k}.
\end{aligned}$$

Return to Exercise 7

Exercise 8.

Here, we have $\underline{a} = 2\underline{i} + \underline{j} - \underline{k}$ and $\underline{b} = 3\underline{i} - 6\underline{j} + 2\underline{k}$

$$\begin{aligned}
\underline{a} \times \underline{b} &= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & 1 & -1 \\ 3 & -6 & 2 \end{vmatrix} \\
&= \underline{i}\begin{vmatrix} 1 & -1 \\ -6 & 2 \end{vmatrix} - \underline{j}\begin{vmatrix} 2 & -1 \\ 3 & 2 \end{vmatrix} + \underline{k}\begin{vmatrix} 2 & 1 \\ 3 & -6 \end{vmatrix} \\
&= \underline{i}[(1)(2)-(-1)(-6)] - \underline{j}[(2)(2)-(-1)(3)] + \underline{k}[(2)(-6)-(1)(3)] \\
\underline{a} \times \underline{b} &= \underline{i}[2-6] - \underline{j}[4+3] + \underline{k}[-12-3] \\
&= -4\underline{i} - 7\underline{j} - 15\underline{k}.
\end{aligned}$$

Return to Exercise 8

Exercise 9.

Here, we have $\underline{a} = 3\underline{i} + 4\underline{j} - 3\underline{k}$ and $\underline{b} = \underline{i} + 3\underline{j} + 2\underline{k}$

$$\begin{aligned}
\underline{a} \times \underline{b} &= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 3 & 4 & -3 \\ 1 & 3 & 2 \end{vmatrix} \\
&= \underline{i} \begin{vmatrix} 4 & -3 \\ 3 & 2 \end{vmatrix} - \underline{j} \begin{vmatrix} 3 & -3 \\ 1 & 2 \end{vmatrix} + \underline{k} \begin{vmatrix} 3 & 4 \\ 1 & 3 \end{vmatrix} \\
&= \underline{i}[(4)(2)-(-3)(3)] - \underline{j}[(3)(2)-(-3)(1)] + \underline{k}[(3)(3)-(4)(1)] \\
\underline{a} \times \underline{b} &= \underline{i}[8+9] - \underline{j}[6+3] + \underline{k}[9-4] \\
&= 17\underline{i} - 9\underline{j} + 5\underline{k}.
\end{aligned}$$

Return to Exercise 9

Exercise 10.

Here, we have $\underline{a} = \underline{i} + 2\underline{j} - \underline{k}$ and $\underline{b} = 3\underline{i} + 3\underline{j} + \underline{k}$

$$\begin{aligned}
\underline{a} \times \underline{b} &= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 2 & -1 \\ 3 & 3 & 1 \end{vmatrix} \\
&= \underline{i}\begin{vmatrix} 2 & -1 \\ 3 & 1 \end{vmatrix} - \underline{j}\begin{vmatrix} 1 & -1 \\ 3 & 1 \end{vmatrix} + \underline{k}\begin{vmatrix} 1 & 2 \\ 3 & 3 \end{vmatrix} \\
&= \underline{i}[(2)(1)-(-1)(3)] - \underline{j}[(1)(1)-(-1)(3)] + \underline{k}[(1)(3)-(2)(3)] \\
\text{i.e. } \underline{a} \times \underline{b} &= \underline{i}[2+3] - \underline{j}[1+3] + \underline{k}[3-6] \\
&= 5\underline{i} - 4\underline{j} - 3\underline{k}.
\end{aligned}$$

Return to Exercise 10

Exercise 11.

Here, we have $\underline{a} = 2\underline{i} + 4\underline{j} + 2\underline{k}$ and $\underline{b} = \underline{i} + 5\underline{j} - 2\underline{k}$

$$\begin{aligned}
\underline{a} \times \underline{b} &= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & 4 & 2 \\ 1 & 5 & -2 \end{vmatrix} \\
&= \underline{i} \begin{vmatrix} 4 & 2 \\ 5 & -2 \end{vmatrix} - \underline{j} \begin{vmatrix} 2 & 2 \\ 1 & -2 \end{vmatrix} + \underline{k} \begin{vmatrix} 2 & 4 \\ 1 & 5 \end{vmatrix} \\
&= \underline{i}[(4)(-2)-(2)(5)] - \underline{j}[(2)(-2)-(2)(1)] + \underline{k}[(2)(5)-(4)(1)] \\
\underline{a} \times \underline{b} &= \underline{i}[-8-10] - \underline{j}[-4-2] + \underline{k}[10-4] \\
&= -18\underline{i} + 6\underline{j} + 6\underline{k}.
\end{aligned}$$

Return to Exercise 11

Exercise 12.

Here, we have $\underline{a} = 3\underline{i} - 4\underline{j} + \underline{k}$ and $\underline{b} = 2\underline{i} + 5\underline{j} - \underline{k}$

$$\underline{a} \times \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 3 & -4 & 1 \\ 2 & 5 & -1 \end{vmatrix}$$

$$= \underline{i}\begin{vmatrix} -4 & 1 \\ 5 & -1 \end{vmatrix} - \underline{j}\begin{vmatrix} 3 & 1 \\ 2 & -1 \end{vmatrix} + \underline{k}\begin{vmatrix} 3 & -4 \\ 2 & 5 \end{vmatrix}$$

$$= \underline{i}[(-4)(-1) - (1)(5)] - \underline{j}[(3)(-1) - (1)(2)] + \underline{k}[(3)(5) - (-4)(2)]$$

$$\underline{a} \times \underline{b} = \underline{i}[4 - 5] - \underline{j}[-3 - 2] + \underline{k}[15 + 8]$$

$$= -\underline{i} + 5\underline{j} + 23\underline{k}.$$

Return to Exercise 12

Exercise 13.

Here, we have $\underline{a} = 2\underline{i} - 3\underline{j} + \underline{k}$ and $\underline{b} = 2\underline{i} + \underline{j} + 4\underline{k}$

$$\begin{aligned}
\underline{a} \times \underline{b} &= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & -3 & 1 \\ 2 & 1 & 4 \end{vmatrix} \\
&= \underline{i}\begin{vmatrix} -3 & 1 \\ 1 & 4 \end{vmatrix} - \underline{j}\begin{vmatrix} 2 & 1 \\ 2 & 4 \end{vmatrix} + \underline{k}\begin{vmatrix} 2 & -3 \\ 2 & 1 \end{vmatrix} \\
&= \underline{i}[(-3)(4)-(1)(1)] - \underline{j}[(2)(4)-(1)(2)] + \underline{k}[(2)(1)-(-3)(2)] \\
\underline{a} \times \underline{b} &= \underline{i}[-12-1] - \underline{j}[8-2] + \underline{k}[2+6] \\
&= -13\underline{i} - 6\underline{j} + 8\underline{k}.
\end{aligned}$$

Return to Exercise 13

Exercise 14.

Here, we have $\underline{a} = 2\underline{i} + 3\underline{k} = 2\underline{i} + 0\underline{j} + 3\underline{k}$ and $\underline{b} = \underline{i} + 2\underline{j} + 4\underline{k}$

$$\begin{aligned}
\underline{a} \times \underline{b} &= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & 0 & 3 \\ 1 & 2 & 4 \end{vmatrix} \\
&= \underline{i} \begin{vmatrix} 0 & 3 \\ 2 & 4 \end{vmatrix} - \underline{j} \begin{vmatrix} 2 & 3 \\ 1 & 4 \end{vmatrix} + \underline{k} \begin{vmatrix} 2 & 0 \\ 1 & 2 \end{vmatrix} \\
&= \underline{i}[(0)(4)-(3)(2)] - \underline{j}[(2)(4)-(3)(1)] + \underline{k}[(2)(2)-(0)(1)] \\
\underline{a} \times \underline{b} &= \underline{i}[0-6] - \underline{j}[8-3] + \underline{k}[4-0] \\
&= -6\underline{i} - 5\underline{j} + 4\underline{k}.
\end{aligned}$$

Return to Exercise 14